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METRICS AND SPECIAL KÄHLER GEOMETRY ON THE MODULI SPACES OF HIGGS BUNDLES AND HITCHIN SYSTEMS

ZHENXI HUANG

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The notions of Hitchin systems and Higgs bundles (also called Higgs pairs) were introduced by N. Hitchin in 1987. They rapidly formed a subject lying on the crossroads of representation theory, symplectic geometry and algebraic geometry. In this research area, the main objects that attract mathematicians' attention are the moduli spaces of Higgs bundles $\mathcal{M}_{n,d}$ (the moduli space of Higgs bundles is a space parameterising the collection of all Higgs bundles). These moduli spaces have many good properties that make them interesting objects worthy of study. They are symplectic manifolds. For example, the moduli space of the Higgs bundles with rank one and degree zero is the cotangent bundle of the Jacobian variety of a Riemann surface. These moduli spaces are also equipped with Riemannian hyperkähler metrics g_{hk} which cannot be written explicitly in general, but can be approximated by another metric, g_{sf}, called the semi-flat metric. Roughly, $g_{hk} = g_{sf} + \{\text{correction}\}$. In fact, Gaiotto *et al.* conjectured what the 'correction' should be in 2013 [5]. To find the semi-flat metric of $\mathcal{M}_{n,d}$, we first investigate the integrable system (Hitchin system) $h : \mathcal{M}_{n,d} \to B$, where the map h is known as the Hitchin map. The Lagrangian fibres of the Hitchin systems are Jacobian (or Prym) varieties of spectral curves of a given compact Riemann surface. The semi-flat metric of $\mathcal{M}_{n,d}$ is induced by a special Kähler metric on B through the Lagrangian fibration.

In the thesis, we compute these metrics as explicitly as possible and give close approximations to them. In Chapter 1, we recall the basic notions of Hitchin systems [6–8].

In Chapters 2, 3 and 4, we give formulas for $\mathrm{SL}_n(\mathbb{C})$- and $\mathrm{GL}_n(\mathbb{C})$-Hitchin systems. To do that, we treat the regular part of the Hitchin systems as complex torus bundles

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with fibres the Jacobian varieties (or Prym varieties) of the spectral curves. These fibres are Lagrangian submanifolds and the base B^{reg} has a special Kähler structure. Then we apply the results of Freed [4] for special Kähler manifolds and results of Hitchin [9] for complex Lagrangian submanifolds to get two sets of complex affine coordinate systems on B^{reg}. By using these coordinates, we get some formulas for these metrics. For the $\mathrm{GL}_n(\mathbb{C})$-Hitchin system,

$$g_{sk} = \frac{1}{2}\sum_{i,j}^{g_s}(\mathrm{Im}(\tau_{ij})(ds_1^i \otimes ds_1^j + ds_2^i \otimes ds_2^j)),$$

$$g_{sf} = \sum_{i,j,k} g_{ijk} ds_k^i \otimes ds_k^j + g_{ijk}^{-1} d\lambda_k^i \otimes d\lambda_k^j,$$

where s_m^i, λ_n^j are the real and imaginary parts of the affine coordinates and τ_{ij} are the periods of the spectral curves. One may notice that the spectral curves play the key role in the geometry of Hitchin systems.

In Chapter 5, we study the variation of the complex structures of the spectral curves, derive a new residue formula for the Donagi–Markman cubic,

$$\frac{\partial \tau_{jk}}{\partial \lambda_i} = 2\pi i \sum_{l=1}^{m} \mathrm{Res}\left(\frac{\omega_i\omega_j\omega_k\lambda\frac{\partial}{\partial\lambda}}{\theta}; \widetilde{p}_l\right),$$

which measures the variation of the period of a spectral curve, and give a new description of the special Kähler geometry on B.

In Chapter 6, we discuss the relation between special Kähler geometry and topological recursion [2, 3]. That leads to a conclusion that once we know the periods of one of the spectral curves, we can derive the special Kähler geometry of the base everywhere else. The resulting paper [1] is joint work with David Baraglia.

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ZHENXI HUANG, School of Mathematical Science, University of Adelaide,
Adelaide, South Australia 5005, Australia
e-mail: huangzhendong2011@163.com

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