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A CLOSED-FORM ANALYTICAL SOLUTION FOR THE VALUATION OF CONVERTIBLE BONDS WITH CONSTANT DIVIDEND YIELD

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Abstract

In this paper, a closed-form analytical solution for pricing convertible bonds on a single underlying asset with constant dividend yield is presented. A closed-form analytical formula has apparently never been found for American-style convertible bonds (CBs) of finite maturity time although there have been quite a few approximate solutions and numerical approaches proposed. The solution presented here is written in the form of a Taylor's series expansion, which contains infinitely many terms, and thus is completely analytical and in a closed form. Although it is only for the simplest CBs without call or put features, it is nevertheless the first closed-form solution that can be utilised to discuss convertibility analytically. The solution is based on the homotopy analysis method, with which the optimal converting price has been elegantly and temporarily removed in the solution process of each order, and consequently, the solution of a linear problem can be analytically worked out at each order, resulting in a completely analytical solution for the optimal converting price and the CBs' price.

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1. Introduction

Convertible bonds (CBs) are complex in nature and widely-used hybrid financial instruments. They are different from bonds and stocks, and yet with some combining characteristics of bonds and options. During the life of a convertible bond, the holder can choose to convert the bond into the stock of the issuing company or financial institution with a pre-specified conversion price, or hold the bond till maturity to receive coupons and the principal prescribed in the purchase agreement.

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A theoretical framework for pricing CBs was initiated by Ingersoll [8] and Brennan and Schwartz [3]. They used the contingent claims approach and took the firm value as the underlying variable. Brennan and Schwartz [4] later investigated the effect of stochastic interest rates and found that the effect of a stochastic term structure on convertible bond prices is so small that it can be ignored for empirical purposes. In 1986, McConnel and Schwartz [13] developed a valuation model, using the stock value as the underlying stochastic variable.

Because of their hybrid nature, the valuation of CBs can be much more complicated than that of simple options, especially when an additional complexity such as callability and putability or the issue of default risk of the issuer is added to the valuation task. Nyborg [14] presented a closed-form solution for most basic convertible bonds, that is, those that are non-callable and non-putable, but where conversion is only allowed at maturity. If conversion is allowed at any time prior to expiry, we say that the CB is of American style, and there are only numerical approaches to deal with this in the literature. Examples of numerical approaches include the finite element approach (see [1]), the finite volume approach (see [19]) and the finite difference approach (see [16]).

Since most traded CBs are of American style in today's financial markets, it is extremely desirable that an analytical formula be added to the literature of pricing CBs. In this paper, an analytical closed-form solution is presented for the first time to price CBs without callability and putability but with conversion being of American style. The essential difficulty for this problem lies in the fact that once conversion is allowed to take place prior to expiry, there is an optimal value of the underlying asset, at which the holder of the CB should convert the CB into the underlying asset. Mathematically, like the problem of valuing American options, the problem becomes highly nonlinear because the problem has been turned into a free boundary value problem (as happens with, for example, Stefan problems of melting ice [7]). Thus, valuing CBs with American-style conversion is very different from the valuation of CBs with European-style conversion as the latter is essentially still a linear problem.

The explicit and closed-form analytical solution presented in this paper is an extension of Zhu [17], who presented a closed-form solution for the valuation of American options by constructing a Taylor's series expansion of the unknown option price and the unknown optimal exercise price based on the homotopy analysis method. The terminology "closed-form" has been used in the literature of financial derivatives' pricing theory in different ways. Here we use the definition given by Gukhal [6]. That is, by being a "closed-form" solution, it is meant that the solution can be written in terms of a set of standard and generally accepted mathematical functions and operations. A solution in the form of an infinite series expansion is certainly in a closed form by this definition. By "explicit", we mean that the solution for the unknown function (or functions) can be determined explicitly in terms of all the inputs to the problem. It is within this context that we interpret other authors' comments that such a solution does not exist in the literature.

This paper is organised into four sections. In Section 2, the valuation problem is first formulated into a differential system. In Section 3, a closed-form solution to the differential system is followed by some numerical examples presented in Section 4. Some concluding remarks are given in the last section.

2. The formation of the problem

Let $V(S, t)$ denote the value of a convertible bond, S be the price of the underlying asset and t be the current time. Then, under the Black-Scholes framework (see [2]), the value of a convertible bond V should satisfy the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0, \tag{2.1}$$

where r is the risk-free interest rate, σ is the volatility of the underlying asset price and D_0 is the rate of continuous dividend paid to the underlying asset. In this paper, r and σ are assumed to be constant.

Equation (2.1) needs to be solved together with a set of appropriate boundary conditions and the terminal condition.

The terminal condition of a CB is slightly more complicated than that of an option. Because of the holder's right of conversion and the issuer's guaranteed amount of redemption, there is a jump from the condition

$$V(S, T^-) = \max\{nS, Z\}, \tag{2.2}$$

imposed right before the expiry time T, to the condition

$$V(S, T) = Z, \tag{2.3}$$

imposed right at the expiry time T when the CB has been redeemed by the issuer. In Equations (2.2) and (2.3), n is the conversion ratio and Z is the principal (also called face value or par value). Mathematically, such a jump represents a singularity and it is usually hard to deal with. However, investors would always use condition (2.2) to maximise their profit. Therefore, (2.2) should always be used to value a CB for any time prior to its expiry. In other words, the valuation problem of CBs can be mathematically conducted in two time zones: a Zone 1 that includes all the time up to but not including the expiry time and a Zone 2 that has one single point on the time axis with $t = T$. In Zone 2 we already know the value of the CB through (2.3) and the remaining task is to value the CB in Zone 1.

If a CB can only be "converted" or "redeemed" at expiry, the boundary condition for large underlying asset values must be placed at infinity, that is,

$$\lim_{S \to \infty} \frac{V(S, t)}{S} = n, \tag{2.4}$$

just like that in the problem of European options. However, since most of the convertible bonds issued are of American style, conversion is allowed at any time prior to the expiry of the CB, just like American options. For these American-style CBs, the boundary condition at infinity should be replaced by two conditions:

$$\begin{cases} V(S_c(t), t) = nS_c(t), \\ \dfrac{\partial V}{\partial S}(S_c(t), t) = n, \end{cases} \tag{2.5}$$

where $S_c(t)$ is a moving boundary which needs to be found as part of the solution. This paper focuses on the valuation of CBs with American-style conversion since the valuation of those with European-style conversion is trivial.

The price of a CB can be bounded above by a call feature sold to the issuer. The call option allows the issuer to purchase back the bond if the underlying asset value becomes too high. The price of a CB can also be bounded below by a "put" option that allows the holder to sell the CB back to the issuer in case the underlying asset value becomes too low. While the call feature would lower the price of a CB, the put feature increases the price of a CB in comparison with the CB without. The valuation of CBs with a call feature is not significantly more complicated than that without simply because the call option is in the hands of the issuer; the bond holder is only obliged to deliver if the bond is called. This obligation underpins the movement of the free boundary to an upper limit, beyond which the imposition of the upper bound on the CB price becomes effective. On the other hand, the "put" feature gives the holder a right to either hold the bond or exercise the right to sell the bond. Thus, the "put" feature requires a second free boundary be introduced to the problem, in addition to the free boundary associated with the conversion, $S_c(t)$, as mentioned in the previous paragraph.

In this paper, we focus on CBs without "put" or "call" features. The valuation problem with a call feature is currently being worked out and the results are to be presented in a forthcoming paper.

In the absence of default issues, the boundary condition at $S = 0$ for convertible bonds without "put" features is

$$V(0, t) = Ze^{-r(T-t)} + \sum_i K_i e^{-r(t_i - t)}, \tag{2.6}$$

where K_i represents discrete coupon payments to the CB's holder by the issuer and t_i is the time at which the ith coupon will be paid ($t_i > t$). Financially, such a boundary condition implies that the CB would be of the same value as a regular bound when the stock price is very low. This is true, except that the bond behaves like a risky bond if the issuer's credit risk is taken into consideration. When risky bonds default, their value becomes zero. Therefore, if defaultability has to be taken into consideration for risky CBs, this boundary condition has to be altered.

Since the presence of discrete coupon payments introduces no additional difficulty other than making the solution process a little bit more tedious as far as our solution procedure to be presented in the next section is concerned, we shall concentrate only on the cases with zero coupon payments. In other words, we set all K_is in Equation (2.6) to zero from now on.

The CB valuation problem is now completely defined by a differential system composed of Equations (2.1), (2.2), (2.5) and (2.6). To solve this system more efficiently and consistently, we first normalise the system by introducing dimensionless variables as follows:

$$V' = V/Z, \quad S' = S/Z, \quad \tau' = \tau\sigma^2/2 = (T-t)\sigma^2/2.$$

Then omitting all primes for the sake of simplicity, the normalised system can be easily derived as

$$\begin{cases} -\dfrac{\partial V}{\partial \tau} + S^2\dfrac{\partial^2 V}{\partial S^2} + (\gamma - \beta)S\dfrac{\partial V}{\partial S} - \gamma V = 0, \\ V(0,\tau) = e^{-\gamma\tau}, \\ V(S_c(\tau),\tau) = nS_c(\tau), \\ \dfrac{\partial V}{\partial S}(S_c(\tau),\tau) = n, \\ V(S,0) = \max\{nS, 1\}, \end{cases} \tag{2.7}$$

in which γ $(= 2r/\sigma^2)$ is the risk-free interest rate relative to the volatility of the underlying asset price and β $(= 2D_0/\sigma^2)$ is the dividend yield rate relative to the volatility of the underlying asset price. Such a normalisation not only provides mathematical efficiency in the solution procedure but also has some financial advantage in that CBs of different face values and under different currencies can be easily compared.

The normalised differential system (2.7) shows that the solution will be a four-parameter family. That is, the solution of the system depends only on four parameters: the relative interest rate, γ, the conversion ratio, n, the relative dividend payment rate, β and the dimensionless time to expiry, $\tau_{\exp} = T\sigma^2/2$. It should be noted that the introduction of time to expiration τ as the difference between the expiration time T and the current time t results in the change of the terminal condition (2.2) to an initial condition in (2.7).

3. A closed-form analytical solution

To find a closed-form analytical solution for the differential system (2.7), we follow Zhu's method in [17] and introduce a transform

$$x = -\ln(S/S_f(\tau)) \tag{3.1}$$

to shift the moving boundary conditions to fixed boundary conditions. After performing the coordinate transformation and some algebraic manipulations, the differential system (2.7) can be written as

$$\begin{cases} \dfrac{\partial V}{\partial \tau} - \dfrac{\partial^2 V}{\partial x^2} + (\gamma - \beta - 1)\dfrac{\partial V}{\partial x} + \gamma V = -\dfrac{1}{S_c(\tau)}\dfrac{dS_c}{d\tau}\dfrac{\partial V}{\partial x}, \\ V(x, 0) = 1, \\ V(0, \tau) = nS_c(\tau), \\ \dfrac{\partial V}{\partial x}(0, \tau) = -nS_c(\tau), \\ \lim\limits_{x\to\infty} V(x, \tau) = e^{-\gamma\tau}. \end{cases} \tag{3.2}$$

The non-zero boundary conditions at infinity could pose a problem later if we want to adopt the solution strategy employed by Zhu [17]. So, a simple shift of the vertical axis by a time-dependent amount of $e^{-\gamma\tau}$ can be carried out and the transform

$$U(x, \tau) = V(x, \tau) - e^{-\gamma\tau} \tag{3.3}$$

changes (3.2) to

$$\begin{cases} \dfrac{\partial U}{\partial \tau} - \dfrac{\partial^2 U}{\partial x^2} + (\gamma - \beta - 1)\dfrac{\partial U}{\partial x} + \gamma U = -\dfrac{1}{S_c(\tau)}\dfrac{dS_c}{d\tau}\dfrac{\partial U}{\partial x}, \\ U(x, 0) = 0, \\ U(0, \tau) - nS_c(\tau) = -e^{-\gamma\tau}, \\ \dfrac{\partial V}{\partial x}(0, \tau) + nS_c(\tau) = 0, \\ \lim\limits_{x\to\infty} V(x, \tau) = 0. \end{cases} \tag{3.4}$$

Now we notice that the nonlinearity in (3.4) is explicitly concentrated in the nonhomogeneous term of the governing differential equation while the boundary conditions defined on a moving boundary have been replaced by a set of boundary conditions defined on a *fixed* boundary. We can take advantage of this when the homotopy analysis method is applied to solve a nonlinear system with fixed boundary conditions.

The homotopy analysis method originated from the homotopic deformation in topology and was initially suggested by Ortega and Rheinboldt [15]. Recently, it has been successfully used to solve a number of heat transfer problems (see [9] and [12]) and fluid flow problems (see [11] and [10]). The essential concept of the method is to construct a continuous "homotopic deformation" through a series expansion of the unknown function. The series solution of the unknown function is of infinitely many terms, but is nevertheless of closed form. By a "closed-form" solution, we mean that it can be written in terms of functions and mathematical operations from a given generally accepted set and theoretically can be calculated to any desired degree of accuracy. By this definition, a solution explicitly written in terms of a convergent infinite series is certainly a closed-form solution. However, in practice, calculation of the actual values of the unknown function at a point in space (the underlying asset price here) and time requires truncation of the infinite series to a finite one, just as performed for the calculation of many other standard mathematical functions. Therefore, the fact that the realisation of our closed-form analytical solution in numerical values requires truncation of the series solution does not devalue the analyticity of the solution itself. The key point to determine if a homotopic solution is truly analytic or not is whether or not an analytical solution can be found at each order. If an analytical solution can be constructed at each order like we present in this paper rather than computed numerically like in [9] and [12], where each order of the equations was solved numerically through the boundary element method, a truly closed-form analytical solution is obtained. As far as the accuracy is concerned, unlike those unavoidable "discretisation errors" associated with a finite difference method or finite element method, there are no "discretisation errors" at each order for a truly closed-form analytical solution. Theoretically speaking, one should be able to achieve machine accuracy if a sufficient number of terms is included in the summation process, as then all the numerical errors will result only from "truncation errors" when real numbers are stored in a computer with a finite number of digits.

We now construct two new unknown functions $\bar{U}(x, \tau, p)$ and $\bar{S}_c(\tau, p)$ that satisfy the following differential system:

$$\begin{cases} (1-p)\mathscr{L}[\bar{U}(x,\tau,p)-\bar{U}_0(x,\tau)] = -p\{\mathscr{A}[\bar{U}(x,t,p),\bar{S}_c(\tau,p)]\}, \\ \bar{U}(x,0,p) = (1-p)\bar{U}_0(x,0), \\ \bar{U}(0,\tau,p) - n\bar{S}_c(\tau,p) = -e^{-\gamma\tau}, \\ \partial\bar{U}(0,\tau,p)/\partial x + n\bar{S}_c(\tau,p) = (1-p)\left[e^{-\gamma\tau} + \partial\bar{U}_0(0,\tau)/\partial x + \bar{U}_0(0,\tau)\right], \\ \lim\limits_{x\to\infty} \bar{U}(x,\tau,p) = 0, \end{cases} \tag{3.5}$$

where $\mathscr{L}$ is a differential operator defined as

$$\mathscr{L} = \partial/\partial\tau - \partial^2/\partial x^2 + (\gamma-\beta-1)\partial/\partial x + \gamma,$$

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and $\mathscr{A}$ is a functional defined as

$$\mathscr{A}[\bar{U}(x,\tau,p),\bar{S}_c(\tau,p)] = \mathscr{L}(\bar{U}) + \frac{1}{\bar{S}_c(\tau,p)}\frac{\partial \bar{S}_c}{\partial \tau}(\tau,p)\frac{\partial \bar{U}}{\partial x}(x,\tau,p).$$

When $p = 0$, we have

$$\begin{cases} \mathscr{L}[\bar{U}(x,\tau,0)] = \mathscr{L}[\bar{U}_0(x,\tau)], \\ \bar{U}(x,0,0) = \bar{U}_0(x,0), \\ \bar{U}(0,\tau,0) - n\bar{S}_c(\tau,0) = -e^{-\gamma\tau}, \\ \dfrac{\partial \bar{U}}{\partial x}(0,\tau,0) + n\bar{S}_c(\tau,0) = e^{-\gamma\tau} + \dfrac{\partial \bar{U}_0}{\partial x}(0,\tau) + \bar{U}_0(0,\tau), \\ \lim\limits_{x\to\infty} \bar{U}(x,\tau,0) = 0. \end{cases} \tag{3.6}$$

Clearly, the solution of differential system (3.6) is

$$\begin{cases} \bar{U}(x,\tau,0) = \bar{U}_0(x,\tau), \\ \bar{S}_c(\tau,0) = \dfrac{1}{n}\left[e^{-\gamma\tau} + \bar{U}_0(0,\tau)\right] = \bar{S}_0(\tau), \end{cases} \tag{3.7}$$

so long as the initial guess $\bar{U}_0(x,\tau)$ satisfies the condition $\lim_{x\to\infty} \bar{U}_0(x,\tau) = 0$. One should notice that other than this condition, theoretically, there are no other requirements for the initial guess $\bar{U}_0(x,\tau)$ to satisfy. However, if we choose a function that has already satisfied an additional condition $\mathscr{L}\bar{U}_0(x,\tau) = 0$, we should expect a faster convergence of the series.

On the other hand, if $p = 1$, the differential system (3.5) becomes

$$\begin{cases} \mathscr{L}[\bar{U}(x,\tau,1)] = -\dfrac{1}{\bar{S}_c(\tau,1)}\dfrac{\partial \bar{S}_c}{\partial \tau}(\tau,1)\dfrac{\partial \bar{U}}{\partial x}(x,\tau,1), \\ \bar{U}(x,0,1) = 0, \\ \bar{U}(0,\tau,1) - n\bar{S}_c(\tau,1) = -e^{-\gamma\tau}, \\ \dfrac{\partial \bar{U}}{\partial x}(0,\tau,1) + n\bar{S}_c(\tau,1) = 0, \\ \lim\limits_{x\to\infty} \bar{U}(x,\tau,1) = 0. \end{cases} \tag{3.8}$$

Comparing (3.8) and (3.4), it is obvious that the solution we seek is simply

$$\begin{cases} U(x,\tau) = \bar{U}(x,\tau,1), \\ S_c(\tau) = \bar{S}_c(\tau,1). \end{cases} \tag{3.9}$$

The two unknown functions $\bar{U}(x, \tau, 1)$ and $\bar{S}_c(\tau, 1)$ can now be found by expanding them as two Taylor's series expansions of p,

$$\bar{U}(x, \tau, p) = \sum_{m=0}^{\infty} \frac{\bar{U}_m(x, \tau)}{m!} p^m, \tag{3.10}$$

$$\bar{S}_c(\tau, p) = \sum_{m=0}^{\infty} \frac{\bar{S}_m(\tau)}{m!} p^m, \tag{3.11}$$

where $\bar{U}_m$ is the mth-order partial derivative of $\bar{U}(x, \tau, p)$ with respect to p and then evaluated at $p = 0$,

$$\bar{U}_m(x, \tau) = \left.\frac{\partial^m}{\partial p^m} \bar{U}(x, \tau, p)\right|_{p=0},$$

and $\bar{S}_m$ is the mth-order partial derivative of $\bar{S}_c(\tau, p)$ with respect to p and then evaluated at $p = 0$,

$$\bar{S}_m(\tau) = \left.\frac{\partial^m}{\partial p^m} \bar{S}_c(\tau, p)\right|_{p=0}.$$

To find all the coefficients in the above Taylor's expansions, we need to derive a set of governing partial differential equations and appropriate boundary and initial conditions for the unknown functions $\bar{U}_m(x, \tau)$ and $\bar{S}_m(\tau)$. They can be derived from differentiating each equation in system (3.5) with respect to p and then setting p equal to zero. After this process, we obtain

$$\begin{cases} \mathscr{L}[\bar{U}_1(x, \tau)] = -\mathscr{L}[\bar{U}_0(x, \tau)] + \mathscr{A}'(x, \tau, 0), \\ \bar{U}_1(x, 0) = -\bar{U}_0(x, 0), \\ \bar{U}_1(0, \tau) - n\bar{S}_1(\tau) = 0, \\ \dfrac{\partial \bar{U}_1}{\partial x}(0, \tau) + n\bar{S}_1(\tau) = -\left[\bar{U}_0(0, \tau) + \dfrac{\partial \bar{U}_0}{\partial x}(0, \tau) + e^{-\gamma\tau}\right], \\ \lim\limits_{x\to\infty} \bar{U}_1(x, \tau) = 0, \end{cases} \tag{3.12}$$

and

$$\begin{cases} \mathscr{L}[\bar{U}_m(x, \tau)] = m \left.\dfrac{\partial^{m-1}\mathscr{A}'}{\partial p^{m-1}}\right|_{p=0}, \\ \bar{U}_m(x, 0) = 0, \\ \bar{U}_m(0, \tau) - n\bar{S}_m(\tau) = 0, \quad \text{if } m \geq 2, \\ \dfrac{\partial \bar{U}_m}{\partial x}(0, \tau) + n\bar{S}_m(\tau) = 0, \\ \lim\limits_{x\to\infty} \bar{U}_m(x, \tau) = 0. \end{cases} \tag{3.13}$$

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In Equations (3.12) and (3.13), $\mathscr{A}'(x, \tau, p)$ is the negative value of the second term of $\mathscr{A}(x, \tau, p)$, that is,

$$\mathscr{A}'(x, \tau, p) = -\frac{1}{\bar{S}_c(\tau, p)} \frac{\partial \bar{S}_c}{\partial \tau}(\tau, p) \frac{\partial \bar{U}}{\partial x}(x, \tau, p).$$

This term needs to be calculated recursively. With the development of modern symbolic calculation packages, such as Maple and Mathematica, such recursive calculation becomes simple and straightforward. As demonstrated by Zhu and Hung [18], the fast development of modern symbolic calculation packages now enables applied mathematicians and engineers to develop solution approaches that would otherwise not be possible without symbolic manipulation capabilities. The solution procedure presented here is another example. The expression of $\mathscr{A}'(x, \tau, p)$ is lengthy and there is no need to write out its explicit form due to its recursive nature. However, the recurrent calculation of this term can be easily realised in a symbolic calculation package such as Maple.

After eliminating $\bar{S}_m(\tau)$ from the two boundary conditions at $x = 0$ in (3.12) and (3.13), we can write (3.12) and (3.13) in a general form

$$\begin{cases} \mathscr{L}[\bar{U}_m(x, \tau)] = f_m(x, \tau), \\ \bar{U}_m(x, 0) = \psi_m(x), \\ \dfrac{\partial \bar{U}_m}{\partial x}(0, \tau) + \bar{U}_m(0, \tau) = \phi_m(\tau), \\ \lim\limits_{x \to \infty} \bar{U}_m(x, \tau) = 0, \end{cases} \tag{3.14}$$

with $f_m(x, \tau)$, $\psi_m(x)$ and $\phi_m(\tau)$ being expressed respectively as

$$f_m(x, \tau) = \begin{cases} -\mathscr{L}[\bar{U}_0(x, \tau)] + \mathscr{A}'(x, \tau, 0), & \text{if } m = 1, \\ m \dfrac{\partial^{m-1} \mathscr{A}'}{\partial p^{m-1}}\bigg|_{p=0}, & \text{if } m \geq 2, \end{cases} \tag{3.15}$$

$$\psi_m(\tau) = \begin{cases} -\bar{U}_0(x, 0), & \text{if } m = 1, \\ 0, & \text{if } m \geq 2, \end{cases} \tag{3.16}$$

$$\phi_m(\tau) = \begin{cases} -\left[\bar{U}_0(0, \tau) + \dfrac{\partial \bar{U}_0}{\partial x}(0, \tau) + e^{-\gamma\tau}\right], & \text{if } m = 1, \\ 0, & \text{if } m \geq 2. \end{cases} \tag{3.17}$$

The elimination of $\bar{S}_m(\tau)$ is the key to our success in eventually working out an analytical solution for this highly nonlinear problem through solving a sequence of infinitely many linear partial differential systems. Upon performing a transformation

$$\bar{U}_m(x, \tau) = e^{(\gamma-\beta-1)x/2 + [-(\gamma+1)^2/4 + \beta(\gamma-\beta/2+1)/2]\tau} \bar{V}_m(x, \tau), \tag{3.18}$$

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we can rewrite (3.14) in the form of a standard diffusion equation as

$$\begin{cases} \dfrac{\partial \bar{V}_m}{\partial \tau} - \dfrac{\partial^2 \bar{V}_m}{\partial x^2} = e^{-\frac{1}{2}(\gamma-\beta-1)x+\left[\frac{1}{4}(\gamma+1)^2-\frac{\beta}{2}(\gamma-\frac{\beta}{2}+1)\right]\tau} f_m(x,\tau), \\ \bar{V}_m(x,0) = e^{-\frac{1}{2}(\gamma-\beta-1)x}\psi_m(x), \\ \dfrac{\partial \bar{V}_m}{\partial x}(0,\tau) + \dfrac{1}{2}(\gamma-\beta+1)\bar{V}_m(0,\tau) = e^{\left[\frac{1}{4}(\gamma+1)^2-\frac{\beta}{2}(\gamma-\frac{\beta}{2}+1)\right]\tau}\phi_m(\tau), \\ \lim\limits_{x\to\infty} \bar{V}_m(x,\tau) = 0. \end{cases} \tag{3.19}$$

A closed-form solution of Equation (3.19) at each order (that is, with each m) can now be found by splitting the linear problem into three problems, a technique frequently used in solving linear partial differential equations. The solution of the first problem, which involves a homogeneous differential equation and homogeneous boundary condition at $x = 0$ but arbitrary initial condition, can be easily worked out by utilising the Laplace transform technique while the solution of the second problem, which also involves a homogeneous differential equation and zero initial condition but a non-homogeneous boundary condition at $x = 0$ can be found in [5]. The solution of the third problem, in which the differential equation is non-homogeneous but both the boundary condition at $x = 0$ and the initial condition are homogeneous, can be worked out by using Duhamel's theorem (see [5]) once the solution of the first problem is found. Without going through the lengthy solution procedures, the final analytic solution of Equation (3.14) is given here explicitly in terms of three single integrals and two double integrals as:

$$\begin{aligned}
&\bar{V}_m(x,\tau) \\
&= \frac{1}{\sqrt{\pi}}\Bigg\{e^{-(\gamma-\beta-1)x/2}\int_{-x/2\sqrt{\tau}}^{x/2\sqrt{\tau}} \psi_m(2\sqrt{\tau}\,\xi + x)e^{-(\gamma-\beta-1)\sqrt{\tau}\xi-\xi^2}\,d\xi \\
&\quad + \int_{x/2\sqrt{\tau}}^{\infty}\left[e^{-(\gamma-\beta-1)x/2}\psi_m(2\sqrt{\tau}\,\xi + x) + e^{(\gamma-\beta-1)x/2}\psi_m(2\sqrt{\tau}\,\xi - x)\right] \\
&\quad \times e^{-(\gamma-\beta-1)\sqrt{\tau}\xi-\xi^2}\,d\xi\Bigg\} + (\gamma-\beta+1)\sqrt{\tau}e^{(\gamma-\beta-1)x/2+(\gamma-\beta+1)^2\tau/4} \\
&\quad \times \int_{\frac{x}{2\sqrt{\tau}}}^{\infty} \psi_m(2\sqrt{\tau}\xi - x)e^{-2(\gamma-\beta)\sqrt{\tau}\xi}\operatorname{erfc}\left(\xi - \frac{(\gamma-\beta+1)}{2}\sqrt{\tau}\right)d\xi \\
&\quad - \frac{2}{\sqrt{\pi}}e^{[-(\gamma+1)^2/4-\beta(\gamma-\beta/2+1)/2]\tau}\int_0^{\infty} e^{-(\gamma-\beta+1)\eta/2} \\
&\quad \times \int_{(x+\eta)/2\sqrt{\tau}}^{\infty} \phi_m\left(\tau - \frac{(x+\eta)^2}{4\xi^2}\right)e^{-([-(\gamma+1)^2/4-\beta(\gamma-\beta/2+1)/2](x+\eta)^2/4\xi^2)-\xi^2}\,d\xi\,d\eta
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{\tau}\left\{\frac{e^{\left[-\frac{1}{4}(\gamma+1)^2+\frac{1}{2}\beta(\gamma-\frac{\beta}{2}+1)\right]\eta}}{\sqrt{\pi}}\left[e^{\frac{1}{2}(\gamma-\beta-1)x}\int_{-\frac{x}{2\sqrt{\tau-\eta}}}^{\frac{x}{2\sqrt{\tau-\eta}}} f_m(2\sqrt{\tau-\eta}\,\xi+x,\eta)\right.\right.\\
&\times e^{(\gamma-\beta-1)\sqrt{\tau-\eta}\,\xi-\xi^2}d\xi+\int_{\frac{x}{2\sqrt{\tau-\eta}}}^{\infty}\left[e^{\frac{1}{2}(\gamma-\beta-1)x}f_m(2\sqrt{\tau-\eta}\,\xi+x,\eta)\right.\\
&\left.\left.+e^{-\frac{1}{2}(\gamma-\beta-1)x}f_m(2\sqrt{\tau-\eta}\,\xi-x,\eta)\right]e^{(\gamma-\beta-1)\sqrt{\tau-\eta}\,\xi-\xi^2}d\xi\right]\\
&+(\gamma-\beta+1)\sqrt{\tau-\eta}\,e^{-\frac{1}{2}(\gamma-\beta-1)x+\frac{(\gamma-\beta+1)^2}{4}\tau+\left[-\frac{1}{2}(\gamma+1)^2+\beta(\gamma-\frac{\beta}{2}+1)\right]\eta}\\
&\times\int_{\frac{x}{2\sqrt{\tau-\eta}}}^{\infty} f_m(2\sqrt{\tau-\eta}\,\xi-x,\eta)e^{-2\sqrt{\tau-\eta}\,\xi}\\
&\left.\times\operatorname{erfc}\left(\xi-\frac{(\gamma-\beta+1)}{2}\sqrt{\tau-\eta}\right)d\xi\right\}d\eta,
\end{aligned}
\tag{3.20}
$$

where $\operatorname{erfc}(x)$ denotes the complementary error function.

Upon finding the coefficients $\bar{U}_m(x,\tau)$ from Equations (3.18) and (3.20), $\bar{S}_m(\tau)$ can be easily found from the third equation of Equations (3.12) and (3.13), that is,

$$
\bar{S}_m(\tau)=\frac{1}{n}\bar{U}_m(0,\tau). \tag{3.21}
$$

Then the final solution of our original problem (3.4) can be written, by virtue of (3.10) and (3.11), in terms of a series of infinitely many terms as

$$
\begin{cases}
U(x,\tau)=\bar{U}(x,\tau,1)=\displaystyle\sum_{m=0}^{\infty}\frac{\bar{U}_m(x,\tau)}{m!},\\
S_c(\tau)=\bar{S}_c(\tau,1)=\displaystyle\sum_{m=0}^{\infty}\frac{\bar{S}_m(\tau)}{m!}.
\end{cases}
\tag{3.22}
$$

The summation process begins with an initial guess $U_0(x,\tau)$. As shown in Equations (3.6) and (3.7), the initial guess can be virtually any continuous function defined on $x\in[0,\infty)$. However, for the present CB problem, we can choose the solution for a European option with continuous yield dividend as the initial guess. Just like the nice initial choice used by Zhu [17] who outlined the three major advantages when an elegant European-style counterpart of the financial derivative to be valued is used in conjunction with the homotopy analysis method, here choosing the solution for a European option with continuous yield dividend as our initial guess also significantly simplifies the problem. First of all, the dimensionless solution

$$
U_E(x,\tau)=e^{-x-\beta\tau}N(d_{10})-e^{-\gamma\tau}N(d_{20}) \tag{3.23}
$$

with

$$d_{10} = \frac{(\gamma + 1 - \beta)\tau - x}{\sqrt{2\tau}} \quad \text{and} \quad d_{20} = \frac{(\gamma - 1 - \beta)\tau - x}{\sqrt{2\tau}}$$

satisfies the equation

$$\mathscr{L}[\bar{U}_0(x, \tau)] = 0, \tag{3.24}$$

and therefore $f_1(x, \tau)$ in Equation (3.15) is further simplified as the first term disappears if $u_E(x, \tau)$ is set to be equal to $U_0(x, \tau)$. Secondly, because of the transform Equation (3.1), we are actually only using part of the solution (3.23). At $\tau = 0$, the part we actually used is the U_E in the range $S \in [0, Z)$ or in terms of x, $x \in [0, \infty)$. Within this range,

$$\bar{U}_0(x, 0) = 0, \tag{3.25}$$

which has considerably simplified the solution (3.20) because $\psi_1(x)$ in Equation (3.16) vanishes, resulting in the integral involving ψ_m in Equation (3.20) being entirely eliminated. Finally, even the boundary condition for $\bar{U}_0(x, \tau)$ as x approaches infinity is also satisfied because that is the boundary condition that the value of a European call option with continuous dividend payment must satisfy. This can be easily verified in Equation (3.23).

These advantages have led to a reasonable convergence rate; about 30 terms are needed to reach a convergent solution with an accuracy up to the 3rd decimal place. This is about one third of the terms needed when Liao [9] combined the homotopy analysis method with boundary element techniques to solve a nonlinear heat transfer problem. On the other hand, if other initial guesses are taken, numerical experiments show that it could take considerably longer to reach the same level of accuracy, although eventually a convergent solution can still be found.

4. Examples and discussions

A convertible bond example is now used to illustrate some calculated results obtained from using the newly-derived analytical solution. To help readers who may not be used to discussing financial problems with dimensionless quantities, all results, unless otherwise stated, are now converted back to dimensional quantities in this section before they are graphed and presented.

The example is based on a basic convertible bond with conversion being allowed at any time prior to expiry. The bond's parameters are:

- Strike price $X = \$100$,
- Risk-free annual interest rate $r = 10\%$,

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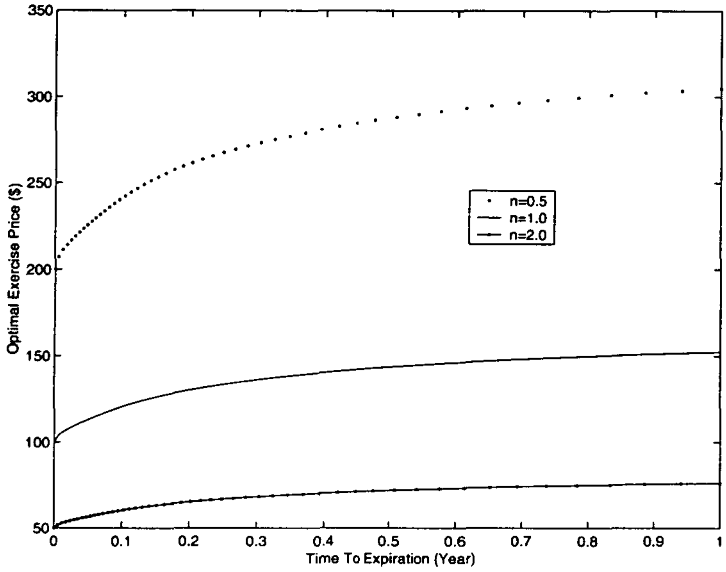

FIGURE 1. Optimal exercise prices for three different conversion ratios.

- Rate of continuous dividend payment $D_0 = 7\%$,
- Volatility $\sigma = 0.4$,
- Time to expiration $T = 1$ (year).

In terms of the dimensionless variables, the three parameters involved are $\gamma = 1.25$, $\beta = 0.875$ and $\tau_{\text{exp}} = 0.08$.

There are many choices for the numerical computation of the integrals involved in the closed-form analytical solution Equations (3.14), (3.16) and (3.17). All the results presented in this paper were calculated with a variable grid spacing in time and equal grid spacing in the dimensionless stock price. The symbolic calculation package Maple 9 was used to carry out the recursive computation of $f_m(x, \tau)$ in (3.15) for $m \geq 2$. Numerical integration with a compound Simpson's rule was performed for the spatial integration and the simple trapezoidal rule was used for the temporal integration. Because the integrals involving an infinite upper limit converge extremely fast, only a small finite number are needed to replace the infinite upper limit; beyond this finite limit the integrand is virtually zero and contributes almost nothing to the result of the integration.

The results of the analytical series solution were obtained when the solution became convergent after 30 terms were summed. Depicted in Figure 1 are the results of the optimal exercise prices, S_c, for three different conversion ratios. As expected, all optimal exercise prices increase monotonically with time to expiry, $\tau = T - t$, or decrease with time t. However, as the conversion ratio becomes large, the $S_c(\tau)$

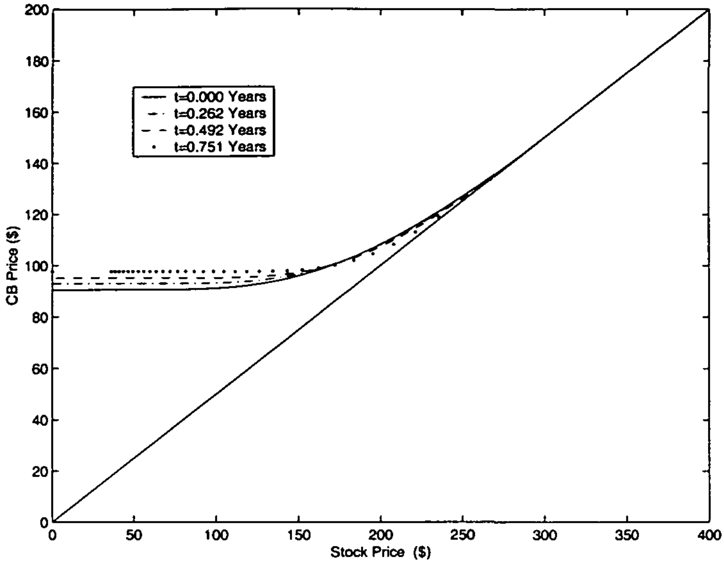

FIGURE 2. The prices of the convertible bond with a conversion ratio of 0.5 at four different time instants.

curve becomes flatter. Of course, when the time approaches the expiration time T of the option, the optimal exercise prices all approach the strike price divided by the conversion ratio, respectively, because of the arbitrage-free assumption, based on which the governing differential system is derived.

With the closed-form analytic solution, we can graph the value of the convertible bond versus the stock price at a fixed time. Depicted in Figures 2–4 are the prices of the convertible bond with three different conversion ratios as a function of the underlying asset value S at four instants, $t = 0$ (years), $t = 0.262$ (years), $t = 0.492$ (years) and $t = 0.751$ (years), respectively. Clearly, one can observe that all the price curves smoothly land on the straight line, which represents the intrinsic value of the CB for each case. This smooth landing demonstrates how well the boundary conditions prescribed on the moving boundary in (2.5) are satisfied.

In this example, the summations in (3.18) were carried out up to 30 terms when a convergent optimal exercise price was found for the case with conversion ratios of half and two, but only up to 27 terms for the case with conversion ratio of one. In all these cases, any further inclusion of more terms in the solution resulted in a contribution in the order of 10^{-3}. The convergence of our results as m is increased can be clearly seen in Figure 5, in which the dimensionless $\bar{V}_m(x, \tau)$ values for the case with the half conversion ratio are plotted for $m = 25$ to $m = 30$. As m increases, the magnitude of $\bar{V}_m(x, \tau)$ decreases. When m becomes greater than 30, the remainder of the series becomes insignificant as the computed result has more or less reached such a level that

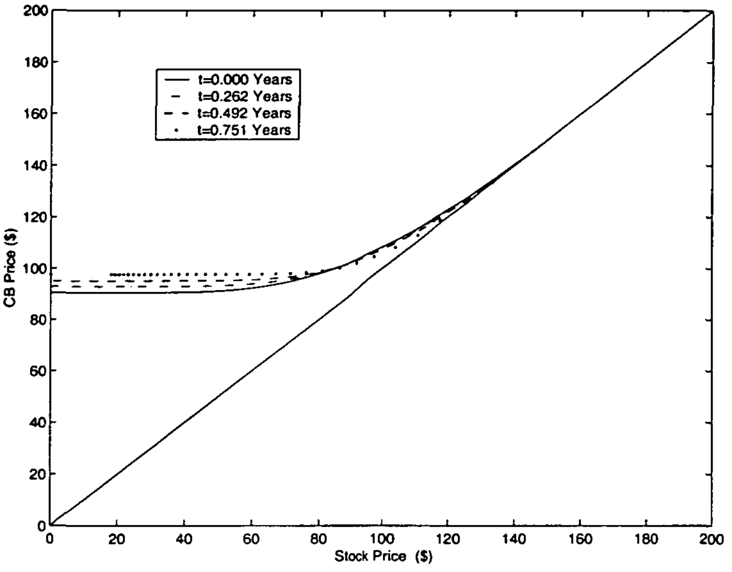

FIGURE 3. The prices of the convertible bond with a unit conversion ratio at four different time instants.

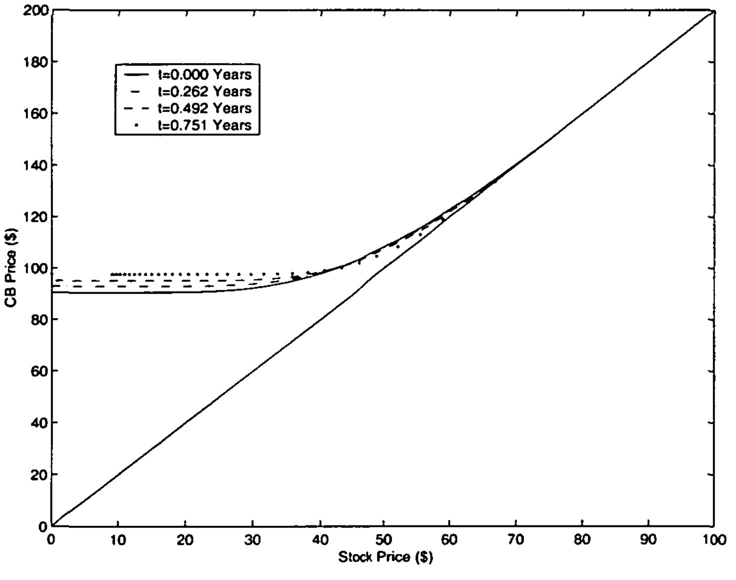

FIGURE 4. The prices of the convertible bond with a conversion ratio of two at four different time instants.

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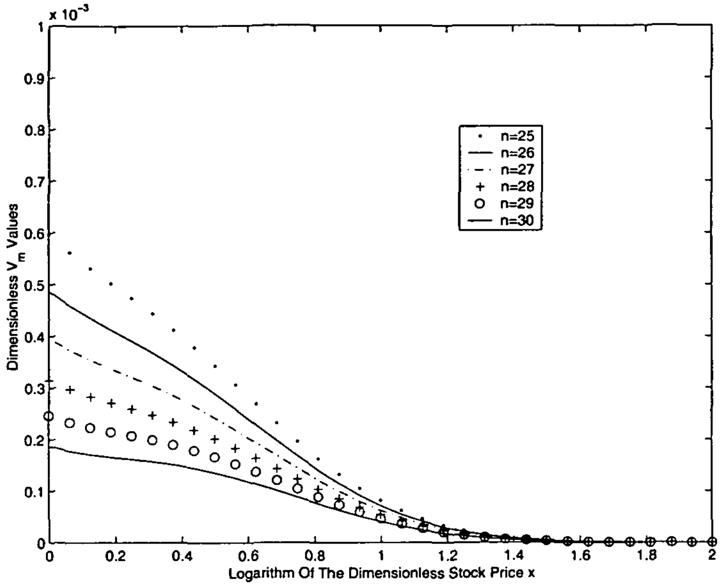

FIGURE 5. An illustration of convergence for the case with a conversion ratio of 0.5.

they are hardly distinguishable when they are plotted out. The behaviour of $\bar{V}_m(x, \tau)$ for the other two cases is very similar and the corresponding graphs are thus omitted.

5. Conclusions

Making use of the concept of homotopic deformation in topology, the nonlinear problem of valuing a convertible bond with the American style of conversion is solved analytically and a closed-form solution of the well-known Black-Scholes equation is obtained for the first time. It is shown that the optimal conversion price, which is the key difficulty in the valuation of American-style convertible bonds, can be expressed explicitly in a closed form in terms of four input parameters; the risk-free interest rate, the continuous dividend yield, the volatility and the time to expiration. This closed-form analytical solution can be used to validate other numerical solutions designed for more complicated cases where no analytical solutions exist.

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