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On the Milnor Fiber of a Quasi-ordinary Surface Singularity

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Abstract. We verify a generalization of (3.3) from [Lê73] proving that the homotopy type of the Milnor fiber of a reduced hypersurface singularity depends only on the embedded topological type of the singularity. In particular, using [Zariski68, Lipman83, Oh93, Gau88] for irreducible quasi-ordinary germs, it depends only on the normalized distinguished pairs of the singularity. The main result of the paper provides an explicit formula for the Euler-characteristic of the Milnor fiber in the surface case.

1 Introduction

We consider the germ of an irreducible quasi-ordinary surface singularity at the origin in $\mathbf{C}^3$, defined by the equation $f(x, y, z) = 0$. Thus, we suppose that the projection $\pi\colon (F, 0) \to (\mathbf{C}^3, 0)$ on the first two coordinates (x, y) restricted to $(F, 0) = (\{f = 0\}, 0)$ is a finite map onto $(\mathbf{C}^2, 0)$ with (reduced) discriminant $(\Delta, 0)$ contained in $(\{xy = 0\}, 0)$. The main goal of this note is the computation of the Euler-characteristic $\chi(F_\epsilon)$ of the Milnor fiber F_ϵ of f. We recommend as general references for hypersurface singularities the books [Milnor68] and [AGV88]. The Milnor fiber F_ϵ is defined as follows. For any germ $f\colon (\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$, we fix a sufficiently small closed ball B_r in $\mathbf{C}^{d+1}$ of radius r, then $F_\epsilon := f^{-1}(\epsilon) \cap B_r$ $(0 < \epsilon \ll r)$. For $\chi(F_\epsilon)$ we will use the simplified notation $\chi(f)$. Our main result is the following

Theorem *Assume that $f\colon (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ is an irreducible quasi-ordinary singularity represented in a "normalized" coordinate system (cf. 2.7). Then $\chi(f) = \chi(f|_{y=0})$. More precisely, the Euler-characteristic of the Milnor fiber of f is exactly the Euler-characteristic of the Milnor fiber of the plane curve singularity $(x, z) \to f(x, 0, z)$.*

The importance of the above theorem can be illuminated by the following qualitative results as well. First we recall that the embedded topological type of an irreducible plane curve singularity is completely determined by its Puiseux pairs. Moreover, these pairs determine $\chi(F_\epsilon)$ as well.

The point is that for any higher dimensional irreducible quasi-ordinary singularity f, one can define the generalization of the Puiseux pairs: they are called the normalized distinguished tuples of f. (In fact, they are defined via a parametrization ζ of F, but it turns out that they are independent of ζ and depend only on the analytic type of $(F, 0)$.) For details, see [Lipman65, Lipman83, Lipman88].

Using Zariski's result on saturation of local rings, one can prove that these pairs determine the embedded topological type of f (*i.e.*, the homeomorphism type of

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the pair $(F, 0) \subset (\mathbf{C}^{d+1}, 0)$, *cf.* [Zariski68] and [Lipman88], or [Oh93] for a different proof.) Moreover, using Lipman's results [Lipman88], Gau in [Gau88] proves that from the embedded topological type one can recover the normalized distinguished tuples of f. In addition, a generalization of Theorem 3.3 of [Lê73] (*cf.* also [Teissier74], and see (2.4) in our Section 2) says that for any reduced hypersurface singularity, the homotopy type of the Milnor fiber depends only on the embedded topological type of the singularity. Therefore, it is natural to ask for explicit formulae for the homotopical invariants of F_ϵ in terms of the distinguished tuples of f. In the next section we will present a more precise version of our theorem which provides $\chi(F_\epsilon)$ in the surface case in terms of the normalized distinguished pairs (*cf.* 2.8).

The precise definition of the distinguished pairs will be given in Section 2. Since the main result is formulated for surface singularities, in order to avoid an unnecessary flood of notations, we give the notations in this case only. The general situation can be found *e.g.* in [Lipman88].

The proof starts in Section 3, where we generalize A'Campo's formula for $\chi(F_\epsilon)$ [A'Campo75] (in terms of the embedded resolution). In fact, the proof of the main theorem is based on the following fact: the expression $\chi(f) - \chi(f|_{y=0})$ is "stable" with respect to all the types of blow ups which appear in the resolution process of f. In order to verify this, we need the "splitting property" (3.3) (which itself is proved by our generalized A'Campo-type theorem). The verification of the "stability" is in Section 4.

We end the introduction with the following remark. Upon inspecting the literature of quasi-ordinary singularities, one is very surprised to discover that the amount of results about the smoothing invariants is incredibly small. The point is that although the strong assumption about F (namely, the existence of the projection π with normal crossing discriminant) provides a nice list of analytic and geometric properties of F, it is surprisingly difficult to relate these properties to any kind of property of the level sets F_ϵ ($\epsilon \neq 0$). The present note is the first in a series of articles in which the authors plan to analyze the smoothing invariants (like the zeta function of the monodromy of f, *etc.*). We remark that, while a central purpose of the definition of quasi-ordinary singularities is to generalize features of plane curve singularities to higher dimensions, our main theorem applies only to surfaces (though see note added at the end of the paper). Also the result has no analog for curves.

2 Quasi-Ordinary Singularities and Their Topological Type

In this section we recall the definition of the normalized distinguished pairs associated with an irreducible quasi-ordinary singularity $(F, 0) \subset (\mathbf{C}^3, 0)$, and we reformulate our theorem in terms of these pairs.

(2.1) By the very definition of irreducible hypersurface quasi-ordinary singularities, there exists coordinates such that f can be expressed as a pseudo-polynomial in the third variable z: $f(x, y, z) = z^m + g_1(x, y)z^{m-1} + \cdots + g_m(x, y)$, where g_i are power series, and the (reduced) discriminant of $\pi\colon (\{f = 0\}, 0) \to (\mathbf{C}^2, 0)$ induced by $(x, y, z) \mapsto (x, y)$ is contained in $(\{xy = 0\}, 0)$. Moreover, there exists a parametrization of $F = \{f = 0\}$ by a fractional power series $\zeta = H(x^{1/n}, y^{1/n})$, where $H(s, t)$ is a power series and n is a suitable natural number depending on f. (This means that

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the parametrization $(\mathbf{C}^2, 0) \to (F, 0)$ is given by $x = s^n$, $y = t^n$, $z = H(s,t)$.)

The conjugates of ζ are obtained by multiplying $x^{1/n}$ and $y^{1/n}$ by n-th-roots of unity; the number of different conjugates $\{\zeta_i\}_i$ of ζ is precisely the degree m of the covering π, and

$$f(x, y, z) = \prod_{i=1}^{m} (z - \zeta_i).$$

The roots ζ_i are called the *branches* of f. There is a finite subset $\{(\lambda_i, \mu_i)\}_{i=1}^{s}$ of the set of exponents $\{(\lambda_i, \mu_i)\}_i$ of the monomials appearing in ζ with non-zero coefficient, called the *distinguished pairs*, which play a role similar to Puiseux pairs for plane curve singularities. Namely, by unique factorization of the discriminant one has:

$$\zeta_i - \zeta_j = x^{u_{ij}/n} y^{v_{ij}/n} \epsilon_{ij}(x^{1/n}, y^{1/n}) \quad \text{with } \epsilon_{ij}(0,0) \neq 0,$$

and the set of exponents $\{(u_{ij}/n, v_{ij}/n)\}_{i,j}$ constitute the set of distinguished pairs.

Say that $(\lambda, \mu) \leq (\lambda', \mu')$ if $\lambda \leq \lambda'$ and $\mu \leq \mu'$. Then the distinguished pairs are ordered:

(1) $(0,0) < (\lambda_1, \mu_1) < (\lambda_2, \mu_2) < \cdots < (\lambda_s, \mu_s)$; and (after permuting x and y if necessary) we can assume:

(2) $(\lambda_1, \ldots, \lambda_s) > (\mu_1, \ldots, \mu_s)$ (lexicographically).

Let $c_{\lambda,\mu}$ be the coefficient of the term in ζ having exponent pair (λ, μ). The distinguished pairs generate all the other exponent pairs in the following sense:

(3) If $c_{\lambda,\mu} \neq 0$ then $(\lambda, \mu) \in \mathbf{Z} \times \mathbf{Z} + \sum_{(\lambda_i,\mu_i) \leq (\lambda,\mu)} \mathbf{Z}(\lambda_i, \mu_i)$ (and $(\lambda_j, \mu_j) \notin \mathbf{Z} \times \mathbf{Z} + \sum_{(\lambda_i,\mu_i) < (\lambda_j,\mu_j)} \mathbf{Z}(\lambda_i, \mu_i)$).

The parametrization ζ of F, in general, is not unique. Moreover, different parametrizations could produce different sets of distinguished pairs. But if we consider only the "normalized" parametrizations, then the corresponding pairs are independent of the choice of parametrization. A parametrization is said to be normalized if in addition to (1)–(3) we have

(4) If $c_{\lambda,\mu} \neq 0$, then λ and μ are not both integers. (In the literature this condition is not usually included, and as we see in Example 4.3 is not really necessary.)

(5) If $\mu_1 = 0$ then $\lambda_1 > 1$.

Given a parametrization

$$\zeta = cx^a y^b + \text{ higher order terms } \in \mathbf{C}\{x^{1/n}, y^{1/n}\} \quad (c \neq 0)$$

then if ζ is normalized, it follows from (3) and (4) that $(\lambda_1, \mu_1) = (a, b)$. If ζ is not normalized and $(\lambda_1, \mu_1) \not\leq (\lambda, \mu)$ with $c_{\lambda,\mu} \neq 0$ then $(\lambda, \mu) \in \mathbf{Z} \times \mathbf{Z}$.

(2.2) If a parametrization is not normalized, then by a change of variables ϕ: $(x, y, z) \to (x', y', z')$ one can transform ζ into a normalized parametrization $\zeta' = \zeta \circ \phi$. Basically, there are two cases to consider.

(a) If $c_{\lambda,\mu} \neq 0$ with $(\lambda, \mu) \in \mathbf{Z} \times \mathbf{Z}$, then we make a change of variables $(x', y', z') = (x, y, z - c_{\lambda,\mu} x^\lambda y^\mu)$. In this way ζ' will have no term with exponent (λ, μ). This applied repeatedly assures the validity of (4). Moreover, this type of transformation preserves the distinguished pairs of the parametrization.

(b) If (5) is not satisfied then first one has to make a transformation of type $(x, y, z) \to (z, y, x)$ (and express f as a pseudo-polynomial in $z' = x$), then, if it is necessary, one has to perform a finite number of times the transformation described in (a). (This sequence of transformations is called "inversion", for details see [Lipman65].) In this case the distinguished pairs will be changed (*cf.* also Lipman's Table presented in paragraph (4.2)). In fact the transformation $(x, y, z) \to (z, y, x)$ (and f rewritten as a pseudo-polynomial in $z' = x$) changes the distinguished pairs $\{(\lambda_i, \mu_i)\}_{i=1}^s$ into the new pairs (the first of which may be integral, hence non-distinguished):

$$\left\{ \left((\lambda_i + 1 - \lambda_1)/\lambda_1, \mu_i \right) \right\}_{i=1}^{s}.$$

If the first new pair $(1/\lambda_1, \mu_1)$ is in $\mathbf{Z} \times \mathbf{Z}$, then we eliminate this pair (and any other previously non-distinguished pair which now becomes integral) by a sequence of transformations of type (a). All the other pairs will survive as distinguished pairs; for this notice that $(\lambda_2 + 1 - \lambda_1)/\lambda_1, \mu_2)$ cannot become integral unless (λ_2, μ_2) depends integrally on (λ_1, μ_1), which is forbidden by (3) above.

(2.3) As we already mentioned in the introduction, in any dimension, for any irreducible hypersurface quasi-ordinary singularity $f\colon (\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$ the information codified in the set of normalized distinguished tuples is the same as the information codified in the embedded topological type of f, *i.e.*, in the homeomorphism type of the pair $L \subset S^{2d+1}$, where $S^{2d+1} = \partial B_r$, and $L = f^{-1}(0) \cap S^{2d+1}$ is the link of f.

On the other hand, we can verify the following general result. For isolated singularities it was proved in [Lê73] (*cf.* also [Yau89]); the topological invariance of the Milnor number for isolated singularities was first noticed in [Teissier74].

Proposition 2.4 *Denote by $f\colon (\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$ a reduced hypersurface singularity, and let F_ϵ be its Milnor fiber, and m_{geom} the geometric monodromy acting on F_ϵ (defined up to an isotopy). Then:*

- *(a) The homotopy type of $(F_\epsilon, m_{\mathrm{geom}})$ can be recovered from the embedded topological type of f.*
- *(b) F_ϵ is connected.*
- *(c) If f is irreducible then $\pi_1(F_\epsilon) = [G, G]$, where $G = \pi_1(S^{2d+1} \setminus L, *)$.*

Proof The proof is similar to the proof of Theorem 3.3 in [Lê73]. For the convenience of the reader, we give the details as well. F_ϵ is a manifold with boundary $\partial F_\epsilon = f^{-1}(\epsilon) \cap S^{2d+1}$. Let F_ϵ^0 be the open Milnor fiber $F_\epsilon \setminus \partial F_\epsilon$. Then there is a (Milnor) fibration $\arg := f/|f|\colon S^{2d+1} \setminus L \to S^1$ with fiber (diffeomorphic to) F_ϵ^0 [Milnor68]. The fibration is completely characterized (up to a homotopy) by the induced map $\arg_1\colon \pi_1(S^{2d+1} \setminus L, *) \to \pi_1(S^1) = \mathbf{Z}$. Indeed, since $\arg_1$ is onto (see below), F_ϵ^0 can be homotopically identified with the covering space of $S^{2d+1} \setminus L$ associated with $\arg_1$ (or with the subgroup $\ker(\arg_1)$). Moreover, the corresponding Galois action is exactly $\mathbf{Z}$, and its generator $1_{\mathbf{Z}}$ can be identified with m_{geom}.

On the other hand, $\arg_1$ can be completely determined from (S^{2d+1}, L). Indeed by Alexander duality $H_1(S^{2d+1} \setminus L, \mathbf{Z}) = H^{2d-1}(L, \mathbf{Z})$. Consider the irreducible decomposition $f = f_1 f_2 \cdots f_t$ of f. Let $L_i = f_i^{-1}(0) \cap S^{2d-1}$, then obviously $L = \bigcup_i L_i$. Moreover, $H^{2d-1}(L_i, \mathbf{Z}) = \mathbf{Z}$, and by a Mayer-Vietoris argument $H^{2d-1}(L, \mathbf{Z}) = \mathbf{Z}^t$.

The isomorphism $H_1(S^{2d+1} \setminus L, \mathbf{Z}) = \mathbf{Z}^t$ is realized as follows: for any smooth point p of L_i take a local transversal slice of L_i in S^{2d+1} and fix a small oriented circle in it going around L_i in S^{2d+1}. Its homology class is the i-th base element of $\mathbf{Z}^t$. This shows that $\arg_1$ is the composed map $\pi_1(S^{2d+1} \setminus L, *) \xrightarrow{h} H_1(S^{2d+1} \setminus L, \mathbf{Z}) = \mathbf{Z}^t \xrightarrow{s} \mathbf{Z}$ where h is the Hurewicz map and $s\big((\alpha_1, \ldots, \alpha_t)\big) = \sum_i \alpha_i$. This shows that $\arg_1$ is onto, hence F^0_ϵ is connected. If f is irreducible, then $\arg_1 = h$, and $\pi_1(F^0_\epsilon) = [G, G]$ by the long homotopy exact sequence.

(2.5) If we start with $f = f_1^{m_1} \cdots f_t^{m_t}$, then in the above proof $s\big((\alpha_1, \ldots, \alpha_t)\big) = \sum_i \alpha_i m_i$, hence F^0_ϵ has $\gcd(m_1, \ldots, m_t)$ connected components, and the homotopy type of F^0_ϵ can be recovered from the type of (S^{2d+1}, L) and the set of integers $(m_1, \ldots, m_t)$ (*i.e.*, from the "multilink" $L \subset S^{2d+1}$) (see also [Dimca92], page 76].)

(2.6) The discussion (2.3) and Proposition 2.4 show that, for any irreducible quasi-ordinary singularity, there should be an explicit formula for $\chi(F_\epsilon)$ in terms of the distinguished pairs of f. The goal of the next part is the presentation of this formula.

(2.7) Now we return to an irreducible quasi-ordinary surface singularity f: $(\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$. We assume that it is represented in some coordinate system (x, y, z) which admits a normalized parametrization ζ.

First assume that Sing $F = \varnothing$. Then f has no normalized distinguished pairs, and $f = z$. Hence $m = \deg_z(f) = 1$. We also define $g(x, z) := z$, and $\tilde{m} = deg_z(g) = 1$.

If Sing $F \neq \varnothing$, then denote the normalized distinguished pairs by $\{(\lambda_i, \mu_i)\}_{i=1}^s$.

In this paragraph we make more precise the connection between the curve singularity $f(x, 0, z)$ and the parametrization ζ.

First assume that $\mu_1 = 0$. Then by (5) $\lambda_1 > 1$, and by (1) all λ_i are non-zero. Suppose $\mu_1 = \mu_2 = \cdots = \mu_k = 0$, and $\mu_{k+1} > 0$. Then, again by (1) $\mu_{k+i} > 0$ for all $i \geq 1$. In this case $H(s, t)$ (*cf.* 2.1) has a decomposition $H(s, t) = K(s) + L(s, t)$, where K and L are power series and $L(s, 0) = 0$. Recall that $\zeta = H(x^{1/n}, y^{1/n})$; then we define the fractional power series $\varphi = K(x^{1/n})$ and $\psi = L(x^{1/n}, y^{1/n})$. Let $\tilde{m}$ be the number of conjugates $\{\varphi_j\}_{j=1}^{\tilde{m}}$ of φ. We emphasize that even if ζ (respectively φ) has some terms whose exponents are not distinguished pairs, the number of conjugates m (respectively $\tilde{m}$) of ζ (respectively of φ) depends only on the distinguished pairs $\{(\lambda_i, \mu_i)\}_{i=1}^s$ (respectively $\{(\lambda_i, 0)\}_{i=1}^k$). (Indeed, m, respectively $\tilde{m}$, is the number of different conjugates of the "simplified parametrization" $\tilde{\zeta} = \sum_{i=1}^s x^{\lambda_i} y^{\mu_i}$, respectively of $\tilde{\varphi} = \sum_{i=1}^k x^{\lambda_i}$.)

Since each conjugate of ζ may be obtained as a conjugate of φ plus a conjugate of ψ, it is clear that $\tilde{m}$ divides m. Let's consider the curve singularity given by the conjugates of φ:

$$g(x, z) = \prod_{j=1}^{\tilde{m}} (z - \varphi_j).$$

If $\mu_1 \neq 0$, then by definition, $\varphi \equiv 0$, $\tilde{m} = 1$ and $g(x, z) \equiv z$.

With these notations one has: $g(x, z) = f(x, 0, z)_{\mathrm{red}}$, $f(x, 0, z) = g(x, z)^{m/\tilde{m}}$ (and φ is a parametrization of $\{g = 0\}$). Moreover, the set $\{\lambda_i\}_{i=1}^k$ provides exactly the set of Puiseux pairs of g.

If $\mu_1 \neq 0$ then $\pi^{-1}(\{xy = 0\})$ is the union of the x and the y-axis of $\mathbf{C}^3$. If $\mu_1 = 0$, then $\pi^{-1}(\{x = 0\})$ is the y-axis of $\mathbf{C}^3$, but $\pi^{-1}(\{y = 0\})$ is the curve $\{g = 0\}$. The integer $\tilde{m}$ also has the following interpretation: for a given point $(x_0, 0)$ with $x_0 \neq 0$, the fiber $\pi^{-1}(x_0, 0)$ consists of $\tilde{m}$ points (counted without multiplicities).

Now we formulate our main result in terms of the normalized distinguished pairs of f:

Theorem 2.8 *Consider f and g as in (2.7). Then*

$$\chi(f) = \chi(g) \cdot \frac{m}{\tilde{m}}; \textit{i.e.}\, \chi(f) = \chi(f|_{y=0}).$$

In particular, $\chi(f)$ depends only on the integer m and the subset $\{(\lambda_i, 0)\}_{i=1}^k$ of the set of normalized distinguished pairs.

Moreover, in the "symmetric case", i.e. if either Sing $F = \varnothing$, *or $\mu_1 \neq 0$, one has:*

$$f|_{y=0} = f|_{x=0} = z^m,$$

hence $\chi(f) = m$.

For the computation of $\chi(g)$ in terms of $\{\lambda_i\}_{i=1}^k$, see *e.g.* [BK86]. In a normalized coordinate system as in (2.7)–(2.8), we sometimes say that the y-axis is "distinguished."

The proof of (2.8) is given in the next two sections.

3 The First Part of the Proof. General Facts About $\chi(F_\epsilon)$

(3.1) We start with the following generalization of A'Campo's formula [A'Campo75]. Fix the following data:

- an arbitrary analytic germ $h\colon (\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$.
- a local analytic divisor $(V, 0) \subset (\mathbf{C}^{d+1}, 0)$.
- an analytic subset $(S, 0) \subset (V, 0) \cup (\{h = 0\}, 0)$.

Let B_r be a sufficiently small ball in $\mathbf{C}^{d+1}$ centered at the origin. We write $\{h = \epsilon\}$ for the Milnor fiber $h^{-1}(\epsilon) \cap B_r$ $(0 < \epsilon \ll r)$.

Assume that $\phi\colon X \to B_r$ is a birational modification such that

(1) $\phi^{-1}(\{h = 0\} \cup V)$ is a normal crossing divisor.
(2) ϕ is an isomorphism above $B_r \setminus S$.

Let E be the total transform of $\{h = 0\} \cup V$, $\{E_i\}_i$ the irreducible components of E, and, $m_{E_i}(h)$ the vanishing order of $h \circ \phi$ along E_i.

Proposition 3.2 *With the above notations, one has:*

$$\chi(\{h = \epsilon\} \setminus V) = \sum_i m_{E_i}(h) \cdot \chi\Big(E_i \Big\backslash \bigcup_{j \neq i} E_j\Big).$$

The proof is similar to the proof of the classical case [A'Campo75] (see also [AGV88], Theorem 3.10), and it is left to the reader.

In the next corollary we will use the following notation. If f, g are hypersurface singularities $(\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$ then $f|_{g=0} \colon (\{g = 0\}, 0) \to (\mathbf{C}, 0)$ denotes the restriction of f, and $\chi(f|_{g=0})$ is the Euler-characteristic of its Milnor fiber (*i.e.*, $\chi(\{f = \epsilon\} \cap \{g = 0\} \cap B_r)$ with $0 < \epsilon \ll r$).

Corollary 3.3 **(The Splitting Property)** *Let $f, g \colon (\mathbf{C}^{d+1}, 0) \to (\mathbf{C}, 0)$ be two germs of analytic functions. Then*

$$\chi(fg) = \chi(f) + \chi(g) - \chi(f|_{g=0}) - \chi(g|_{f=0}).$$

Proof Fix an embedded resolution ϕ of the divisor $(\{fg = 0\}, 0) \subset (\mathbf{C}^{d+1}, 0)$ which has the property that it is an isomorphism above $\mathbf{C}^{d+1} \setminus S$ for some $S \subset \operatorname{Sing}\{fg = 0\}$. Let E be the total transform of $\{fg = 0\}$. By applying (3.2) for $h = f$ and $V = \{g = 0\}$, we obtain $\chi(f) - \chi(f|_{g=0}) = \sum_i m_{E_i}(f) \cdot \chi(E_i \setminus \cup_{j \neq i} E_j)$.

One can write a similar identity for $\chi(g) - \chi(g|_{f=0})$, and for $\chi(fg)$ (with $h = fg$ and $V = \varnothing$). The corollary then follows from $m_{E_i}(fg) = m_{E_i}(f) + m_{E_i}(g)$.

Remarks 3.4 (a) Notice that (3.3) has no analogue for the zeta function associated with the monodromy action of f.

(b) We want to emphasize the following fact regarding A'Campo-type theorems. In order to be able to apply the classical formula [A'Campo75], or (3.2), we need to know the structure and topology of the exceptional divisors in an embedded resolution. But typically what we know is only an algorithm (and even this can be very involved) which gives the steps of the resolution (*i.e.*, tells what to blow up in the next step), without giving a global overview of the result of the resolution process. In such cases it is extremely helpful to have instead a formula for $\chi(F_\epsilon)$ in terms of a partial resolution.

Theorem 3.5 **([GLM97])** *Let $\phi \colon X \to B_r$ be an arbitrary birational modification such that ϕ is an isomorphism above the complement of $\{f = 0\}$. Let $\mathcal{S}$ be an analytic stratification of the total transform of $\{f = 0\}$ such that along each stratum Ξ of $\mathcal{S}$ the Euler-characteristic of the Milnor fiber of $f \circ \phi$ at $x \in \Xi$ does not depend on $x \in \Xi$. Denote this number by χ_Ξ. Then:*

$$\chi(f) = \sum_{\Xi \in \mathcal{S}} \chi_\Xi \cdot \chi(\Xi).$$

This theorem will be applied in the resolution process of a quasi-ordinary singularity. The computation is in the next section.

We end this section with the following list of Euler-characteristic computations associated with the different plane sections of $\{f = 0\}$.

Lemma 3.6 *Let $f \colon (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$, $f(x, y, z) = z^m + g_1(x, y) z^{m-1} + \cdots$ be an irreducible quasi-ordinary singularity, such that the discriminant of the projection π: $(F, 0) \to (\mathbf{C}^2, 0)$ is contained in $\{xy = 0\}$. Assume that $(F, 0)$ admits a parametrization $z = H(s, t) = cs^a t^b + \cdots$, $x = s^n$, $y = t^n$ $(c \neq 0)$ having the form $H(s, t) = cs^a t^b(\mathrm{unit})$, but which is possibly not normalized. Assume that $a \neq 0$. Then*

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(1) $\chi(f|_{x=0}) = \chi(f|_{x=y=0}) = m$.

(2) $\chi(x|_{f=0}) = \chi(x|_{f=y=0})$ $(= \tilde{m} = \big((f|_{y=0})_{\mathrm{red}}, x\big)_0)$.
In fact $\{x = \epsilon\} \cap \{f = y = 0\}$ *is a strong deformation retract of* $\{x = \epsilon\} \cap \{f = 0\}$

(3) $\chi(y|_{f=0}) = \chi(y|_{f=x=0}) = \chi(y|_{f=z=0}) = 1$.
In fact $\{y = \epsilon\} \cap \{f = 0\}$ *is contractible and* $\{y = \epsilon\} \cap \{f = x = 0\}$ *is a strong deformation retract of* $\{y = \epsilon\} \cap \{f = 0\}$.

(4) If $b \neq 0$, *then* $\chi(z|_{f=0}) = 0$; *in fact there is a topological covering* $F_\epsilon(H) \to \{z = \epsilon\} \cap \{f = 0\}$, *where* $F_\epsilon(H)$ *is the Milnor fiber of* H. *In this case,* $\{z = \epsilon\} \cap \{f = y = 0\} = \varnothing$, *hence* $\chi(z|_{f=y=0}) = 0$ *as well.*
If $b = 0$, *then* $\chi(z|_{f=0}) = \chi(z|_{f=y=0}) = \big((f|_{y=0})_{\mathrm{red}}, z\big)_0$.
In fact $\{z = \epsilon\} \cap \{f = y = 0\}$ *is a strong deformation retract of* $\{z = \epsilon\} \cap \{f = 0\}$.

(5) If $b \neq 0$, *then* $\chi(f|_{z=0}) = 0$; *and* $\{f = \epsilon\} \cap \{z = y = 0\} = \varnothing$, *hence* $\chi(f|_{z=y=0}) = 0$ *as well.*
If $b = 0$, *then* $\chi(f|_{z=0}) = \chi(f|_{z=y=0}) = (f, z, y)_0$.
In fact $\{f = \epsilon\} \cap \{z = y = 0\}$ *is a strong deformation retract of* $\{f = \epsilon\} \cap \{z = 0\}$.

(In the above, in all cases $\epsilon \neq 0$ *and all spaces are considered in the ball* B_r*; and* $(\ ,\ )_0$ *denotes the intersection multiplicity.)*

Proof For (1), notice that $f|_{x=0} = f|_{x=y=0} = z^m$ because of the condition $a \neq 0$. For (2), observe that the Milnor fiber of $x|_{f=0}$ is just the inverse image under the projection π of the disk $\Delta = \{(\epsilon, y)\}$. Since π is a topological covering over the set $\{xy \neq 0\}$ and the punctured disk $\Delta^* = \Delta \setminus (0,0)$ lies in this region, $\pi^{-1}(\Delta^*)$ must be a union of disjoint punctured disks E_i. The closure of each of these punctured disks must be a disk, whence $\pi^{-1}(\Delta)$ is a finite union of disks, any two of which intersect at most at a single point of the (finite) set $\pi^{-1}(\epsilon, 0)$. Clearly each point in $\pi^{-1}(\epsilon, 0)$ belongs to at least one E_i. Thus each point of $\pi^{-1}(\epsilon, 0)$ is the center of a non-empty collection of disks which intersect only at that point, which shows that $\pi^{-1}(\epsilon, 0)$ is a strong deformation retract of $\pi^{-1}(\Delta)$. The cardinality of $\pi^{-1}(\epsilon, 0)$ is precisely the z degree of $(f|_{y=0})_{\mathrm{red}}$. Part (3) of the lemma is the same as part (2) except that $(f|_{x=0})_{\mathrm{red}}$ has degree one, *i.e.* $(f|_{x=0})_{\mathrm{red}} = z$. For part (4) and $b \neq 0$, the set $\{z = \epsilon\} \cap \{f = 0\}$ must lie above $\{xy \neq 0\}$, *i.e.* on the portion of $(F, 0)$ for which the parametrization $(x, y, z) = (s^n, t^n, z)$ is a covering space. But the pull-back is $F_\epsilon(H)$, which has the form $\{cs^a t^b(\mathrm{unit}) = \epsilon\}$, a collection of disjoint punctured disks; hence $\chi\big(F_\epsilon(H)\big) = 0$. On the other hand if $b = 0$, the situation and the argument are the same as in part (3) with the roles of x and z reversed. Finally, for part (5), $\{f = \epsilon\} \cap \{z = 0\}$ is given by $\{cx^a y^b(\mathrm{unit}) = \epsilon\}$, and this has Euler characteristic zero if $b \neq 0$. If $b = 0$ then it is clear that the Euler characteristic is unchanged when y is restricted.

The splitting property (3.3), part (3) of Lemma 3.6, and the main theorem have the following consequence.

Corollary 3.7 *Assume that* $f\colon (\mathbf{C}^3, 0) \to (\mathbf{C}, 0)$ *is an irreducible quasi-ordinary singularity, and the* y *axis is distinguished (cf. 2.8). Then* $\chi(yf) = 0$.

Remark The main Theorem 2.8 is actually equivalent to the statement $\chi(yf) = 0$. This shows that in some sense it may be more natural to consider the object yf rather than just f.

4 The Second Part of the Proof. The Resolution of f

We will apply the following fundamental facts:

Fact 4.1 Every surface $(F, 0) \subset (B_r, 0)$ (or $F \subset \mathbf{C}^3$) has an embedded resolution obtained by blowing up smooth centers within the singular locus, *i.e* we may assume that we have a finite sequence

$$(*)\qquad \begin{array}{ccccccccc} M_n & \xrightarrow{r_n} & \cdots & \longrightarrow & M_2 & \xrightarrow{r_2} & M_1 & \xrightarrow{r_1} & \mathbf{C}^3 \\ \cup & & & & \cup & & \cup & & \cup \\ F_n & \xrightarrow{r_n} & \cdots & \longrightarrow & F_2 & \xrightarrow{r_2} & F_1 & \xrightarrow{r_1} & F_0 = F \end{array}$$

such that r_i blows up a smooth center in F_{i-1}, F_n is smooth, and F_n has normal crossing with the exceptional divisors of $r_1 \circ \cdots \circ r_n$. Moreover we may assume that the center of each blow up is a point or an equimultiple curve.

There are many proofs of this fact, see *e.g.* [Hironaka64, Abhyankar98, BM91, Villamayor89].

In addition we have the following result of Lipman [Lipman65].

Fact 4.2 If an irreducible quasi-ordinary surface F is blown up at a point or a smooth equimultiple curve inside the singular locus, then the strict transform F' of F is again quasi-ordinary. (Note that this is not the case for quasi-ordinary varieties of higher dimension.)

Lipman also provides rules of transformation by which the (possibly not normalized) distinguished pairs of F' may be read from those of F. We summarize those rules in Table 1.

In the resolution of a surface quasi-ordinary singularity, in $(*)$ each morphism r_i is either a quadratic transformation (blow up of a point) or a monoidal transformation (blow up of a smooth equimultiple line, locally a disc) with center inside $\mathrm{Sing}(F_{i-1})$.

It is proved in [BMc00] that there exists a resolution process $(*)$ which depends only on the normalized distinguished pairs of (the original) F, but at each step i the type of modification r_i required depends on the history of the process (*i.e.*, on the steps $r_{i-1}, r_{i-2}, \ldots, r_1$). We will not use this complicated algorithm.

Instead, we will examine the effect of *each possible* type of transformation on the local equation for F_{i-1}. For convenience, we drop the subscript $i-1$ and we write f for the local equation of F_{i-1}. The blow up of F_{i-1} (in the case of a blow up of a point) is covered by charts U_x, U_y, U_z (defined by the local equations x, y, and z); we denote the strict transform of f in each chart by f_x, f_y and f_z.

We will assume that f is written in some coordinate system (x, y, z) in which f admits a normalized parametrization. By the above notation, f_x, f_y and f_z are the germs of the strict transform of f considered at the points $[1 : 0 : 0]$, $[0 : 1 : 0]$

Transformation	Pairs of resulting branch
Inversion	$\frac{\lambda_i+1-\lambda_1}{\lambda_1}, \mu_i$
Monoidal Transformation	
Center (x, z)	$\lambda_i - 1, \mu_i$
Center (y, z)	$\lambda_i, \mu_i - 1$
Quadratic Transformation	
"Transversal Case" $(\lambda_1 + \mu_1 \geq 1)$	
Direction $(1 : 0 : 0)$	$\lambda_i + \mu_i - 1, \mu_i$
Direction $(0 : 1 : 0)$	$\lambda_i, \lambda_i + \mu_i - 1$
"Non-Transversal Case" $(\lambda_1 + \mu_1 < 1)$	
Direction $(1 : 0 : 0)$	$\lambda_i + \frac{(1+\mu_i)(1-\lambda_1)}{\mu_1} - 2, \frac{1+\mu_i}{\mu_1} - 1$
Direction $(0 : 1 : 0)$	$\mu_i + \frac{(1+\lambda_i)(1-\mu_1)}{\lambda_1} - 2, \frac{1+\lambda_i}{\lambda_1} - 1$
Direction $(0 : 0 : 1)$	$\frac{\lambda_i(1-\mu_1)+\mu_i\lambda_1}{1-\lambda_1-\mu_1}, \frac{\lambda_i\mu_1+\mu_i(1-\lambda_1)}{1-\lambda_1-\mu_1}$

Table 1

and $[0 : 0 : 1]$ respectively. In order to avoid burdensome notation, we will continue to denote by (x, y, z) the new local coordinates: *e.g.* at the point $[1 : 0 : 0]$ these are obviously equal to (x, xy, xz) in terms of the coordinates of f. We call this new coordinate system at the point $[1 : 0 : 0]$ the "natural coordinate system" of f_x. For f_y and f_z the discussion is similar. If we blow up an axis, then we use similar notations.

We notice that if $\zeta = cx^{\lambda_1}y^{\mu_1} + \cdots$ is a normalized parametrization of f, and we blow up a center, then the leading term of a (possibly not normalized) parametrization of the strict transform of f in the "natural coordinate system" can be read from the above table by taking $i = 1$ (*i.e.* the leading term is exactly the first pair provided by the table). In the non-transversal case, x and y-directions, a permutation of the variables is needed to normalize the new parametrization (see Table 2 below).

Then we will apply Theorem 3.5 for the modification r_i. In particular there are only two components of the 2-dimensional stratum. One of them is $E^o = E \setminus \{\text{strict transform of } \{f = 0\}\}$, where E is the exceptional divisor created at this step r_i. The contribution on the left hand side of the formula (3.5) is $\chi(E^o)m_0(f)$, where $m_0(f)$ is the multiplicity of f along E. (In general $m_0(f) \neq m$; where m is the degree of f.)

The other component is supported by the strict transform St of f, but (as we will now show) $\chi(\mathrm{St} \setminus E)$ $(=\chi(F \setminus \mathrm{Center})) = 0$ hence it gives no contribution. The center is either the origin or the w-axis ($w = x$ or y); if it's the origin, let w be the smooth component of the pre-image of the discriminant (which must exist). Decompose $F \setminus \mathrm{Center}$ as the disjoint union of $A = (F \setminus \mathrm{Center}) \cap \{w \neq 0\}$ and $B = (F \setminus \mathrm{Center}) \cap \{w = 0\}$. Then A is a fiber bundle over a punctured disk (namely the punctured local w-axis), hence $\chi(A) = 0$. Similarly, B is a punctured local curve, hence $\chi(B) = 0$ as well.

One can verify that in all cases (see the list of cases below) the one dimensional strata are either punctured discs D^*'s or $\mathbf{C}^*$'s, hence their Euler characteristics vanish. Therefore, the theorem reads as

$$\chi(f) = m_0(f)\chi(E^o) + \text{ a sum corresponding to the zero-dimensional strata.}$$

At each point of the zero-dimensional strata the local equation is the total transform of f. We split these local equations using the splitting property (*cf.* 3.3). Then, for any f expressed in a normalized coordinate system, we define $D(f) := \chi(f) - \chi(f|_{y=0})$ and we verify that $D(f)$ is stable, *i.e.*

$$\text{(S)} \qquad D(f) = \sum_{\dim \Xi = 0} D(f_\Xi).$$

We emphasize that the expression $D(f_\Xi)$ must also be computed in the normalized coordinate system for the germ f_Ξ and that the corresponding local coordinate "y" is distinguished in the sense of (2.8).

On the other hand, $D(f_\Xi)$ can often be computed in the natural local coordinate system.

Examples 4.3 (a) Given the germ $\zeta = c_\lambda x^\lambda + \cdots$, not normalized, we claim that $D(f) = \chi(f) - \chi(f|_{y=0})$. Indeed, it is easy to check that the normalization transformations preserve the germ $\{f|_{y=0}\}$, *i.e.* $\{f|_{y=0}\} = \{f|_{y'=0}\}$ where (x', y', z') are the normalized coordinates.

(b) If f can be normalized by a sequence of transformations of type 2.2(a) alone, *i.e.* $(x', y', z') = (x, y, z - c_{a,b}x^a y^b))$, where at each stage $\zeta = c_{a,b}x^a y^b u(x, y)$ and $u(x, y)$ is a unit, then again $D(f)$ can be computed in the non-normalized coordinates (x, y, z). As in (a), the germs $\{f|_{y=0}\}$ and $\{f|_{y'=0}\}$ are equal if $ab \neq 0$, and isomorphic otherwise.

Finally, notice that if f is smooth, then $D(f) = \chi(f) - \chi(f|_{y=0}) = 1 - 1 = 0$, hence (S) by induction will imply that $D(f) = 0$ for the original singularity $F = F_0$. The stability formula (S) will be verified for all the possible transformations.

Now we verify the stability (S). Hence we start with a quasi-ordinary singularity f represented in a coordinate system (x, y, z) with a normalized parametrization (satisfying 2.1(1)–(5)) $\zeta = cx^{\lambda_1} y^{\mu_1} + \cdots$, where $\lambda_1 \geq \mu_1 \geq 0$.

Case I. Blowing Up an Axis

First assume that the center C is the y-axis.

Since (by our assumption, *cf.* 4.1) the center C is an equi-multiple curve, we must have $\lambda_1 \geq 1$.

We distinguish three cases:

(A) $\mu_1 = 0$;
(B) $\mu_1 \neq 0, \lambda_1 > 1$;

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(C) $\mu_1 \neq 0, \lambda_1 = 1$.

(A) Since $\mu_1 = 0$ we have $\lambda_1 > 1$ by the normalization property 2.1(5). The blow up of C is constructed by gluing two charts U_x and U_y together. The strict transform St of f is contained in U_x (this follows from $\lambda_1 > 1$). Then $E \approx D \times \mathbf{P}^1$, where D is the disc given by the new y-axis (*i.e.* $D \approx C$), and the strict transform St intersects E along D. Since $\lambda_1 > 1$, $m_0(f) = m$. There is only one zero-dimensional strata (namely the origin of D); the strict transform of f will be denoted by f_x. Then:

$$\chi(f) = m\chi(E^o) + \chi(x^m f_x).$$

Here $E^o = E \setminus \mathrm{St} \approx D \times \mathbf{C}$, hence $\chi(E^o) = 1$. We write a similar formula for $f|_{y=0}$: set $\tilde{E} := E \cap \{y = 0\} \approx \mathbf{P}^1$ and $\tilde{E}^0 = \tilde{E} \setminus \mathrm{St} \approx \mathbf{C}$. Then:

$$\chi(f|_{y=0}) = m\chi(\tilde{E}^0) + \chi(x^m f_x|_{y=0}).$$

Now, by the splitting property (3.3) (and from the fact that for any germ h one has $\chi(h^m) = m\chi(h)$):

$$\chi(x^m f_x) = m + \chi(f_x) - m\chi(x|_{f_x=0}) - \chi(f_x|_{x=0}).$$

There is a similar relation for $\chi(x^m f_x|_{y=0})$. Subtracting them one has:

$$\begin{aligned} D(f) = \chi(f_x) - \chi(f_x|_{y=0}) - m[\chi(x|_{f_x=0}) - \chi(x|_{f_x=y=0})] \\ - [\chi(f_x|_{x=0}) - \chi(f_x|_{x=y=0})]. \end{aligned} \tag{4.4}$$

We emphasize that in general at this step it can happen that f_x, expressed in the new variables (x, y, z) is not normalized. Moreover, we have to identify the distinguished axis for f_x as well. In order to do this, we will find the parametrization of f_x. From Table (4.2) we see that if $\zeta = cx^{\lambda_1} + \cdots$ is the (normalized) parametrization of f then $\zeta_x = cx^{\lambda_1 - 1} + \cdots$ is the parametrization of f_x in the new natural local coordinates. If $\lambda_1 - 1 > 1$ then ζ_x is already normalized (and the y axis is distinguished), thus $D(f_x) = \chi(f_x) - \chi(f_x|_{y=0})$. On the other hand if $\lambda_1 - 1 < 1$, we apply Example 4.3 to conclude again $D(f_x) = \chi(f_x) - \chi(f_x|_{y=0})$. Moreover, Lemma 3.6 works as well, because $a = \lambda_1 - 1 \neq 0$ and $\zeta_x = x^{-1}\zeta$ guarantees that the first term of ζ_x divides all the others.

In particular, $\chi(x|_{f_x=0}) = \chi(x|_{f_x=y=0})$ (by 3.6(2)), and $\chi(f_x|_{x=0}) = \chi(f_x|_{x=y=0})$ (by 3.6(1)).

Hence (4.4) gives $D(f) = D(f_x)$.

(B) If $\mu_1 \neq 0$ but $\lambda_1 > 1$, the above argument works with a small modification. In this case $\zeta_x = cx^{\lambda_1 - 1}y^{\mu_1} + \cdots$, which is already normalized up to a permutation of x and y. On the other hand, this is the "symmetric case" (*cf.* 2.8), hence either the x- or the y-axis is distinguished. Moreover, in (3.6) we can freely permute the x and y axes. In particular $\chi(x|_{f_x=0}) = \chi(x|_{f_x=y=0})$, and $\chi(f_x|_{y=0}) = \chi(f_x|_{x=0}) = \chi(f_x|_{x=y=0})$. Hence (4.4) can be reduced to

$$D(f) = \chi(f_x) - \chi(f_x|_{y=0}) - \chi(f_x|_{x=0}) + \chi(f_x|_{x=y=0}).$$

where the last three Euler characteristics are equal. Since either x or y is distinguished, $D(f) = \chi(f_x) - \chi(f_x|_{w=0}) = D(f_x)$ (where w is x or y).

(C) Assume that $\mu_1 \neq 0$ and $\lambda_1 = 1$. Then $E \approx D \times \mathbf{P}^1$ as above, but $E \cap \mathrm{St}$ is a curve singularity containing the center O of D (and whose intersection with $\tilde{E} = \{O\} \times \mathbf{P}^1$ is exactly O). By additivity $\chi(E \setminus \mathrm{St}) = \chi(D^* \times \mathbf{P}^1 \setminus \mathrm{St}) + \chi(\tilde{E} \setminus \mathrm{St})$. But $\chi(D^* \times \mathbf{P}^1 \setminus \mathrm{St}) = 0$ because the first projection $D^* \times \mathbf{P}^1 \setminus \mathrm{St} \to D^*$ is a topological fiber bundle over D^*. Since $\tilde{E} \setminus \mathrm{St} \approx \mathbf{C}$ one obtains that $\chi(E^o) = \chi(\tilde{E}^o) = 1$. Moreover, the argument (A) works again as far as (4.4).

In this case $\zeta = cx^1 y^{\mu_1} + \cdots$, therefore $\zeta_x = cy^{\mu_1} + \cdots$. After permuting x and y, we apply the same argument as in (A) (for the case $\lambda_1 - 1 < 1$). Therefore $D(f_x) = \chi(f_x) - \chi(f_x|_{x=0})$.

Moreover, (permuting the x axis and the y axis in (3.6)) we obtain: $\chi(x|_{f_x}) = \chi(x|_{f_x=y=0})$ (3.6(3)), and $\chi(f_x|_{y=0}) = \chi(f_x|_{x=y=0})$ (3.6(1)). Therefore $D(f) = D(f_x)$ again.

Finally, *suppose we blow up the x axis instead of the y axis*. Since the center must be equimultiple, $\mu_1 \geq 1$. If $\mu_1 > 1$ then $\zeta_y = cx^{\lambda_1} y^{\mu_1 - 1} + \cdots$ is already normalized. This is also the symmetric case (*cf.* 2.8) so we have $\chi(f|_{y=0}) = \chi(f|_{x=0})$ and $\chi(f_y|_{y=0}) = \chi(f_y|_{x=0})$. Therefore $D(f) = \chi(f) - \chi(f|_{x=0})$. We repeat argument (A) as far as (4.4), treating x as the distinguished variable; no normalization is required and the last two expressions vanish by (3.6) exactly as before. Thus $D(f) = D(f_y)$.

If $\mu_1 = 1$ then $\lambda_1 > 1$ (since $(\lambda_1, \mu_1) \notin \mathbf{Z} \times \mathbf{Z}$) and once again $\zeta_y = cx^{\lambda_1} + \cdots$ is already normalized. Here we (as usual) take y as the distinguished axis for f_y. We repeat the argument of case (A) (with no normalization) but use the argument of (C) to show that $\chi(E^o) = \chi(\tilde{E}^o) = 1$.

Case II. Blowing Up a Point

Again there are a few cases to consider, depending on the "size" of (λ_1, μ_1), the first characteristic pair. If $\lambda_1 + \mu_1 \geq 1$ (the "transverse case" in Lipman's language), then the strict transform lives in only two of the three special directions. If $\lambda_1 + \mu_1 < 1$ then the strict transform lives in all three special directions. These possibilities are then multiplied by the condition that φ (*cf.* 2.7) and/or $\varphi_x, \varphi_y, \varphi_z$ are identically zero or not (*i.e.* if $\mu_1(f)$ and/or $\mu_1(f_x)$, *etc.* vanish or not)). We summarize all the cases along with the forms of the parametrizations ζ_* of f and the strict transforms f_x, f_y, f_z in Table 2. To obtain the first characteristic pair of each ζ_* we use the table from paragraph (4.2); each ζ_* is expressed in the natural local coordinates. For simplicity we drop the subscripts and coefficients, *e.g.* $(\lambda, \mu) = (\lambda_1, \mu_1)$.

Cases (A)–(B)–(C) ($\lambda_1 + \mu_1 \geq 1$)

Obviously $E = \mathbf{P}^2$. In cases (A)–(B), $E \cap \mathrm{St} = E \cap \{\text{strict transform of } \{z = 0\}\}$, *i.e.* $E^o \approx \mathbf{P}^2 \setminus \mathbf{C}^1 \approx \mathbf{C}^2$. Therefore $\chi(E^o) = 1$. In case (C), $E^o = \mathbf{P}^2 \setminus \{z^m = x^{\lambda m} y^{\mu m}\}$ (recall $\lambda + \mu = 1$). We verify that in this case $\chi(E^o) = 1$ as well. Indeed, letting $h(x, y, z)$ be the homogeneous polynomial $z^m - x^{\lambda m} y^{\mu m}$ we have $\chi(\mathbf{P}^2 \setminus \{h = 0\}) = \chi(\mathbf{P}^2) - \chi(\{h = 0\})$. Since h is a projective curve of degree m, its Euler characteristic is $-m(m-3) + \sum \mu_i$, where μ_i are the Milnor numbers of h at each of its singular

(A) $\lambda+\mu>1, \zeta=x^\lambda+\cdots, \zeta_x=x^{\lambda-1}+\cdots, \zeta_y=x^\lambda y^{\lambda-1}+\cdots,$
$\mu=0.$

(B) $\lambda+\mu>1, \zeta=x^\lambda y^\mu+\cdots, \zeta_x=x^{\lambda+\mu-1}y^\mu+\cdots, \zeta_y=x^\lambda y^{\lambda+\mu-1}+\cdots,$
$\lambda\mu\neq 0.$

(C) $\lambda+\mu=1, \zeta=x^\lambda y^\mu+\cdots, \zeta_x=y^\mu+\cdots, \zeta_y=x^\lambda+\cdots,$
$\lambda\mu\neq 0$

(D) $\lambda+\mu<1, \zeta=x^\lambda y^\mu+\cdots, \zeta_x=x^{1-\lambda-\mu/\mu}z^{1/\mu}+\cdots, \zeta_y=y^{1-\lambda-\mu/\lambda}z^{1/\lambda}+\cdots,$
$\lambda\mu\neq 0, \quad \zeta_z=x^{\lambda/1-\lambda-\mu}y^{\mu/1-\lambda-\mu}+\cdots.$

Table 2

points. It is easy to check (using $\lambda\mu\neq 0$) that $\sum\mu_i=(m-1)(m-2)$. Thus $\chi(E^o)=3+m(m-3)-(m-1)(m-2)=1$.

Since in all cases (A)–(C) $\lambda_1+\mu_1\geq 1$ we see that $m_0(f)=m$. Therefore

$$\chi(f)=m+\chi(x^m f_x)+\chi(y^m f_y).$$

On the other hand, $\tilde{E}^o=E^o\cap\{y=0\}\approx\mathbf{C}^1$ in all these cases, hence $\chi(f|_{y=0})=m+\chi(x^m f_x|_{y=0})$. Using the splitting property for χ, one gets:

$$\begin{aligned}D(f)&=m-m\chi(x|_{f_x=0})-m\cdot\chi(y|_{f_y=0})+m\cdot\chi(x|_{f_x=y=0})\\&\quad+\chi(f_x)-\chi(f_x|_{x=0})+\chi(f_y)-\chi(f_y|_{y=0})-\chi(f_x|_{y=0})+\chi(f_x|_{x=y=0}).\end{aligned}$$

We claim that $D(f_y)=\chi(f_y)-\chi(f_y|_{y=0})$ in all three cases. In case (A), f_y is already normalized, so this is true by definition. In case (B), f_y is only normalized up to a permutation of x and y, but $\chi(f_y|_{y=0})=\chi(f_y|_{x=0})$ since this is the "symmetric case" (*cf.* 2.8). Finally, in case (C), Example 4.3 gives $D(f_y)=\chi(f_y)-\chi(f_y|_{y=0})$ as well. Moreover, in all three cases $\chi(y|_{f_y=0})=1$ by Lemma 3.6(3). Now for each case we collect several identities.

(A) Applying (4.3) and (3.6) directly gives the following:

$$D(f_x)=\chi(f_x)-\chi(f_x|_{y=0}), \chi(f_x|_{x=0})=\chi(f_x|_{x=y=0}), \chi(x|_{f_x=0})=\chi(x|_{f_x=y=0}).$$

(C) Similarly, applying (4.3) and (3.6) with the roles of x and y reversed, we have:

$$D(f_x)=\chi(f_x)-\chi(f_x|_{x=0}), \chi(f_x|_{y=0})=\chi(f_x|_{x=y=0}), \chi(x|_{f_x=0})=\chi(x|_{f_x=y=0}).$$

(B) Since ζ_x is normalized up to a permutation of x and y, we obtain the identities of (A) or (C), depending on which of x or y is distinguished for f_x.

Applying these to the expression for $D(f)$, in all three cases $D(f)=D(f_x)+D(f_y)$.

Case (D) ($\lambda_1 + \mu_1 < 1$)

Note that since f is assumed to be normalized, $\lambda_1\mu_1 \neq 0$. As before, $E = \mathbf{P}^2$, but $E^0 = \mathbf{P}^2 \setminus \{\text{strict transform of } xy = 0\} \approx \mathbf{C} \times \mathbf{C}^*$. Therefore $\chi(E^o) = 0$. The multiplicity $m_0(f)$ of f now is *not* m, but $\nu = (\lambda_1 + \mu_1)m$. Therefore:

$$\chi(f) = \chi(x^\nu f_x) + \chi(y^\nu f_y) + \chi(z^\nu f_z).$$

Moreover, the exceptional curve of the induced resolution of $f|_{y=0}$ is $\mathbf{P}^1$ with two special points situated on it. Hence:

$$\chi(f|_{y=0}) = \chi(x^\nu f_x|_{y=0}) + \chi(z^\nu f_z|_{y=0}).$$

The contribution of f_x to $D(f)$ is:

$$\chi(f_x) - \nu\chi(x|_{f_x=0}) - \chi(f_x|_{x=0}) - \chi(f_x|_{y=0}) + \nu\chi(x|_{f_x=y=0}) + \chi(f_x|_{x=y=0}).$$

By Table 2, the parametrization of f_x is $\zeta_x = x^{1-\lambda_1-\mu_1/\mu_1} z^{1/\mu_1} + \cdots$. To normalize we need only make the permutations $z \to x$, $x \to y$, $y \to z$ and transformations of type 2.2(a). Making the substitutions and applying 3.6(3), (5) and Example 4.3(b), the total f_x contribution to $D(f)$ is exactly $D(f_x)$.

The contribution of f_y to $D(f)$ is

$$\chi(y^\nu f_y) = \nu + \chi(f_y) - \nu\chi(y|_{f_y=0}) - \chi(f_y|_{y=0}).$$

Again by Table 2, we see that f_y is normalized by permuting x and z and making transformations 2.2(a). In particular the set $\{y = 0\}$ is invariant under this change. Then by 3.6(3) and 4.3(b) we have $\chi(y|_{f_y}) = 1$ and $D(f_y) = \chi(f_y) - \chi(f_y|_{y=0})$, *i.e.* the contribution of f_y to $D(f)$ is exactly $D(f_y)$. Finally, the contribution of f_z to $D(f)$ is

$$\chi(f_z) - \nu\chi(z|_{f_z=0}) - \chi(f_z|_{z=0}) - \chi(f_z|_{y=0}) + \nu\chi(z|_{f_z=y=0}) + \chi(f_z|_{z=y=0}).$$

By Table 2, f_z is normalized by transformations 2.2(a) alone. Then $D(f_z) = \chi(f_z) - \chi(f_z|_{y=0})$ by 4.3(b), and the remaining contributions cancel out by 3.6(4) and (5). Combining these calculations, we have shown that $D(f) = D(f_x) + D(f_y) + D(f_z)$.

This ends the proof of the main theorem.

Note Added on August 1, 2001 Using different techniques (based on the thesis of P. González Pérez) the authors have since been able to extend the results of this paper to calculate the analogous formula for the zeta function. The new result holds in all dimensions and for reducible quasi-ordinary singularities as well.

References

[Abhyankar98] S. S. Abhyankar, *Resolution of Singularities of Embedded Algebraic Surfaces*. Springer Monographs in Math., 2nd Edition, 1998.

[A'Campo75] N. A'Campo, *La fonction zeta d'une monodromie*. Comment. Math. Helvetici, **50** (1975), 233–248.

[AGV88] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps*. Vol. I and Vol. II, Birkhäuser, 1988.

[BMc00] C. Ban and L. J. McEwan, *Canonical resolution of a quasi-ordinary surface singularity*. Canad. J. Math. (6) **52**(2000), 1149–1163.

[BM91] E. Bierstone and P. Milman, *A simple constructive proof canonical resolution of singularities*. Effective Methods in Algebraic Geometry, Progr.Math. **94**(1991), 11–30.

[BK86] E. Brieskorn and H. Knörrer, *Plane Algebraic Curves*. Birkhäuser, Boston 1986.

[Dimca92] A. Dimca, *Singularities and Topology of Hypersurfaces*. Universitext, Springer-Verlag, 1992.

[Gau88] Y-N. Gau, *Embedded topological classification of quasi-ordinary singularities*. Mem. Amer. Math. Soc. **388**(1988).

[GLM97] S. M. Gusein-Zade, I. Luengo and A. Melle-Hernández, *Partial resolutions and the zeta-function of a singularity*. Comment. Math. Helv. **72**(1997), 244–256.

[Hironaka64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero I, II*. Ann. of Math. **79**(1964), 109–326.

[Lê73] Tráng Lê Dũng, *Topologie des singularités des hypersurfaces complexes*. Astérisque **7–8**(1973), 171–182.

[Lipman65] J. Lipman, *Quasi-ordinary singularities of embedded surfaces*. Thesis, Harvard University, (1965).

[Lipman83] ———, *Quasi-ordinary singularities of surfaces in* $\mathbf{C}^3$. Proc. Symp. Pure Math. (2) **40**(1983), 161–172.

[Lipman88] ———, *Topological invariants of quasi-ordinary singularities*. Mem. Amer. Math. Soc. **388**(1988).

[Milnor68] J. Milnor, *Singular points of complex hypersurfaces*. Annals of Math. Studies **61**, Princeton Univ. Press, 1968.

[Oh93] K. Oh, *Topological types of quasi-ordinary singularities*. Proc. Amer. Math. Soc. (1) **117**(1993), 53–59.

[Teissier74] B. Teissier, *Deformation á type topologique constant I, II*. In Séminarie Douady-Verdier 1971–72, Astérisque, **16**(1974), 215–249.

[Villamayor89] O. Villamayor, *Constructiveness of Hironaka's resolution*. Ann. Scient. École Norm. Sup. (4) **22**(1989), 1–32.

[Yau89] S.-T. S. Yau, *Topological types of isolated hypersurface singularities*. Contemp. Math. **101**(1989), 303–321.

[Zariski68] O. Zariski, *Studies in equisingularity, III. Saturation of local rings and equisingularity*. Amer. J. Math. **90**(1968), 961–1023; reprinted in Collected papers **4**, 96–158.

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