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SOLUTIONS OF HOMOGENEOUS ELLIPTIC EQUATIONS

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We consider an elliptic partial differential equation with constant coefficients and zero on the right hand side. It is well known [1] that every solution of such an equation can be approximated uniformly on each compact set by a sum of products of polynomials and exponential functions which satisfy the equation. Furthermore, if one assumes that the polynomial operator is homogeneous, then the approximation can be made with polynomials alone. It is our purpose to show, in the latter case, when the number of variables is two, that each solution can be written as an infinite series in certain specific polynomials. Our method is to factor the polynomial and build up the solution in terms of solutions of first degree equations.

We shall denote by $P=P(s_1,\ldots,s_n)$ a polynomial in n complex variables. The corresponding partial differential equation is

$$P\left(i\frac{\partial}{\partial x}\right)u = P\left(i\frac{\partial}{\partial x_1},\ldots,i\frac{\partial}{\partial x_n}\right)u = 0.$$

We shall always assume that the polynomial is *elliptic* which means that the solution u is real analytic. For simplicity we consider only global solutions although it is not hard to see how our results can be generalized to the case of local solutions. Thus the functions u in this paper, will always be infinitely differentiable, complex valued functions on $\mathbf{R}^n$.

We say that a polynomial P of degree m is *homogeneous* if $P(\lambda s_1,\ldots,\lambda s_n)=\lambda^m P(s_1,\ldots,s_n)$ for all complex numbers λ. If P is any polynomial of degree m, we can always write $P=P_0+P_1$ where P_0 is homogeneous of degree m and the degree of P_1 is strictly less than m. We call P_0 the *principal part* of P. An important characterization of ellipticity is the following [1]:

The polynomial P is elliptic if and only if its principal part P_0 has the property that for any $\sigma\in\mathbf{R}^n$, $P_0(\sigma)=0$ if and only if $\sigma=0$.

If λ is a complex number, its conjugate will be denoted $\bar{\lambda}$ and i in the sequel will always stand for $\sqrt{-1}$.

PROPOSITION 1. *Let P, Q be polynomials. Then the product, PQ is elliptic if and only if both P and Q are.*

Proof. Let P_0, Q_0 be the principal parts of P, Q respectively. Then $P_0\,Q_0$ is the

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principal part of PQ. Then if PQ is elliptic and $P_0(\sigma)=0$ for $\sigma \in \mathbf{R}^n$, then $P_0(\sigma)Q_0(\sigma)=0$ so $\sigma=0$. Thus P is elliptic, and similarly, Q is elliptic.

Conversely if P and Q are elliptic and $P_0(\sigma)Q_0(\sigma)=0$, then either $P_0(\sigma)=0$ or $Q_0(\sigma)=0$ so $\sigma=0$. Thus PQ is elliptic.

PROPOSITION 2. *If $P(s_1, s_2)$ is a homogeneous polynomial of degree m, then P can be factored into linear factors of the form, $\alpha s_1+\beta s_2$.*

Proof. Let λ be any complex number with $P(1, \lambda)=0$. Then

$$P(s_1, \lambda s_1) = s_1^m P(1, \lambda) = 0.$$

Thus $s_2-\lambda s_1$ is a factor and the rest follows by induction.

PROPOSITION 3. *If $P(s_1, s_2)=\alpha s_1+\beta s_2$, then P is elliptic if and only if α, β are real linearly independent.*

Proof. Let $\alpha=\alpha_1+i\alpha_2$, $\beta=\beta_1+i\beta_2$ where α_1, α_2, β_1, β_2 are real. P is elliptic if and only if the following system of equations has no non-trivial solution:

$$\alpha_1\sigma_1+\beta_1\sigma_2 = 0$$
$$\alpha_2\sigma_1+\beta_2\sigma_2 = 0$$

But this is equivalent to the fact that (α_1, α_2), (β_1, β_2) are linearly independent.

Let $A: \mathbf{C}^n \to \mathbf{C}^n$ be an invertible linear transformation whose matrix with respect to the usual basis is (a_{kj}) and all a_{kj} are real. Let $A^T=(a^T_{kj})$, $a^T_{kj}=a_{jk}$, be the transpose of A. We wish to apply such a transformation to a polynomial. The next result describes the effect on the solution of the corresponding partial differential equation.

PROPOSITION 4. *Let P, Q be polynomials with $Q=P\circ A^T$. Then $Q(i\,\partial/\partial x)u=0$ if and only if $P(i\,\partial/\partial x)(u\circ A)=0$.*

Proof. Let $v=u\circ A$, and suppose that $P(i\,\partial/\partial x)v=0$. Since

$$\frac{\partial v}{\partial x_j}(x) = \sum_{k=1}^{n} a_{kj}\frac{\partial u}{\partial x_k}(A(x)) = \sum_{k=1}^{n} a^T_{jk}\frac{\partial u}{\partial x_k}(A(x)),$$

then

$$\left(i\frac{\partial}{\partial x}\right)v(x) = A^T\left(i\frac{\partial}{\partial x}\right)u(A(x))$$

and

$$0 = P\left(i\frac{\partial}{\partial x}\right)v(x) = P\circ A^T\left(i\frac{\partial}{\partial x}\right)u(A(x)) = Q\left(i\frac{\partial}{\partial x}\right)u(A(x)).$$

Since this is true for all x and A is invertible, it follows that $Q(i\,\partial/\partial x)u(x)=0$ for all x, i.e. $Q(i\,\partial/\partial x)u=0$.

To prove the converse, one applies the above argument to A^{-1}.

PROPOSITION 5. *Let* $P(s_1, s_2)=\alpha s_1+\beta s_2$ *be elliptic. Then there exists an invertible real linear transformation, A such that* $P \circ A^T$ *is the polynomial* s_1+is_2.

Proof. It suffices to let A^T be the matrix which transforms the vectors (α_1, β_1), (α_2, β_2) into the vectors $(1, 0)$ and $(0, 1)$. This is always possible since by Proposition 3 and our assumption of ellipticity, the two vectors are linearly independent. Thus we will have,

$$\begin{aligned} P \circ A^T(s_1, s_2) &= (\alpha_1+i\alpha_2)(a_{11}s_1+a_{21}s_2)+(\beta_1+i\beta_2)(a_{12}s_1+a_{22}s_2) \\ &= (a_{11}\alpha_1+a_{12}\beta_1)s_1+(a_{21}\alpha_1+a_{22}\beta_1)s_2 \\ &\qquad\qquad +i[(a_{11}\alpha_2+a_{12}\beta_2)s_1+(a_{21}\alpha_2+a_{22}\beta_2)s_2] \\ &= s_1+is_2 \end{aligned}$$

THEOREM 1. *Let* $P(s_1, s_2)=(\alpha s_1+\beta s_2)^k$ *be elliptic. Let* $d=\alpha_1\beta_2-\alpha_2\beta_1$ *where* $\alpha=\alpha_1+i\alpha_2$, $\beta=\beta_1+i\beta_2$, *and let* $\gamma=1/d(\beta_1-i\alpha_1)$, $\delta=1/d(\beta_2-i\alpha_2)$. *Then* $Pu=0$ *if and only if*

$$u(x_1, x_2) = \sum_{j=0}^{k-1} (\bar{\delta}x_1-\bar{\gamma}x_2)^j \sum_{n=0}^{\infty} a_n^j(\delta x_1-\gamma x_2)^n, \tag{1}$$

where each $\{a_n^j\}$, $j=0, \ldots, k-1$, *is an infinite sequence of complex numbers satisfying,*

$$\lim_{n\to\infty} |a_n^j|^{1/n} = 0.$$

Proof. First we assume that $\alpha=1$, $\beta=i$. Then (1) becomes

$$u(x_1, x_2) = \sum_{j=0}^{k-1} (x_1-ix_2)^j \sum_{n=0}^{\infty} a_n^j(x_1+ix_2)^n.$$

Let $Q(s_1, s_2)=s_1+is_2$, so that $P=Q^k$ and $Q(i\,\partial/\partial x)u=i\,\partial u/\partial x_1-\partial u/\partial x_2$. Thus $Q(i\,\partial/\partial x)u=0$ is the Cauchy-Riemann equation, so if we let

$$f_j(x_1, x_2) = \sum_{n=0}^{\infty} a_n^j(x_1+ix_2)^n,$$

then $Q(i\,\partial/\partial x)f_j=0$. Also, $Q(i\,\partial/\partial x)(x_1-ix_2)^j=2ij(x_1-ix_2)^{j-1}$. Further, if g is any C^∞-function and f is analytic, then

$$Q(gf) = gQf+fQg = fQg$$

so

$$P(gf) = Q^k(gf) = fQ^kg.$$

But if $g(x_1, x_2)=(x_1-ix_2)^j$ with $j<k$ then $Q^kg=0$ and hence, $Pu=0$ if u has the given form.

For the converse, we proceed by induction on k. For $k=1$, the conclusion follows from the power series expansion of entire functions. Suppose the result is true for k and let $Q^{k+1}u=0$. Then $Q^k(Qu)=0$ so we have, by assumption,

$$Qu(x_1, x_2) = \sum_{j=0}^{k-1} (x_1-ix_2)^j \sum_{n=0}^{\infty} a_n^j(x_1+ix_2)^n = f(x_1, x_2).$$

Now let

$$u^0(x_1, x_2) = -\sum_{j=1}^{k} \frac{i}{2j}(x_1 - ix_2)^j \sum_{n=0}^{\infty} a_n^{j-1}(x_1+ix_2)^n.$$

Then employing the above relations,

$$Qu^0(x_1, x_2) = -\sum_{j=1}^{k} \frac{i}{2j} Q(x_1 - ix_2)^j \sum_{n=0}^{\infty} a_n^{j-1}(x_1+ix_2)^n$$

$$= \sum_{j=1}^{k} (x_1 - ix_2)^{j-1} \sum_{n=0}^{\infty} a_n^{j-1}(x_1+ix_2)^n = f(x_1, x_2).$$

Thus we can conclude that $u = u^0 + u^1$ where $Qu^1 = 0$, that is

$$u^1(x_1, x_2) = \sum_{n=0}^{\infty} b_n(x_1+ix_2)^n.$$

Adding the expressions for u^0 and u^1, one obtains the desired expression for u.

Finally we apply Propositions 4 and 5 to obtain (1). To see this, we compute,

$$A^T = \begin{pmatrix} \frac{\beta_2}{d} & -\frac{\alpha_2}{d} \\ -\frac{\beta_1}{d} & \frac{\alpha_1}{d} \end{pmatrix}.$$

So by Proposition 4, we see that $x_1 + ix_2$ must be replaced by

$$\frac{1}{d}(\beta_2 x_1 - \beta_1 x_2 - i\alpha_2 x_1 + i\alpha_1 x_2) = \delta x_1 - \gamma x_2$$

and similarly for the terms involving $x_1 - ix_2$.

LEMMA 1. *Let $P_1(s) = s_1 + is_2$, $P_2(s) = \alpha s_1 + \beta s_2$ where (α, β), $(1, i)$ are linearly independent in $\mathbf{C}^2$. Let k_1 be a non-negative integer. Let f be a solution of $P_2 f = 0$. Then there exists a solution g of $P_2 g = 0$ with $P_1^{k_1} g = f$.*

Proof. Applying Theorem 1, we can write

$$f(x_1, x_2) = \sum_{n=0}^{\infty} a_n(\delta x_1 - \gamma x_2)^n, \lim_{n\to\infty} |a_n|^{1/n} = 0,$$

where δ and γ are defined in Theorem 1. First we observe that $i\delta + \gamma \neq 0$. For if $i\delta + \gamma = 0$, then $(\alpha, \beta) = (\alpha, i\alpha) = \alpha(1, i)$, which is contrary to our assumption. Furthermore, since $\lim_{n\to\infty} |a_n|^{1/n} = 0$, it is easy to see that

$$\lim_{n\to\infty} |a_{n-k_1}|^{1/n} \left| \frac{(n-k_1)!}{n!} \right|^{1/n} = 0.$$

So we may define

$$g(x_1, x_2) = (i\delta+\gamma)^{-k_1} \sum_{n=k_1}^{\infty} a_{n-k_1} \frac{(n-k_1)!}{n!} (\delta x_1 - \gamma x_2)^n,$$

and conclude from Theorem 1 that $P_2 g = 0$. Also,

$$P_1^{k_1} g = \sum_{n=k_1}^{\infty} a_{n-k_1} (\delta x_1 - \gamma x_2)^{n-k_1} = f.$$

LEMMA 2. *In the context of Lemma* 1, *let w be a solution of $P_2^{k_2} w = 0$ for some non-negative integer $k_2 \le k_1$. Then there exists a solution v of $P_2^{k_2} v = 0$ such that $P_1^{k_1} v = w$.*

Proof. Applying Theorem 1, we can write

$$w = \sum_{j=0}^{k_2-1} (\bar{\delta} x_1 - \bar{\gamma} x_2)^j f_j,$$

where $P_2 f_j = 0$. If v has the form

$$v = \sum_{j=0}^{k_2-1} (\bar{\delta} x_1 - \bar{\gamma} x_2)^j g_j,$$

where $P_2 g_j = 0$, then it follows by Theorem 1 that $P_2^{k_2} v = 0$. Thus we need only determine the functions, $g_0, \ldots, g_{k_2-1}$. This we do by utilizing the condition that $P_1^{k_1} v = w$ must hold. Thus we must have

$$\begin{aligned} P_1^{k_1} v &= \sum_{\mu=0}^{k_1} \sum_{j=\mu}^{k_2-1} \binom{k_1}{\mu} \frac{j!}{(j-\mu)!} (i\bar{\delta}+\bar{\gamma})^{\mu} (\bar{\delta} x_1 - \bar{\gamma} x_2)^{j-\mu} P_1^{k_1-\mu} g_j \\ &= \sum_{j=0}^{k_2-1} (\bar{\delta} x_1 - \bar{\gamma} x_2)^j f_j, \end{aligned}$$

where $P_1^0 g_j = g_j$. Rearranging terms and equating coefficients of $(\bar{\delta} x_1 - \bar{\gamma} x_2)^j$ on both sides of this equation, we obtain the system of equations:

$$P_1^{k_1} g_{k_2-1} = f_{k_2-1},$$

$$P_1^{k_1} g_{k_2-\nu} + \sum_{\mu=1}^{\min(\nu-1, k_1)} C_{\mu,\nu} P_1^{k_1-\mu} g_{k_2-\nu+\mu} = f_{k_2-\nu}, \quad \nu = 2, \ldots, k_2,$$

where the $C_{\mu,\nu}$ are complex numbers. Now the first equation can be solved for g_{k_2-1} by Lemma 1. Substituting this result in the second equation with $\nu=2$, we may apply Lemma 1 again and solve for g_{k_2-2}. Repeating this process, we obtain the desired $g_0, \ldots, g_{k_2-1}$.

LEMMA 3. *Let $P_\nu(s) = (\alpha^\nu s_1 + \beta^\nu s_2)$, $\nu = 1, \ldots, r$ and assume that the pairs (α^ν, β^ν), (α^μ, β^μ), $\nu \ne \mu$ are linearly independent in $\mathbf{C}^2$. Let $P = P_1^{k_1} \ldots P_r^{k_r}$ and suppose u is a solution of $Pu = 0$. Then we can write $u = u_1 + \cdots + u_r$ where $P_\nu^{k_\nu} u_\nu = 0$, $\nu = 1, \ldots, r$.*

Proof. Without loss of generality, we may assume that $k_1 \ge k_2 \ge \cdots \ge k_r$, since P is invariant under a permutation of its factors. We use induction on r. The result is trivial for $r=1$, so we assume that it holds for $r-1$. Without loss of generality, we may assume that $(\alpha^1, \beta^1)=(1, i)$. Now since $Pu=0$, it follows that $P_2^{k_2}\ldots P_r^{k_r}(P_1^{k_1}u)=0$. So by our induction hypothesis, we can write

$$P_1^{k_1}u = w_2+\cdots+w_r, \text{ where } P_\nu^{k_\nu}w_\nu = 0, \nu = 2,\ldots, r.$$

Then by Lemma 2, we have for each $\nu=2,\ldots, r$, a solution u_ν of $P_\nu^{k_\nu}u_\nu=0$ such that $P_1^{k_1}u_\nu=w_\nu$. Now if we write $u_1=u-(u_2+\cdots+u_r)$, then $u=u_1+u_2+\cdots+u_r$ where $P_\nu^{k_\nu}u_\nu=0$ for $\nu=2,\ldots, r$, and

$$P_1^{k_1}u_1 = P_1^{k_1}u-(P_1^{k_1}u_2+\cdots+P_1^{k_1}u_r) = (w_2+\cdots+w_r)-(w_2+\cdots+w_r) = 0.$$

THEOREM 2. *Let P be a homogeneous elliptic polynomial in two variables. Let r be the number of distinct linear factors of P and let their corresponding multiplicities be $k_1,\ldots, k_r$, arranged in decreasing order. Then there are complex numbers $\gamma^1,\ldots,\gamma^r$, $\delta^1,\ldots,\delta^r$ such that $P(i\,\partial/\partial x)u=0$ if and only if*

$$u(x_1, x_2) = \sum_{\nu=1}^{r}\sum_{j=0}^{k_\nu-1}(\bar{\delta}^\nu x_1-\bar{\gamma}^\nu x_2)^j\sum_{n=0}^{\infty}a_n^{j,\nu}(\delta^\nu x_1-\gamma^\nu x_2)^n \tag{2}$$

where each sequence $\{a_n^{j,\nu}\}$ satisfies $\lim_{n\to\infty}|a_n^{j,\nu}|^{1/n} = 0$.

Proof. By hypothesis,

$$P(s_1, s_2) = \prod_{\nu=1}^{r}(a^\nu s_1+\beta^\nu s_2)^{k_\nu} = \prod_{\nu=1}^{r}P_\nu(s_1, s_2), \text{ say.}$$

By Proposition 1 each P_ν is elliptic. Then we obtain γ^ν, δ^ν from α^ν, β^ν as in Theorem 1 and write all solutions of $P_\nu u=0$ from Theorem 1. Finally, we apply Lemma 3 to obtain all solutions of $Pu=0$ in the form of (2).

It can be shown from Theorem 2 that if one equips the space of solutions of a homogeneous elliptic equation in two variables with the compact open topology, then this space has a Schauder basis and is in fact isomorphic to the space of entire functions. The details will be presented in a subsequent paper.

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