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EVANS-KURAMOCHI EXHAUSTION FUNCTIONS ON NON-ALGEBROID RIEMANN SURFACES

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1. Let $\mathfrak{A}$ denote a non-compact parabolic Riemann surface, and let $D \subset \mathfrak{A}$ be compact and such that each frontier point of D is contained in a continuum that is also contained in D. Under these conditions, Kuramochi [2] (see also [3]) has established the existence of a function u on $\mathfrak{A} - D$ satisfying

(a) $u \geq 0$, harmonic on $\mathfrak{A} - D$,

(b) u vanishes continuously on $fr\ D$, the frontier of D,

(c) u tends to $+\infty$ at the ideal boundary of $\mathfrak{A}$.

Any such function will be called an (Evans-Kuramochi) *exhaustion function* on $\mathfrak{A} - D$. An exhaustion function u on $\mathfrak{A} - D$ will be said to satisfy the *k-condition* if and only if there exists an integer k such that the number of components of the level loci $\{u = s\}$ is bounded above by k independent of s, $0 < s < +\infty$.

In [4] it was shown that an extension of the Denjoy-Carleman-Ahlfors theorem in subharmonic form can be obtained for any surface admitting an exhaustion function satisfying the k-condition for some k. Any n-sheeted algebroid Riemann surface over the finite plane falls under this classification. In this paper we construct a non-algebroid surface admitting an exhaustion function satisfying the k-condition, thus answering in the affirmative a question raised in [4]. As we shall see, the desired surface is closely related to that constructed by Heins in [1, pp. 297-299].

2. Following Heins [1], we begin by constructing a non-algebroid surface given as an explicit covering surface of the extended plane. Thus, let $\{a_n\}_{n=0}^{\infty}$ denote a sequence of positive reals such that $a_0 > e$ and $\inf a_n/a_{n-1} > 1$. Let $\{b_{4n+2}\}_{n=0}^{\infty}$ be such that $a_{4n+1} < b_{4n+2} < a_{4n+2}$ and let each segment

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$[a_{4n+1}, a_{4n+2}]$ be subdivided into an odd number $(>1+a_{4n+2})$ of subsegments. Define

E^1, the finite plane less the slits $[a_{2n}, a_{2n+1}]$ and every alternate subsegment of $[a_{4n+1}, a_{4n+2}]$ starting with the second (all n);

E^2, the region E^1 less the slits $[-b_{4n+2}, -a_{4n+1}]$ (all n);

σ_n, the extended plane less the slit $[-b_{4n+2}, -a_{4n+1}]$.

Let $\mathfrak{A}$ denote the Riemann surface formed by joining copies of E^1 and E^2 along their common slits in the usual way, identifying the upper edges of the slits of E^1 with the corresponding lower edges of the slits of E^2 and vice versa. The remaining free edges of E^2 are identified with the opposite edges of the slits of copies of the corresponding σ_n.

It follows as in [1] that every non-constant meromorphic function on any end Ω of $\mathfrak{A}$ (see [1] for terminology) takes on all values infinitely often with the exception of at most two. Thus $\mathfrak{A}$ is non-algebroid. We wish to establish the following

THEOREM 1. *Given $\{a_n\}_{n=0}^{\infty}$ satisfying the above conditions, there exists $\{b_{4n+2}\}_{n=0}^{\infty}$ satisfying $a_{4n+1} < b_{4n+2} < a_{4n+2}$ and a subdivision of $[a_{4n+1}, a_{4n+2}]$ into an odd number $(>1+a_{4n+2})$ of subsegments (all n) such that the surface $\mathfrak{A}$ constructed as above from these quantities admits an exhaustion function satisfying the k-condition for $k \leq 2$ on $\mathfrak{A} - D$ where D denotes the set of points over $|z| \leq 1$ in the copies of E^1 and E^2.*

Before proceeding to the proof of Theorem 1, we establish the following notation. If $b_2, \cdots, b_{4m+2}$ are given such that $a_{4n+1} < b_{4n+2} < a_{4n+2}$, $0 \leq n \leq m$, and if an odd number $(>1+a_{4n+2})$ of subsegments subdividing $[a_{4n+1}, a_{4n+2}]$, $0 \leq n \leq m$, are given, let

E_m^1 denote the finite plane less the slits $[a_{2n}, a_{2n+1}]$ for $0 \leq n \leq 2m+1$, the slits $[a_{4n}, a_{4n+3}]$ for $n \geq m+1$, and every alternate subsegment starting with the second of $[a_{4n+1}, a_{4n+2}]$ for $0 \leq n \leq m$;

E_m^2 denote the region E_m^1 less the slits $[-b_{4n+2}, -a_{4n+1}]$ for $0 \leq n \leq m$;

σ_n as before, $0 \leq n \leq m$.

Let $\mathfrak{A}_m$, $m \geq 0$ denote the surface formed from the above quantities as in the construction of $\mathfrak{A}$. Finally, let $\mathfrak{A}_{-1}$ denote the surface constructed by copies of E_{-1}^1 and E_{-1}^2 by identifying opposite edges in the usual way where $E_{-1}^1 = E_{-1}^2 =$ the finite plane less the slits $[a_{4n}, a_{4n+3}]$ (all n).

Note that, since $\inf a_n/a_{n-1} > 1$, any such $\mathfrak{A}_m$, $m \geq -1$, has harmonic dimension one in the sense of Heins [1]. If $\mathfrak{A}_m$, $m \geq -1$, is given, let D_m

denote the set of points in $\mathfrak{A}_m$ lying over $|z| \leq 1$ in the copies of E_m^1 and E_m^2. Let p_m denote the point in $\mathfrak{A}_m$ lying over $z = e$ in the copy of E_m^1, and let u_m denote the unique exhaustion function on $\mathfrak{A}_m - D_m$ normalized such that $u_m(p_m) = 1$.

We assert that Theorem 1 is a consequence of

THEOREM 2. *Given $\{a_n\}_{n=0}^{\infty}$ satisfying the conditions of Theorem* 1, *there exists a sequence $\{b_{4n+2}\}_{n=0}^{\infty}$ and a subdivision of $[a_{4n+1}, a_{4n+2}]$ (all n) satisfying the conditions of Theorem* 1 *such that for all m, the normalized exhaustion function u_m on $\mathfrak{A}_m - D_m$ satisfies the following conditions.*

I. $u_m(p) = A_m \log|c(p)| + B_m + H_m(p)$, *if p lies in the joining of E_m^1 and E_m^2 over $|z| \geq r_m = a_{4m+3}$ where c is the natural projection map and*

a) A_m, B_m *are constants*, $A_m > \frac{1}{2}$;

b) H_m *is harmonic*, $|H_m| < \frac{\log r}{4}$, $r = \inf a_n/a_{n-1}$;

c) $|H_m(p)| < \frac{c_m}{|c(p)|} < \frac{(1 - 1/r)\log 2}{4|c(p)|}$ *if* $c(p) \in [a_{4n}, a_{4n+3}]$ *and* $n \geq m + 1$.

II. *For each s, $0 < s < A_m \log r_m + B_m + \frac{\log r}{4}$, the level locus $\{u_m = s\}$ is contained in a relatively compact open subset Ω of $\mathfrak{A}_m - D_m$ less the points in $E_m^1 \cup E_m^2$ over $|z| \geq a_{4m+4}$ where Ω is either*

a) *a region having genus one and connectivity two*,

b) *a plane region having connectivity three, or*

c) *the union of two disjoint doubly connected plane regions.*

III. u_m *satisfies the k-condition for $k \leq 2$.*

We remark that conditions I and II of Theorem 2 imply condition III. In fact, if there exists $s < A_m \log r_m + B_m + \frac{\log r}{4}$ such that $\{u_m = s\}$ has three or more components, then by condition II some subset of these components forms the boundary of a relatively compact subregion of $\mathfrak{A}_m - D_m$, and thus u_m must be identically constant. If $s \geq A_m \log r_m + B_m + \frac{\log r}{4}$, then by condition *I* $s > \max u_m$ in the joining of E_m^1 and E_m^2 over $|z| = r_m$. Therefore, $\{u_m = s\}$ lies in the joining of E_m^1 and E_m^2 over $|z| > r_m$, and the representation given by condition I is valid. Now if p_1 and p_2 are points in the joining of E_m^1 and E_m^2 over $|z| > r_m$ such that $|c(p_2)/c(p_1)| \geq r$, then by condition I we have

$$
\begin{aligned}
u_m(p_2) - u_m(p_1) &= A_m \log |c(p_2)/c(p_1)| + H_m(p_2) - H_m(p_1) \\
&\geq A_m \log |c(p_2)/c(p_1)| - \frac{1}{2} \log r \\
&> 0.
\end{aligned}
$$

It follows that $\{u_m = s\}$ lies over an annular region of the form $R_1 < |z| < R_2$ with $R_2/R_1 < r$. Since $r = \inf a_n/a_{n-1}$, $\{u_m = s\}$ is contained in a set Ω satisfying either condition IIb) or condition IIc). Thus $\{u_m = s\}$ consists of at most two components. Condition III is established.

Proof that Theorem 2 *implies Theorem* 1: Let $\mathfrak{A}$ denote the Riemann surface constructed as in Theorem 1 from the quantities given in Theorem 2. Observe that, for all m, u_m on $\mathfrak{A}_m - D_m$ can be considered as a function defined on $\mathfrak{A} - D$ less the points in the joining of E^1 and E^2 over $[a_{4n+1}, a_{4n+2}]$ for $n \geq m+1$, the points in the copy of E^2 over $[-b_{4n+2}, -a_{4n+1}]$ for $n \geq m+1$, and the points of σ_n for $n \geq m+1$. In particular, for any compact subset K of $\mathfrak{A} - D$, u_m is defined on K if m is sufficiently large. Moreover, $u_m > 0$ and $u_m(p^0) = 1$ (all m) where p^0 denotes the point over $z = e$ in E^1. It follows that $\{u_m | m \geq -1\}$ is normal on $\mathfrak{A} - D$, and thus there exists a subsequence almost uniformly convergent to a harmonic function $u > 0$ on $\mathfrak{A} - D$, $u(p^0) = 1$. It is easily seen that u vanishes continuously at the frontier of D. Since $\mathfrak{A}$ has harmonic dimension one and since $\mathfrak{A}$ has at least one exhaustion function on $\mathfrak{A} - D$, it follows that there is (up to constant multiples) exactly one such exhaustion function, and that function must be u. (It follows from this that the original sequence $\{u_m\}_{m=-1}^{\infty}$ converges almost uniformly to u although this result will not be needed in what follows.)

If u does not satisfy the k-condition for $k \leq 2$, then there exists an $s > 0$, s not a critical level of u, such that $\{u = s\}$ consists of $j(> 2)$ components. Take $\varepsilon > 0$ such that $A = \{s - \varepsilon \leq u \leq s + \varepsilon\}$ contains no critical levels of u. Then A is compact and consists of j components, each conformally equivalent to an annulus with $u = s + \varepsilon$ on one boundary component of each such annulus and $u = s - \varepsilon$ on the other boundary component (cf. [4]). Moreover, for m sufficiently large, u_m is defined on A, and we can assume $|u - u_m| < \varepsilon/2$ on A. But then $\{u_m = s\}$ has at least one component in each of the $j(> 2)$ components of A, a contradiction.

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Before turning to the proof of Theorem 2, it will be convenient to have the following two lemmas at our disposal.

LEMMA 1. *If* $\mathfrak{A}_m$ *is given,* $m \geq -1$, *and if* φ_m *denotes the indirectly conformal map from* $\mathfrak{A}_m$ *onto itself determined by* $\varphi_m(p) = \bar{p}$, $\bar{p}$ *as defined below, then* $u_m \circ \varphi_m = u_m$. *Moreover, in the joining of* E_m^1 *and* E_m^2 *over the slits on the positive real axis, the two determinations of* u_m *agree.*

DEFINITION: If a point p in $\mathfrak{A}_m$ corresponds to a point z in E_m^1, E_m^2 or σ_n, $0 \leq n \leq m$, respectively, then $\bar{p}$ denotes the point in $\mathfrak{A}_m$ corresponding to $\bar{z}$ in E_m^1, E_m^2 or σ_n, $0 \leq n \leq m$, respectively. The obvious modifications are made for points of $\mathfrak{A}_m$ over slits, e.g., if p_1, p_2 denote the two points of $\mathfrak{A}_m$ in the joining of E_m^1 and E_m^2 over a point in (a_{4n}, a_{4n+1}), then $\bar{p}_1 = p_2$ and $\bar{p}_2 = p_1$.

Proof of Lemma 1: Since φ_m is indirectly conformal, $u_m \circ \varphi_m$ is harmonic. Moreover, $u_m \circ \varphi_m(p_m) = 1$. Since $\mathfrak{A}_m$ has harmonic dimension one, it follows that $u_m \circ \varphi_m = u_m$. The remaining assertion of the lemma is an immediate consequence of this property.

LEMMA 2. *Let* h, *harmonic on* $|z| > R$, *be such that* $|h| < M$, $\lim_{z \to \infty} h(z) = 0$. *Then* $|h(z)| \leq \dfrac{2MR}{|z| + R}$ *for* $|z| > R$.

Proof of Lemma 2: The proof follows by a direct application of Harnack's inequalities to the functions $M - h\left(\frac{1}{z}\right)$ and $M + h\left(\frac{1}{z}\right)$ for $|z| < \frac{1}{R}$.

Proof of Theorem 2: The proof is by induction on m. The case $m = -1$ is trivial. Here the normalized exhaustion function u_{-1} on $\mathfrak{A}_{-1} - D_{-1}$ is given by $u_{-1}(p) = \log|c(p)|$ where c is the natural projection. Assume therefore that $\mathfrak{A}_{m-1} (m \geq 0)$ is given such that u_{m-1} satisfies conditions I, II and III of Theorem 2 with $m-1$ replacing m. With this assumption, we show there exists b_{4m+2}, $a_{4m+1} < b_{4m+2} < a_{4m+2}$ and a subdivision of $[a_{4m+1}, a_{4m+2}]$ into an odd number $(> 1 + a_{4m+2})$ of subsegments such that u_m on $\mathfrak{A}_m - D_m$ satisfies the conditions of Theorem 2.

Thus, let $b_{4m+2}(n) = \left(1 - \frac{1}{n}\right) a_{4m+1} + \left(\frac{1}{n}\right) a_{4m+2}$, $n = 1, 2, \cdots$. Let ν, an integer, be such that $a_{4m+2}/2 < \nu < a_{4m+2}$, and let $\delta = (a_{4m+2} - a_{4m+1})/\nu$. Note that $2\nu + 1 > 1 + a_{4m+2}$ and that $\delta > 1 - \frac{1}{r}$, $r = \inf a_n/a_{n-1}$. Let $\alpha_j = a_{4m+1} + j\delta$, $j = 0, \cdots, \nu$, and introduce $\alpha_j(n)$, $j = 0, \cdots, \nu$, and $n = 1, 2, \cdots$, such that

(i) $\alpha_0 < \alpha_0(n) < \alpha_1(n) < \alpha_1$, all n;

(ii) $\alpha_{j-1} < \alpha_j(n) < \alpha_j$, $j = 2, \cdots, \nu$ and all n;

(iii) $\lim_{n\to\infty} \alpha_j(n) = \alpha_j$, $j = 0, \cdots, \nu$.

For each n, the points $\alpha_0, \alpha_0(n), \alpha_1(n), \alpha_1, \cdots, \alpha_\nu(n), \alpha_\nu$ subdivide $[a_{4m+1}, a_{4m+2}]$ into $2\nu+1$ subsegments. Let $\mathfrak{A}_m^n$ denote the Riemann surface constructed in the usual manner with this choice of subintervals for $[a_{4m+1}, a_{4m+2}]$, the subinterval $[-b_{4m+2}(n), -a_{4m+1}]$, a copy of the extended plane slit along $[-b_{4m+2}(u), -a_{4m+1}]$, and the information given from $\mathfrak{A}_{m-1}$. Let u_m^n denote the normalized exhaustion function on $\mathfrak{A}_m^n - D_m^n$.

The functions u_m^n, $n = 1, 2, \cdots$, can be considered as defined on $\mathfrak{A}_{m-1} - D_{m-1}$ less the appropriate subsets (dependent on n). Since $u_m^n > 0$ and $u_m^n(p_{m-1}) = 1$ (all n), the family $\{u_m^n\}_{n=1}^\infty$ is normal on $\mathfrak{A}_{m-1} - D_{m-1}$ less the point of E_{m-1}^2 over $-a_{4m+1}$ and less the points in the joining of E_{m-1}^1 and E_{m-1}^2 over α_j, $j = 0, \cdots, \nu$. Hence there exists a subsequence almost uniformly convergent there to a positive harmonic function u such that $u(p_{m-1}) = 1$. Moreover, it is easily seen that u is bounded in some neighborhood of each of the points deleted from $\mathfrak{A}_{m-1} - D_{m-1}$. Thus u can be extended to a function harmonic on $\mathfrak{A}_{m-1} - D_{m-1}$. Also, u vanishes continuously at $fr\, D_{m-1}$. Since $\mathfrak{A}_{m-1}$ has harmonic dimension one, it follows that $u = u_{m-1}$.

Note that $r_{m-1} < a_{4m}$, and thus the representation of u_{m-1} given by condition I of Theorem 2 is valid in the joining of E_{m-1}^1 and E_{m-1}^2 over $[a_{4m}, a_{4m+3}]$. By Lemma 1, the two determinations of u_{m-1} agree in the joining of E_{m-1}^1 and E_{m-1}^2 over $[a_{4m}, a_{4m+3}]$, and therefore the same is true for H_{m-1}. Thus, if $\alpha \in [a_{4m}, a_{4m+3}]$, let $u_{m-1}(\alpha)$, $H_{m-1}(\alpha)$, respectively, denote the common value of the two determinations of u_{m-1}, H_{m-1}, respectively, over α. If $\alpha, \beta \in [a_{4m}, a_{4m+3}]$ and $\beta > \alpha + \left(1 - \frac{1}{r}\right)$, then

$$
\begin{aligned}
(1) \qquad u_{m-1}(\beta) - u_{m-1}(\alpha) &= A_{m-1} \log(\beta/\alpha) + H_{m-1}(\beta) - H_{m-1}(\alpha) \\
&> A_{m-1} \log\left(1 + \frac{1 - \frac{1}{r}}{\alpha}\right) - \frac{\left(1 - \frac{1}{r}\right)\log 2}{2\alpha} \\
&> M > 0
\end{aligned}
$$

since $\frac{1}{2} < A_{m-1}$, $1 < r$, and $1 < a_{4m} \leq \alpha \leq a_{4m+3}$. In particular, $\min\{|u_{m-1}(\alpha_j)$

$-u_{m-1}(\alpha_k)|: k \neq j\} > M > 0$. We can choose disjoint closed disks Δ_j in $a_{4m+1} \leq |z| \leq a_{4m+2}$ such that $\alpha_j \in \Delta_j$, $j = 0, \cdots, \nu$ and such that, if K_j denotes the points in the joining of E^1_{m-1} and E^2_{m-1} over Δ_j, then

(2) $$\max\{|u_{m-1}(p) - u_{m-1}(\alpha_j)| : p \in K_j\} < \frac{1}{4} M, \ j = 0, \cdots, \nu.$$

Moreover, we can assume $\alpha_j(n) \in \Delta_j$, all j, all n. Let Δ denote a closed disk in E^2_{m-1} containing the point of E^2_{m-1} over $-a_{4m+1}$, Δ lying over $a_{4m+1} < |z| < a_{4m+3}$. If $\varepsilon_m > 0$ is given, there exists a k such that u^k_m is defined on $\mathfrak{A}_{m-1} - D_{m-1} - \Delta - \bigcup_{j=1}^{\nu} K_j$ and such that $|u^k_m - u_{m-1}| < \varepsilon_m$ at the points in the joining of E^1_{m-1} and E^2_{m-1} over $|z| = r \cdot r_m (r_m = a_{4m+3})$ union the points of the frontier of $\Delta \cup (\bigcup_{j=1}^{\nu} K_j)$. We will show that if ε_m is taken sufficiently small and k is chosen as above, then $u^k_m (= u_m)$ on $\mathfrak{A}^k_m - D^k_m (= \mathfrak{A}_m - D_m)$ satisfies the conditions of Theorem 2. Henceforth, write $u^k_m = u_m$, $\mathfrak{A}^k_m = \mathfrak{A}_m$ and $D^k_m = D_m$. We assume, in particular, that

(3) $$0 < \varepsilon_m < \frac{M}{4};$$

(4) $$A_{m-1} \log a_{4m} + B_{m-1} + \frac{\log r}{4} + \varepsilon_m < A_{m-1} \log a_{4m+1} + B_{m-1} - \frac{\log r}{4};$$

(5) $$A_{m-1} \log a_{4m+2} + B_{m-1} + \frac{\log r}{4} < A_{m-1} \log a_{4m+3} + B_{m-1} - \frac{\log r}{4} - \varepsilon_m.$$

Since $A_{m-1} > \frac{1}{2}$ and $\inf a_n/a_{n-1} = r$, the conditions in (4) and (5) can be met. Further restrictions on ε_m will be imposed later.

By the maximum principle, we have $|u_m - u_{m-1}| < \varepsilon_m$ in the joining of E^1_{m-1} and E^2_{m-1} over $|z| = r_m$. Let h_m denote the unique bounded harmonic function defined in the joining of E^1_{m-1} and E^2_{m-1} over $|z| > r_m$ with boundary values $u_m - A_{m-1} \log r_m - B_{m-1} - H_{m-1}$. Note that $|h_m| < \varepsilon_m$, and that h_m has a limit at the ideal boundary of $\mathfrak{A}_{m-1}$ since $\mathfrak{A}_{m-1}$ has harmonic dimension one (cf. [1]). Thus we can write $h_m = f_m + b_m$ where b_m is constant, $|b_m| < \varepsilon_m$, f_m is harmonic, $|f_m| < 2\varepsilon_m$ and f_m tends to 0 at the ideal boundary of $\mathfrak{A}_{m-1}$. The function

$$u_m(p) - A_{m-1} \log r_m - (B_{m-1} + b_m) - (H_{m-1}(p) + f_m(p))$$

is positive harmonic and tends to 0 at the points in the joining of E^1_{m-1} and E^2_{m-1} over $|z| = r_m$, and hence is a multiple of $\log|c(p)/r_m|$ where c is the natural projection map. We have

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$$u_m(p) = A_m \log|c(p)| - (A_m - A_{m-1}) \log r_m + (B_{m-1} + b_m) + (H_{m-1}(p) + f_m(p))$$
$$= A_m \log|c(p)| + B_m + H_m(p), \quad \text{where}$$

(6) $$B_m = -(A_m - A_{m-1}) \log r_m + (B_{m-1} + b_m) \quad \text{and}$$
$$H_m = H_{m-1}(p) + f_m(p), \quad |c(p)| > r_m.$$

Since $|u_m - u_{m-1}| < \varepsilon_m$ in the joining of E^1_{m-1} and E^2_{m-1} over $|z| = r \cdot r_m$, we have

$$|(A_m - A_{m-1}) \log r + b_m + f_m| < \varepsilon_m$$

or

(7) $$|(A_m - A_{m-1}) \log r| < 2\varepsilon_m.$$

Moreover, if F_m denotes the function defined on $|z| > r_m$ as the sum of the two determinations of f_m, then F_m is harmonic, $|F_m| < 4\varepsilon_m$ and $\lim_{z\to\infty} F_m = 0$. It follows by Lemma 2 that

$$|F_m(z)| < \frac{8\varepsilon_m r_m}{|z| + r_m} \quad \text{for} \quad |z| > r_m.$$

However, by Lemma 1, the two determinations of f_m agree in the joining of E^1_{m-1} and E^2_{m-1} over $[a_{4n}, a_{4n+3}]$, $n \geq m+1$. Thus

$$|f_m(z)| < \frac{4\varepsilon_m r_m}{|z| + r_m} \quad \text{for} \quad z \in [a_{4n}, a_{4n+3}], \quad n \geq m+1.$$

It now follows directly that if $\varepsilon_m > 0$ is chosen sufficiently small, then u_m, A_m and H_m satisfy condition I of Theorem 2.

It remains to establish condition II for u_m. Thus, let s be such that $0 < s < A_m \log r_m + B_m + \dfrac{\log r}{4}$.

Case 1. $$0 < s < A_{m-1} \log a_{4m+1} + B_{m-1} - \frac{\log r}{4}.$$

Case 2.

$$A_{m-1} \log a_{4m+1} + B_{m-1} - \frac{\log r}{4} \leq s \leq A_{m-1} \log a_{4m+2} + B_{m-1} + \frac{\log r}{4}.$$

Case 3.

$$A_{m-1} \log a_{4m+2} + B_{m-1} + \frac{\log r}{4} < s < A_m \log r_m + B_m + \frac{\log r}{4}.$$

Note by condition I that, if ε_m is sufficiently small, then $u_{m-1} > A_{m-1} \log a_{4m+1} + B_{m-1} - \frac{\log r}{4} + \varepsilon_m$ in the joining of E^1_{m-1} and E^2_{m-1} over $|z| = a_{4m+1}$. Thus if s is in Case 1 and ε_m is sufficiently small, then $\{s-\varepsilon_m < u_{m-1} < s+\varepsilon_m\}$ is contained in $\mathfrak{A}_{m-1} - D_{m-1}$ less the points in the joining of E^1_{m-1} and E^2_{m-1} over $|z| \geq a_{4m+1}$, a subset of $\mathfrak{A}_{m-1} - D_{m-1}$ which can also be considered as a subset of $\mathfrak{A}_m - D_m$. Moreover, for such s we have $\{u_m = s\} \subset \{s-\varepsilon_m < u_{m-1} < s+\varepsilon_m\}$. Thus, the facts given in conditions I and II for u_{m-1} can be used to assure, if ε_m is sufficiently small, that $\{u_m = s\}$, s in Case 1, satisfies condition II. We omit the details of the proof and turn to Case 3 which, as we shall see, is similar to Case 1. If ε_m is sufficiently small, note that by condition I we have $u_{m-1} < A_{m-1} \log a_{4m+2} + B_{m-1} + \frac{\log r}{4} - \varepsilon_m$ in the joining of E^1_{m-1} and E^2_{m-1} over $|z| = a_{4m+2}$, and note that by condition I, (6) and (7) we have $u_{m-1} > A_m \log r_m + B_m + \frac{\log r}{4} + \varepsilon_m$ in the joining of E^1_{m-1} and E^2_{m-1} over $|z| = r \cdot r_m (< a_{4m+4})$. For s in Case 3, we have then that $\{s - \varepsilon_m < u_{m-1} < s + \varepsilon_m\}$ lies in the joining of E^1_{m-1} and E^2_{m-1} over $a_{4m+2} < |z| < a_{4m+4}$, a subset of $\mathfrak{A}_{m-1} - D_{m-1}$ which can also be considered as a subset of $\mathfrak{A}_m - D_m$ and which, in addition, is a region satisfying condition II(b). Since

$$\{u_m = s\} \subset \{s - \varepsilon_m < u_{m-1} < s + \varepsilon_m\}$$

for s in Case 3, it follows by the above that condition II is satisfied for such s. We turn then to the more interesting Case 2. Note by (4) and (5) that

(8) if s is in Case 2, then $\{u_m = s\}$ lies in the joining of E^1_m and E^2_m over $a_{4m} < |z| < a_{4m+3}$ union σ_m.

Let A denote the set of points in the joining of E^1_m and E^2_m over (a_{4m}, a_{4m+3}), and let $a_s = \min c(A \cap \{u_m = s\})$, $b_s = \max c(A \cap \{u_m = s\})$ where c is the natural projection map and s is in Case 2. Observe that

(9) if $p \in A \cap K_j$ for some j, then by (2) and (3) we have

$$|u_m(p) - u_{m-1}(\alpha_j)| < \frac{M}{2};$$

(10) if $p \in A - \bigcup_{j=0}^{\nu} K_j$, then by (3) we have

$$|u_m(p) - u_{m-1}(p)| < \frac{M}{4}.$$

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Inequalities (9) and (10) together with (1) imply that the interval $[a_s, b_s]$ intersects at most one Δ_j. It follows from this and (8) that, for s in Case 2, $\{u_m = s\}$ is contained in a relatively compact subregion Ω of $\mathfrak{A}_m - D_m$ less the points in the joining of E_m^1 and E_m^2 over $|z| > a_{4m+3}$ where Ω has genus one and connectivity two. Thus, u_m satisfies condition II. The proof of Theorem 2 is complete.

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