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A DENSITY PROBLEM FOR HARDY SPACES OF ALMOST PERIODIC FUNCTIONS

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We construct a counterexample, for $p = 1$, to the conjecture posed by Milaszevitch in 1970: is the space of functions which are analytic in the upper half plane and uniformly almost periodic in its closure dense in the Hardy space H^p $(0 < p < \infty)$ of analytic almost periodic functions?

1. Introduction

Let A_0 denote the space of functions that are analytic in the open upper half plane and uniformly almost periodic in its closure. In this paper we construct an analytic function f which is uniformly almost periodic on any horizontal line in the open half plane in such a way that f is a member of the Hardy space H^1 of almost periodic functions and yet f does not belong to the closure of A_0 in H^1. This provides a counterexample to the conjecture, which we shall refer to as the density problem, posed by Milaszevich [4] in 1970: is A_0 dense in H^p, $0 < p < \infty$ (in analogy with the classical case where the space of bounded analytic functions on the unit disc D is dense in $H^p(D)$)?

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The class of Hardy spaces H^p of almost periodic functions combines many of the properties of the classical Hardy spaces of the upper half plane and the unit disc. H^p consists of functions that are analytic in the upper half plane, uniformly almost periodic on any horizontal line, and yet constrained by a bounded measure on the boundary $\mathbb{R}$ of the upper half plane. This measure arises naturally [2] in the theory of abstract harmonic analysis, where the unit circle is replaced by any locally compact abelian group G and the set of indices over which one forms a "trigonometric series" is taken to be the dual group of G. In our case, the group $\mathbb{R}$ of real numbers endowed with the discrete topology is considered; its dual group can then be identified as $b\mathbb{R}$, the Bohr compactification of $\mathbb{R}$. The natural measure which arises by considering the space of generalized analytic functions on $b\mathbb{R}$ [2] turns out to define precisely the condition that the space be an amalgam of L^p and l^q of $\mathbb{R}$ [3].

2. Notation and preliminary results

We denote by P the open upper half plane and by $\overline{P}$ its closure. The unit disc is denoted D and its boundary is $\partial D = \mathbb{T}$, the unit circle in the complex plane $\mathbb{C}$. Unless otherwise indicated, L^p spaces on subsets of the complex plane are taken with respect to Lebesgue measure on the appropriate subset.

Suppose $0 \le a < b \le \infty$ and define

$$\operatorname{strip}(a, b) = \{x+iy \in \mathbb{C} : a < y < b\} .$$

If f is an harmonic function defined on P was say f is *uniformly almost periodic in the strip* (a, b) if for any $\varepsilon > 0$ there is a $T > 0$ such that any real interval of length T contains τ satisfying

$$|f(z+\tau)-f(z)| < \varepsilon , \quad z \in \operatorname{strip}(a, b) .$$

The above definitions can be extended in the obvious way to half open and closed strips in P.

Given a function f on $\mathbb{C}$ we define the translated and reflected function f_v by

$$f_v(x+iy) = f(v-x+iy) , \quad v \in \mathbb{R} .$$

Let $0 < p < \infty$ and consider the collection of all analytic functions f on P which satisfy the following two conditions:

(2.1) for any $\varepsilon > 0$, f is uniformly almost periodic in the strip $[\varepsilon, \infty)$;

(2.2) $$\|f\|_p^p = \sup_{y>0} \sup_{v\in\mathbb{R}} \frac{1}{\pi} \int_{-\infty}^{\infty} |f(x+iy)|^p \frac{dx}{1+(v-x)^2} < \infty .$$

We define the Hardy space H^p as the space of all analytic functions on P satisfying (2.1) and (2.2).

When $p \geq 1$ the quantity $\|f\|_p$ defines a norm under which the class of functions H^p is a Banach space and it is shown in [2] that these spaces display many of the characteristics of the classical Hardy spaces on the disc and the upper half plane. In particular, if $f \in H^p$ and $1 \leq p < \infty$ then the boundary function

$$f(x) = \lim_{y\to 0} f(x+iy)$$

exists almost everywhere and $f(x+iy)$ can be reproduced as the Poisson integral of $f(x)$. The boundary function f satisfies

(2.3) $$\sup_{v\in\mathbb{R}} \int_{-\infty}^{\infty} |f(x)|^p \frac{dx}{1+(v-x)^2} < \infty ,$$

a fact which can also be seen directly using Fatou's Lemma. In terms of the translated and reflected function f_v we have

$$\sup_{v\in\mathbb{R}} \int_{-\infty}^{\infty} |f(x)|^p \frac{dx}{1+(v-x)^2} = \sup_{v\in\mathbb{R}} \int_{-\infty}^{\infty} |f_v(x)|^p \frac{dx}{1+x^2} .$$

Let $P^{(y)}(t) = y/\pi(y^2+t^2)$, $y, t \in \mathbb{R}$, denote the Poisson kernel and let χ_I denote the characteristic function of a set $I \subseteq \mathbb{R}$. It is shown in [1] that there exist constants C_1 and C_2 such that

$$C_1 \sup_{v\in\mathbb{R}} \int_{v}^{v+1} |f(x)|^p dx \leq \sup_{v\in\mathbb{R}} \int_{-\infty}^{\infty} |f(x)|^p \frac{dx}{1+(v-x)^2}$$

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$$\leqslant C_2 \sup_{v \in \mathbb{R}} \int_v^{v+1} |f(x)|^p dx ,$$

so that by replacing $P^{(1)}(v-t)$ with the box kernel $\chi_{[v,v+1]}$, the condition (2.3) becomes equivalent to the condition

$$\text{(2.4)} \qquad \sup_{v \in \mathbb{R}} \int_v^{v+1} |f(x)|^p dx < \infty .$$

Condition (2.4) is usually expressed [3] by saying that f belongs to the amalgam space $\left(L^p, l^\infty\right)$, with the quantity on the left hand side of (2.4) denoted by $\|f\|_{p,\infty}^p$.

When $u(x+iy)$ is an harmonic function in P satisfying the condition (2.2) and $p \geq 1$, the conjugate function $\tilde{u}$ can be computed explicitly by defining the Hilbert transform on its boundary function $u(x)$. To do this we use the conformal mapping ψ between the unit disc D and the upper half plane given by

$$\psi(z) = -i\left[\frac{z-1}{z+1}\right] , \quad z \in \overline{D} , \quad z \neq -1 .$$

Via this mapping we can define the Hilbert transform Hu_v of each of the functions $u_v(x) = u(v-x)$, since each u_v belonging to $L^p\left(\mathbb{R};\ dx/\left(1+x^2\right)\right)$ is mapped onto a function $u_v \circ \psi$ belonging to $L^p(\mathbf{T})$ and the usual Hilbert transform H is well-defined on this space. Explicitly, Hu_v is defined as $H\left(u_v \circ \psi\right) \circ \psi^{-1}$. Under the conformal mapping ψ however, the translations of u are not preserved; that is, in general $Hu_v \neq (Hu)_v$. But since (the Poisson integrals of) Hu_v and $(Hu)_v$ both represent the imaginary part of the same analytic function their difference can only be a constant; we write $Hu_v - (Hu)_v = c(u, v)$.

Now if $u(x)$ is a continuous bounded function on $\mathbb{R}$ then $\tilde{u}(x)$ is defined by the Hilbert transform and, up to an additive constant, is the boundary function of the conjugate of the harmonic extension of u. It is known [1] that $u \mapsto \tilde{u}$ is a continuous function from L^∞ into BMO, the space of functions of bounded mean oscillation.

PROPOSITION 1. *If* u *is a continuous periodic function on* $\mathbb{R}$ *then* $\tilde{u}$ *is also periodic.*

Proof. Without loss of generality we suppose u to be 2π-periodic. Then u can be identified with a continuous function g defined on $\mathbb{T}$ and g has a Fourier series expansion

$$g(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta} , \quad 0 \le \theta < 2\pi .$$

The conjugate function $\tilde{g}$ is the unique harmonic function satisfying $g + i\tilde{g}$ analytic in D and $\tilde{g}(0) = 0$. It is well known that $\tilde{g}$ has a Fourier series

$$-i \sum_{n\neq 0} \operatorname{sgn} n a_n e^{in\theta}$$

in the sense that the conjugate Poisson integral of g has an expression of the form

$$\tilde{g}\left(re^{i\theta}\right) = -i \sum_{n\neq 0} \operatorname{sgn} n a_n r^{|n|} e^{in\theta}$$

and this tends to a limit almost everywhere on $\mathbb{T}$ as $r \uparrow 1$.

Now let us define the function $\tilde{u} = \tilde{g} \circ \phi$ on P where $\phi(z) = \exp(iz)$. Since ϕ is analytic, $\tilde{u}$ is harmonic and periodic on P . In the limit

$$\tilde{u}(t) = -i \sum_{n\neq 0} \operatorname{sgn} n a_n e^{int} , \quad t \in \mathbb{R} ,$$

defines almost everywhere a periodic function on $\mathbb{R}$ and gives a conjugate function for u since

$$(u+i\tilde{u})(z) = 2 \sum_{n\geq 0} a_n e^{inz}$$

is analytic in P . This completes the proof.

We denote by $\Gamma(t)$ the cone

$$\Gamma(t) = \{z = x+iy \in P : |x-t| < y\} , \quad t \in \mathbb{R} .$$

If u is an harmonic function on P we define the *non-tangential maximal*

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function u^* of u by

$$u^*(t) = \sup_{z \in \Gamma(t)} |u(z)| , \quad t \in \mathbb{R} .$$

The following proposition is proved in [5] and although the result is important for the construction of the counterexample in §3, the techniques of its proof are not and so the proof is omitted.

PROPOSITION 2. *Suppose* u *is harmonic in* P *and* $u^* \in (L^1, l^\infty)$ *. If* $x \to \tilde{u}(x+iy) \in L^\infty$ *for some* $y > 0$ *, then* $\tilde{u} \in (L^1, l^\infty)$ *.*

Note that in this case we have $u + i\tilde{u} \in H^1$, for u^* dominates u and the norm condition for membership of (L^1, l^∞) is precisely that of belonging to H^1 .

A partial answer to the density problem was given by Milaszevitch [4, p. 425] in 1970 and may be stated as follows.

PROPOSITION 3. *Let* $f \in H^p$, $0 < p < \infty$ *. If* f *belongs to the closure of* A_0 *in* H^p *then the function* $t \to f_t$ *is continuous as a function from* $\mathbb{R}$ *into* H^p *.*

Recall that f_t denotes the function f translated and reflected in its real variable by $t \in \mathbb{R}$.

We now proceed to construct a function $f \in H^1$ that does not satisfy the above continuity condition.

3. The construction

The idea behind our counterexample is as follows:- we shall construct a real-valued bounded function $u \in (L^1, l^\infty)$ such that the function $t \to u_t$ is not continuous from $\mathbb{R}$ into (L^1, l^∞) . The function u is to be such that its harmonic extension is uniformly almost periodic on any line $L_y = \{\text{Im } z = y\}$, $y > 0$, and also that the harmonic extension of its conjugate $\tilde{u}$ is uniformly almost periodic on any such line. The maximal function u^* will belong to (L^1, l^∞) and since $\tilde{u}$ is bounded on

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any line L_y , $y > 0$ (being uniformly almost periodic there), this implies $u \in (L^1, l^\infty)$, by Proposition 2. Thus $f = u + i\tilde{u} \in H^1$. But as $t \to f_t$ is not continuous as a function from $\mathbb{R}$ into H^1 we will have by Proposition 3 that f does not belong to the closure of A_0 in H^1 .

To begin, take a periodic progression

$$n_k^{(1)} = kT_1 , \quad k \in \mathbb{Z} ,$$

where $T_1 \geq 10$ is a large integer to be determined later. Let ρ be a C^∞ function with $\|\rho'\|_\infty < \infty$ and such that $\rho \equiv 1$ on $[\frac{1}{4}, \frac{3}{4}]$, $\rho \equiv 0$ outside $[0, 1]$, and ρ is monotonic on each of the intervals $[0, \frac{1}{4}]$ and $[\frac{3}{4}, 1]$. Define ρ_1 periodically by $\rho_1(x) = \rho(x)$, for $x \in [0, T_1]$ and $\rho_1(x+T_1) = \rho_1(x)$, for $x \in \mathbb{R}$. The function u_1 is then determined by the formula

$$u_1(t) = \rho_1(t)\sin(2\pi N_1 t) , \quad t \in \mathbb{R} ,$$

where N_1 is a large integer to be determined later. Note that u_1 is periodic with period T_1 and so its harmonic extension is also periodic.

Proceeding in this fashion, suppose we have defined $u_1, u_2, \ldots, u_{m-1}$. To define u_m we begin by defining the arithmetic progression

$$n_k^{(m)} = a_m + kT_m , \quad k \in \mathbb{Z} ,$$

where $a_m \in \mathbb{Z}$ satisfies

$$a_m \in \bigcap_{i<m} \bigcup_{j=1}^{\infty} \left[\frac{4jT_i}{10}, \frac{6jT_i}{10}\right]$$

and $T_m \geq 10T_{m-1}$ is a large integer to be determined later.

Note that the condition $T_k \geq 10.T_{k-1}$ for each k ensures that it is possible to choose each a_m , that is,

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$$\bigcap_{i<m} \bigcup_{j=1}^{\infty} \left[\frac{4jT_i}{10}, \frac{6jT_i}{10}\right] \neq \emptyset .$$

Define ρ_m periodically by $\rho_m(x) = \rho(x-a_m)$ for $x \in [a_m, a_m+T_m]$ and $\rho_m(x+T_m) = \rho_m(x)$ for any $x \in \mathbb{R}$. Then u_m is defined by the formula

$$u_m(t) = \rho_m(t)\sin(2\pi N_m t) , \quad t \in \mathbb{R} ,$$

where $N_m > N_{m-1}$ is a large integer to be determined later.

Now each u_m is a periodic function on $\mathbb{R}$, $u_m \in L^\infty$, and none of the u_m's overlap; that is, on any interval $[v, v+1]$, $v \in \mathbb{Z}$, at most one u_m is non-zero.

We put

$$u(t) = \sum_{m=1}^{\infty} u_m(t) , \quad t \in \mathbb{R} . \tag{3.1}$$

The Poisson integral extends u naturally to all of P.

LEMMA 1. *The sequences* $(T_m)_{m\geq 1}$ *and* $(N_m)_{m\geq 1}$ *can be chosen so that* u *is uniformly almost periodic on any line* L_y, $y > 0$.

Proof. Fix some line L_{y_0}, $y_0 > 0$. Since each u_m is periodic on this line it is enough to show that the series defining u in (3.1) converges uniformly on L_{y_0}.

We have

$$u_m(x+iy_0) = \int_{-\infty}^{\infty} u_m(t)P^{(y_0)}(x-t)dt .$$

Since the Poisson kernel consists of two monotonic pieces, we can apply Bonnet's form of the Second Mean Value Theorem to each piece. Thus

$$\left|\int_x^{x+1} u_m(t)P^{(y_0)}(x-t)dt\right| \leq \frac{1}{\pi y_0} \cdot \frac{1}{\pi N_m} , \tag{3.2}$$

where we have estimated the integral

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$$\left|\int_x^{x+1} u_m(t)dt\right| = \left|\int_x^{x+1} \rho_m(t)\sin\left(2\pi N_m t\right)dt\right| \le \frac{1}{\pi N_m}$$

in the same fashion.

We now note that

$$\left|\int_{x+1}^{\infty} u_m(t)P^{(y_0)}(x-t)dt\right| \le \frac{1}{\pi}\sum_{N=1}^{\infty}\frac{y_0}{y_0^2+T_m^2N^2}$$

since $|u_m(t)| \le 1$, and that

$$\begin{aligned}\sum_{N=1}^{\infty}\frac{y_0}{y_0^2+T_m^2N^2} &\le \frac{y_0}{y_0^2+T_m^2} + \int_1^{\infty}\frac{y_0}{y_0^2+T_m^2x^2}\,dx \\ &= \frac{y_0}{y_0^2+T_m^2} + \frac{y_0}{T_m}\int_{T_m}^{\infty}\frac{ds}{y_0^2+s^2} \\ &= \frac{y_0}{y_0^2+T_m^2} + \frac{1}{T_m}\int_{T_m/y_0}^{\infty}\frac{dw}{1+w^2} \\ &\le \frac{y_0}{y_0^2+T_m^2} + \frac{y_0}{T_m^2} .\end{aligned}$$

Similar estimates can be given for the intervals $(x-1, x)$ and $(-\infty, x-1)$ so that we obtain

$$\left|\int_{-\infty}^{\infty} u_m(t)P^{(y_0)}(x-t)dt\right| \le \frac{2}{\pi^2 y_0 N_m} + \frac{2y_0}{\pi}\left(\frac{1}{y_0^2+T_m^2} + \frac{1}{T_m^2}\right) .$$

We now select the N_m's and the T_m's to satisfy both the conditions of the construction and the inequality

$$(3.3) \qquad \sum_{m=1}^{\infty}\left[\frac{2}{\pi^2 y_0 N_m} + \frac{2y_0}{\pi}\left(\frac{1}{y_0^2+T_m^2} + \frac{1}{T_m^2}\right)\right] < \infty .$$

This sum is independent of $x \in \mathbb{R}$ but will depend on the line L_{y_0} chosen. With $(T_m)_{m\ge1}$ and $(N_m)_{m\ge1}$ thus chosen, u is the uniform sum of periodic functions on the line L_{y_0} , $y_0 > 0$, and hence u is almost periodic on any line in the upper half plane.

Since the sequence $(N_m)_{m\geq 1}$ is increasing, we also have

LEMMA 2. *The function* $t \to u_t$ *is not continuous from* $\mathbb{R}$ *into* (L^1, l^∞) .

LEMMA 3. $u^* \in (L^1, l^\infty)$.

Proof. This is trivial since $u \in L^\infty$.

LEMMA 4. *The sequences* $(T_m)_{m\geq 1}$ *and* $(N_m)_{m\geq 1}$ *can be chosen so that* $\tilde{u}$ *is uniformly almost periodic on any line* L_y , $y > 0$.

Proof. Note first of all that

$$u(t) = \sum_{m=1}^{\infty} \tilde{u}_m(t) \ , \quad t \in \mathbb{R} \ ,$$

since the conjugation operator is continuous from L^∞ into BMO and the series (3.1) defining u converges absolutely. Fix some line L_{y_0} , $y_0 > 0$; since each $\tilde{u}_m$ is periodic on this line (Proposition 1) it is enough to show that

$$\sum_{m=1}^{\infty} \tilde{u}_m\left(x+iy_0\right)$$

converges uniformly in $x \in \mathbb{R}$.

We begin by studying the boundedness of

$$u_m(t) = \left[\rho_m(t)\sin\left(2\pi N_m t\right)\right]^{\sim} \ , \quad t \in \mathbb{R} \ .$$

Since each u_m is periodic on $\mathbb{R}$ we can consider it as a function on $\mathbb{T}$ by writing $g_m(t) = u_m\left(T_m t/2\pi\right)$, $0 \leq t < 2\pi$. Then we have [6, p. 55],

$$\left|\left[\rho_m\left(\frac{xT_m}{2\pi}\right)\sin\left(T_mN_mx\right)\right]^{\sim} - \rho_m\left(\frac{xT_m}{2\pi}\right)|\sin\left(T_mN_mx\right)|^{\sim}\right|$$

$$= \left|P.V. \int_0^{2\pi} \frac{\rho_m\left(tT_m/2\pi\right)-\rho_m\left(xT_m/2\pi\right)}{\tan\left((t-x)/2\right)} \sin\left(T_mN_mt\right)dt\right|$$

$$\le \frac{KT_m}{2\pi} \|\rho'\|_\infty \cdot \frac{1}{T_mN_m}$$

$$= K\|\rho'\|_\infty/2N_m .$$

To obtain this estimate we have used the Mean Value Theorem applied separately to each of the monotonic pieces of the kernel

$$\frac{\rho_m\left(tT_m/2\pi\right)-\rho_m\left(xT_m/2\pi\right)}{\tan\left((t-x)/2\right)}$$

and the fact that this kernel is bounded by $T_m\|\rho'\|_\infty/2$. The constant K is an upper bound for the number of monotonic pieces of the kernel and is independent of m .

Transferring back to $\mathbb{R}$ we have

$$\tilde{u}_m(t) = \rho_m(t)\cos\left(2\pi N_mt\right) + h_m(t) , \quad t \in \mathbb{R} ,$$

where

$$|h_m(t)| < K\|\rho'\|_\infty/2N_m , \quad t \in \mathbb{R} ,$$

and hence

$$(3.4) \qquad |\tilde{u}_m(t)| \le 1 + K\|\rho'\|_\infty/2N_m , \quad t \in \mathbb{R} .$$

From this we see that if I is any interval of unit length in $\mathbb{R}$ we have

$$\left|\int_I \tilde{u}_m(t)dt\right| = \left|\int_I \left[\rho_m(t)\cos\left(2\pi N_mt\right)+h_m(t)\right]dt\right|$$

$$\le \frac{1}{\pi N_m} + \frac{K\|\rho'\|_\infty}{2N_m} ,$$

where again we have used the Mean Value Theorem applied separately to the (at most) two monotonic pieces of $\rho_m(t)$ over the interval I .

We now write

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$$\tilde{u}_m(x+iy_0) = \int_{-\infty}^{\infty} \tilde{u}_m(t)P^{(y_0)}(x-t)dt$$

and argue as in Lemma 1. We have

$$\left|\int_x^{x+1} \tilde{u}_m(t)P^{(y_0)}(x-t)dt\right| \le \frac{1}{\pi y_0}\left(\frac{1}{\pi N_m} + \frac{K\|\rho'\|_\infty}{2N_m}\right) ,$$

and by (3.4),

$$\left|\int_{x+1}^{\infty} \tilde{u}_m(t)P^{(y_0)}(x-t)dt\right| \le \left(1 + \frac{K\|\rho'\|_\infty}{2N_m}\right)\frac{y_0}{\pi}\left(\frac{1}{y_0^2+T_m^2} + \frac{1}{T_m^2}\right) .$$

Similar estimates hold over the intervals $(x-1, x)$ and $(-\infty, x-1)$ so that finally

$$\left|\int_{-\infty}^{\infty} \tilde{u}_m(t)P^{(y_0)}(x-t)dt\right|$$

$$\le \frac{2}{\pi y_0}\left(\frac{1}{\pi N_m} + \frac{K\|\rho'\|_\infty}{2N_m}\right) + \frac{2y_0}{\pi}\left(1 + \frac{K\|\rho'\|_\infty}{2N_m}\right)\left(\frac{1}{y_0^2+T_m^2} + \frac{1}{T_m^2}\right) .$$

We now select the N_m's and T_m's to satisfy all previous conditions as well as

$$\sum_{m=1}^{\infty} \frac{2}{\pi y_0}\left(\frac{1}{\pi N_m} + \frac{K\ \rho'\ _\infty}{N_m}\right) + \frac{2y_0}{\pi}\left(1 + \frac{K\ \rho'\ _\infty}{2N_m}\right)\left(\frac{1}{y_0^2+T_m^2} + \frac{1}{T_m^2}\right) < \infty .$$

This sum is independent of $x \in \mathbb{R}$ but will depend on the line L_{y_0} chosen. With $(T_m)_{m\ge1}$ and $(N_m)_{m\ge1}$ thus chosen, $\tilde{u}$ is the uniform sum of periodic functions on the line L_{y_0}, and hence is uniformly almost periodic on any line in the upper half plane. This completes the proof.

In conclusion, the function $f = u + i\tilde{u}$ belongs to H^1 but does not belong to the closure of A_0 in H^1.

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