5

Classical (Local) Hardy Inequalities

5.1 Inequalities on $\mathbb{R}^n$

In [94], Hardy proved the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x F(t)\,dt\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty F(x)^p dx \tag{5.1.1}$$

for non-negative functions F on $[0,\infty)$ with $1 < p < \infty$. Landau in [114] showed that the constant $\left(\frac{p}{p-1}\right)^p$ is sharp and that equality is only possible if $F = 0$. On putting $f(x) = \int_0^x F(t)\,dt$, one obtains the more familiar form

$$\int_0^\infty \frac{f(x)^p}{x^p} dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f'(x)^p dx, \tag{5.1.2}$$

satisfied by functions f with $f' \in L_p(0,\infty)$ and $\lim_{x\to 0+} f(x) = 0$.

The analogue of (5.1.2) in $\mathbb{R}^n$ for $n > 1$ is

$$\int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p} dx \le \left|\frac{p}{p-n}\right|^p \int_{\mathbb{R}^n} |\nabla f(x)|^p dx, \tag{5.1.3}$$

where $\nabla f(x) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$, the gradient of f, and with $|\nabla f(x)| = \left(\sum_{i=1}^n |\partial f/\partial x_i|^2\right)^{1/2}$. The inequality (5.1.3) holds for all $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ if $n < p < \infty$ and all $f \in C_0^\infty(\mathbb{R}^n)$ for $1 \le p < n$; see [15], Section 1.2. Since $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ is dense in $\overset{0}{\mathcal{D}}{}_p^1(\mathbb{R}^n \setminus \{0\})$, (5.1.3) is satisfied for all $f \in \overset{0}{\mathcal{D}}{}_p^1(\mathbb{R}^n \setminus \{0\})$ when $n < p < \infty$ and similarly for all $f \in \overset{0}{D}{}_p^1(\mathbb{R}^n)$ for $1 \le p < n$; recall that $\overset{0}{\mathcal{D}}{}_p^1(\Omega)$, is the *homogeneous Sobolev space* defined in Section 2.2, namely, the completion of $C_0^\infty(\Omega)$ with respect to the norm $u \longmapsto \||\nabla u|\|_{p,\Omega}$.

In the case $1 \le p < n$, there is an equivalence between (5.1.3) (for $f \in \overset{0}{\mathcal{D}}{}_p^1(\mathbb{R}^n)$) and an optimal Sobolev inequality

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$$\|f\|_{p*,p} \leq S_{n,p}\|\nabla f\|_p, \;\; f \in \overset{0}{D^1_p}(\mathbb{R}^n), p^* = np/(n-p), \tag{5.1.4}$$

demonstrated by Alvino in [10]; see also [145]. Alvino's constant

$$S_{n,p} = \frac{p}{n-p}\frac{\left[\Gamma(1+n/2)\right]^{1/n}}{\sqrt{\pi}} = \frac{p}{n-p}\left(\frac{n}{\omega_{n-1}}\right)^{1/n}$$

is best possible and is the norm of the embedding $\overset{0}{\mathcal{D}^1_p}(\mathbb{R}^n) \hookrightarrow L_{p*,p}(\mathbb{R}^n)$ which is optimal in the sense that the target space $L_{p*,p}(\mathbb{R}^n)$ is the smallest among all rearrangement-invariant spaces; see the discussion following Theorem 2.1. The equivalence is observed in [41] to be a consequence of the Pólya–Szegö principle and the Hardy–Littlewood inequality by which the left-hand side of (5.1.3) does not increase under radially decreasing symmetrisation and is equal to the left-hand side of (5.1.4) when f is radially decreasing. Alvino actually proved the more general inequality

$$\|f\|_{p*,p} \leq S_{n,p}\|\nabla f\|_{p,q}, \;\; 1 \leq p < n, \;\; 1 \leq q \leq p$$

and this was extended to the full range $1 \leq q \leq \infty$ in [41]. Let $\overset{0}{\mathcal{D}^1_{p,q}}(\mathbb{R}^n)$ denote the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $u \mapsto \|\nabla u\|_{p,q}$. The embedding $\overset{0}{\mathcal{D}^1_{p,q}}(\mathbb{R}^n) \hookrightarrow L_{p*,p}(\mathbb{R}^n)$ is well known in the interpolation theory literature and direct proofs may be found in [11] and [165]. It is then established in [41] that for $1 \leq p < n$, (5.1.3) holds for $f \in \overset{0}{\mathcal{D}^1_p}(\mathbb{R}^n)$ if and only if the Sobolev–Marcinkiewicz embedding inequality

$$\|f\|_{p*,\infty} \leq S_{n,p}\|\nabla f\|_{p,\infty}, \;\; S_{n,p} = \frac{p}{n-p}\left(\frac{n}{\omega_{n-1}}\right)^{1/n} \tag{5.1.5}$$

holds for every $f \in D^1L_{p,\infty}(\mathbb{R}^n) := \{f \in L_{p,\infty}(\mathbb{R}^n) : \|\nabla f\|_{p,\infty} < \infty\}$. In contrast to the Hardy inequality, the best possible constant $S_{n,p}$ in (5.1.5) is attained, an extremal function in $D^1L_{p,\infty}(\mathbb{R}^n)$ being given by

$$\psi(x) = |x|^{-\frac{n-p}{p}}.$$

The Marcinkiewicz space $L_{p*,\infty}(\mathbb{R}^n)$ (also called the *weak*-L_{p*} space) is the smallest rearrangement-invariant space containing ψ.

The (normalised) distance function

$$d_{p*,\infty}(f) := \inf_{a\in\mathbb{R}} \frac{\|f - a\psi\|_{L_{p*,\infty}(\mathbb{R}^n)}}{\|f\|_{L_{p*,p}(\mathbb{R}^n)}}$$

is defined in [45] and there it is shown that for $n \geq 2$ and $1 < p < n$, there exist constants $C = C(n,p)$ and $\alpha = \alpha(n,p)$ such that

$$\left[1 + Cd_{p*,\infty}(f)^{\alpha}\right] \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p}\, dx \leq \int_{\mathbb{R}^n} |\nabla f(x)|^p\, dx. \tag{5.1.6}$$

However, while the Hardy inequality (5.1.3) holds for $p = 1$, the inequality (5.1.6) does not; indeed, for $p = 1$, any spherically symmetric function attains equality in (5.1.3).

There is no valid inequality (5.1.3) for $n = p$, see [15], Section 1.2.5. In the case $n = p = 2$ it is proved in [3], Theorem 4.6, that for all $f \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ satisfying $\int_{1<|x|<2} f(x)\, dx = 0$, there exists a positive constant C such that

$$\int_{\mathbb{R}^2} \frac{|f(x)|^2}{|x|^2(1 + \log^2 |x|)}\, dx < C \int_{\mathbb{R}^2} |\nabla f(x)|^2 dx. \tag{5.1.7}$$

Solomyak had shown earlier in [163] that the logarithmic factor in (5.1.7) is only needed for radial functions and can be removed for functions satisfying

$$\int_{|x|=R} f(x)\, dx = 0$$

for all $R > 0$. This condition is also imposed in the following weighted inequality of Dubinskii from [53], Theorem 2.1, which covers the case $n = p$:

Theorem 5.1 *Let $n \geq 2,\ p > 1$ and suppose that $u \in L_{p,loc}(\mathbb{R}^n \setminus \{0\})$, $\int_{\mathbb{R}^n} |\nabla u(x)|^p |x|^{(p-n)} dx < \infty$ and $\int_{|x|=R} u(x)\, dx = 0$ for all $R > 0$. Then there exists a constant M which depends only on n and p such that*

$$\int_{\mathbb{R}^n} \mu_R(|x|)|u(x)|^p dx \leq M \int_{\mathbb{R}^n} |\nabla u(x)|^p |x|^{p-n} dx, \tag{5.1.8}$$

where for $r > 0$,

$$\mu_R(r) = \min\left\{ \frac{1}{r^n \left|\ln\left(\frac{r}{R}\right)\right|^p}, \frac{1}{r^n} \right\}.$$

An interesting counterpart of (5.1.3) on $\mathbb{R}^2 \setminus \{0\}$ was established in [116] by replacing the gradient ∇ with the magnetic gradient $\nabla + i\mathbf{A}$, where $\mathbf{A}$ is a magnetic potential of Aharonov–Bohm type given in polar co-ordinates $x = (r\cos\theta, r\sin\theta)$ by

$$\mathbf{A}(x) = \frac{\psi(\theta)}{r}(-\sin\theta, \cos\theta),$$

where $\psi \in L^\infty(0, 2\pi)$ and

$$\Psi := \frac{1}{2\pi} \int_0^{2\pi} \psi(\theta) d\theta$$

is the magnetic flux; significant features are that the domain $\mathbb{R}^2 \setminus \{0\}$ is not simply connected and the magnetic field curl $\mathbf{A}(x) = 0$ in $\mathbb{R}^2 \setminus \{0\}$. The resulting Laptev–Weidl inequality is that, for all non-trivial $f \in C_0^\infty\left(\mathbb{R}^2 \setminus \{0\}\right)$,

$$\int_{\mathbb{R}^2} |(\nabla + i\mathbf{A})f(x)|^2\, dx > \min_{k\in\mathbb{Z}} |k - \Psi|^2 \int_{\mathbb{R}^{\nvDash}} \frac{|f(x)|^2}{|x|^2}\, dx, \tag{5.1.9}$$

with sharp constant $\min_{k\in\mathbb{Z}} |k - \Psi|^2$. If the magnetic flux Ψ is an integer, the magnetic Laplace operator $-(\nabla + i\mathbf{A})^2$ is unitarily equivalent in $L_2(\mathbb{R}^2)$ to $-\Delta$ and hence there is no non-trivial Hardy inequality. We shall return to this inequality and discrete versions in Section 5.6.

Our main concern in subsequent sections of this chapter will be with the validity and refinements of a general inequality

$$\int_\Omega \frac{|f(x)|^p}{\delta(x)^p}\, dx \le C(p, \Omega) \int_\Omega |\nabla f(x)|^p dx \tag{5.1.10}$$

for $f \in C_0^\infty(\Omega)$ and Ω an open connected set (domain) in $\mathbb{R}^n$ with non-empty boundary; in (5.1.10), $\delta(x) = \inf\{|x - y| : y \notin \Omega\}$, the distance of x from the boundary of Ω. Since $C_0^\infty(\Omega)$ is dense in $\overset{0}{W}{}^1_p(\Omega)$, it would follow that (5.1.10) holds on $\overset{0}{W}{}^1_p(\Omega)$, and indeed on the larger space $\overset{0}{\mathcal{D}}{}^1_p(\Omega)$.

The production of papers on the Hardy inequality has mushroomed in this century and significant works continue to appear at an accelerating rate. The selection of results deemed to be of particular significance is inevitably personal and some worthy contributions are bound to be omitted. We make an attempt at a comprehensive coverage within these obvious bounds. Some results are stated without proof, but with what we hope is adequate background information and precise references.

In the range $1 < p \le n$, (5.1.10) was proved in [122] to be valid if $\mathbb{R}^n \setminus \Omega$ is *uniformly p-fat*, and valid for $p = n$ if and only if $\mathbb{R}^n \setminus \Omega$ is *uniformly p-fat*. We refer to [122] and [15] for a definition of the *uniformly p-fat property*, but the following examples may help to put it in perspective:

1. A closed set satisfying the interior cone condition is uniformly p-fat for every $p \in (1, \infty)$.
2. The complement of a Lipschitz domain is uniformly p-fat for every $p \in (1, \infty)$. Recall that a domain is Lipschitz if it is a rotation of a set of the form
$$\{x = (x', x_n) = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n : x_n = \Phi(x')\}$$
where $\Phi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function.

The uniform p-fat property of $\mathbb{R}^n \setminus \Omega$ was proved by Lehrbäck in [120] and [121] to be equivalent to the *pointwise q-Hardy inequality for some* $q \in (1, p)$; this notion was introduced by Hajłasz in [93] and is that there exists a positive constant $c(n, q)$, depending only on n and q, such that for all $f \in C_0^\infty(\Omega)$ (extended by zero to all of $\mathbb{R}^n$),

$$\frac{|f(x)|}{\delta(x)} \le c(n,q)\left[\mathbb{M}\left(|\nabla f|^q(x)\right)\right]^{1/q},$$

where $\mathbb{M}f$ is the maximal function defined for $f \in L_{1,loc}(\mathbb{R}^n)$ by

$$\mathbb{M}f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)\, dy.$$

In the range $n < p < \infty$, (5.1.10) was proved in [122] to be valid for all proper open subsets Ω of $\mathbb{R}^n (n \ge 2)$. The weighted inequality in the following theorem includes the case $n < p < \infty$ of [122] and gives the best possible value for the constant $C(p, \Omega)$. It was first proved by Avkhadiev in [7] but alternative proofs have since been given by Chen in [43] and Pinchover and Goel in [148]. The following proof is that in [43].

Theorem 5.2 *Let $\Omega \subsetneq \mathbb{R}^n$, $n \ge 2$, be an arbitrary domain, $1 < p < \infty$ and $\alpha + p > n$. Then for all f such that $|f| \in C_0^\infty(\Omega)$,*

$$\left(\frac{\alpha+p-n}{p}\right)^p \int_\Omega \frac{|f(x)|^p}{\delta(x)^{p+\alpha}}\, dx \le \int_\Omega \frac{|\nabla f(x)|^p}{\delta(x)^\alpha}\, dx, \tag{5.1.11}$$

where the constant is sharp.

Hence, in particular, when $n < p < \infty$, for all $f \in \overset{0}{D}{}_p^1(\Omega)$,

$$\left(\frac{p-n}{p}\right)^p \int_\Omega \frac{|f(x)|^p}{\delta^p(x)}\, dx \le \int_\Omega |\nabla f(x)|^p\, dx, \tag{5.1.12}$$

where the constant is sharp.

Proof An important first step is to show that for any $\beta \ge 2$,

$$\Delta \delta^{2-\beta} \ge (\beta-2)(\beta-n)\delta^{-\beta} \tag{5.1.13}$$

in the sense of distributions in Ω, i.e., for every non-negative $\phi \in C_0^\infty(\Omega)$,

$$\int_\Omega \left\{\delta^{2-\beta}\Delta\phi - (\beta-2)(\beta-n)\delta^{-\beta}\phi\right\} dx \ge 0.$$

To prove this, first observe that for every $a \in \mathbb{R}^n$,

$$u_a(x) := -|x-a|^2 + |x|^2$$

is harmonic in $\mathbb{R}^n$, and that

$$\delta(x) = \min_{a\in\partial\Omega}\{|x-a|\}.$$

Thus

$$-\delta^2(x) + |x|^2 = \max_{a\in\partial\Omega}\{u_a(x)\}$$

and for $0 \le \phi \in C_0^\infty(\Omega)$,

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$$\begin{aligned}\int_\Omega \delta(x)^2\Delta\phi\,dx &= \int_\Omega \left(|x|^2 - \max_{a\in\partial\Omega} u_a(x)\right)\Delta\phi\,dx\\ &\le \int_\Omega \Delta\left(|x|^2 - u_a(x)\right)\phi\,dx\\ &= 2n\int_\Omega \phi\,dx.\end{aligned}$$

This means that $-\Delta\delta^2 \ge -2n$ in the distributional sense and setting $\chi(t) := t^{1-\beta/2}$, we have

$$\begin{aligned}\Delta\chi(\delta^2) &= \chi''(\delta^2)|\nabla\delta^2|^2 + \chi'(\delta^2)\Delta\delta^2\\ &= \beta(\beta-2)\delta^{-\beta}|\nabla\delta|^2 + (1-\beta/2)\delta^{-\beta}\Delta\delta^2\\ &\ge (\beta-2)(\beta-n)\delta^{-\beta},\end{aligned}$$

since $|\nabla\delta| = 1$ a.e. on Ω; this will be proved in Section 5.2 and is a consequence of δ being uniformly Lipschitz and $|\delta(x)-\delta(y)| \le |x-y|$ for $x,\ y\in\Omega$. Therefore (5.1.13) is proved.

Hence, for $|f|\in C_0^\infty(\Omega)$,

$$\begin{aligned}(\beta-2)(\beta-n)\int_\Omega \frac{|f|^p}{\delta^\beta}dx &\le \int_\Omega \Delta\left(\delta^{2-\beta}\right)|f|^p\,dx\\ &= \int_\Omega \delta^{2-\beta}\Delta\left(|f|^p\right)\\ &= -\int_\Omega \nabla\left(\delta^{2-\beta}\right)\cdot\nabla\left(|f|^p\right)dx\\ &= p(\beta-2)\int_\Omega \delta^{1-\beta}|f|^{p-1}\nabla\delta\cdot\nabla|f|\,dx,\end{aligned}$$

and for $\beta > 2$,

$$\begin{aligned}\frac{(\beta-n)}{p}\int_\Omega \frac{|f|^p}{\delta^\beta}\,dx &\le \int_\Omega \delta^{1-\beta}|f|^{p-1}\nabla\delta\cdot\nabla|f|\,dx\\ &\le \left(\int_\Omega |f|^p|\nabla\delta|^{\frac{p}{p-1}}\delta^{-\beta}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla|f||^p\delta^{p-\beta}\right)^{\frac{1}{p}}\\ &\le \left(\int_\Omega |f|^p\delta^{-\beta}dx\right)^{\frac{p-1}{p}}\left(\int_\Omega |\nabla f|^p\delta^{p-\beta}\right)^{\frac{1}{p}},\end{aligned}$$

since $|\nabla|f|| \le |\nabla f|$ a.e. The inequality (5.1.11) follows on putting $\beta = p+\alpha$.

To prove that the constant is sharp, Chen considers

$$\Omega = B_2 := \{x\colon 0 < |x| < 2\},$$

and sets $\gamma_\varepsilon = (\alpha+p-n)/p+\varepsilon,\ \varepsilon > 0$. Let f_ε be a test function with compact support in B_2 and such that $f_\varepsilon(x) = |x|^{\gamma_\varepsilon}$ on $B_1 = \{x\colon 0 < |x| < 1\}$. On using polar co-ordinates, we obtain for small $\varepsilon > 0$,

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$$\int_{B_2} \frac{|f_\varepsilon|^p}{\delta^{p+\alpha}}\,dx = \int_{B_1} \frac{|f_\varepsilon|^p}{\delta^{p+\alpha}}\,dx + O(1)$$
$$= \omega_n \int_0^1 r^{-1+p\varepsilon} dr + O(1)$$
$$= \omega_n (p\varepsilon)^{-1} + O(1).$$

Similarly, we have

$$\int_{B_2} \frac{|\nabla f_\varepsilon|^p}{\delta^\alpha}\,dx = \omega_n \gamma_\varepsilon^p \int_0^1 r^{p(\gamma_\varepsilon - 1)-\alpha+n-1} dr + O(1)$$
$$= \omega_n \gamma_\varepsilon^p (p\varepsilon)^{-1} + O(1).$$

Thus

$$\lim_{\varepsilon\to 0+} \frac{\int_{B_2} \frac{|\nabla f_\varepsilon|^p}{\delta^\alpha}\,dx}{\int_{B_2} \frac{|f_\varepsilon|^p}{\delta^{p+\alpha}}\,dx} = \left(\frac{\alpha+p-n}{p}\right)^p. \tag{5.1.14}$$

Since every f_ε may be approximated by functions in $C_0^\infty(B_2)$ with respect to the norm

$$\left(\int_{B_2} |\cdot|^p/\delta^{p+\alpha}\,dx\right)^{1/p} + \left(\int_{B_2} |\nabla(\cdot)|^p/\delta^\alpha\,dx\right)^{1/p}$$

it follows that the constant in (5.1.11) is sharp. □

Remark 5.3

In [8], Theorem 4, it is proved that if $\mathbb{R}^n \setminus \Omega$ is a compact set, then for any $p \in [1,\infty)$ and $s \in [n,\infty)$,

$$\mu(p,s,\Omega) := \sup_{f\in C_0^\infty(\Omega), f\neq 0} \frac{\int_\Omega \frac{|f(x)|^p}{\delta(x)^s}\,dx}{\int_\Omega \frac{|\nabla f(x)|^p}{\delta(x)^{s-p}}\,dx} = \frac{|s-n|^p}{p^p}.$$

Hence with $s = p \geq n$,

$$\mu(p,\Omega) := \sup_{f\in C_0^\infty(\Omega), f\neq 0} \frac{\int_\Omega \frac{|f(x)|^p}{\delta(x)^p}\,dx}{\int_\Omega |\nabla f(x)|^p dx} = \left(\frac{p-n}{p}\right)^p.$$

The constant $(p-n)^p/p^p$ is therefore sharp in (5.1.12) whenever $\mathbb{R}^n \setminus \Omega$ is compact.

The assertion is false when $\mathbb{R}^n \setminus \Omega$ is unbounded. The case of a half-space Ω provides a counterexample for then $\mu(p,s,\Omega) = |s-1|^p/p^p$.

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5.2 Geometric Properties of Ω

The existence of an inequality (5.1.10) for some positive constant $C(p, \Omega)$ depends on the geometry of Ω and the nature of its boundary. In [72] and [60] a detailed study is made of the class of so-called *Generalised Ridged Domains*, this being a wide class which includes ones with special features, like horns, spirals and domains with fractal boundaries. The study includes an analysis of subsets of Ω which are significant for our present purposes; these are the so called *skeleton* and *ridge*. We refer to [15], Chapter 2 for background information and a detailed discussion of the results relevant to our needs in this chapter.

Let $N(x) := \{y \notin \Omega \colon |x - y| = \delta(x)\}$, and call it the *near set* of x on $\Omega^c := \mathbb{R}^n \setminus \Omega$. The *skeleton* of Ω is defined to be the subset

$$S(\Omega) := \{x \in \Omega \colon \mathrm{card} N(x) > 1\}, \tag{5.2.1}$$

where $\mathrm{card}N(x)$ denotes the cardinality of $N(x)$. Thus if $x \notin S(\Omega)$, there exists a unique $y \in N(x)$ and $\delta(x) = |x - y|$. From [60], Theorem 5.1.5, the function δ is differentiable at x if and only if $x \notin S(\Omega)$ and

$$\nabla\delta(x) = (x - y)/|x - y|, \quad x \in \Omega \setminus S(\Omega); \tag{5.2.2}$$

furthermore, $\nabla\delta$ is continuous on its domain of definition. Therefore, $S(\Omega)$ is the set of points in Ω at which δ is not differentiable. The function δ is uniformly Lipschitz on Ω; for let $x, y \in \Omega$ and choose $z \in \partial\Omega$ such that $\delta(y) = |y - z|$. Then

$$\delta(x) \le |x - z| \le |x - y| + \delta(y),$$

which together with the inequality obtained by reversing x and y yields

$$|\delta(x) - \delta(y)| \le |x - y|.$$

Since a Lipschitz function is differentiable almost everywhere by Rademacher's theorem, it follows that $S(\Omega)$ is of zero Lebesgue measure. Also by (5.2.2) $|\delta(x)| = 1$ a.e. on Ω.

For $x \in \Omega$ and $y \in N(x)$, let

$$\lambda := \sup\{t \in (0, \infty) \colon y \in N(y + t[x - y])\}. \tag{5.2.3}$$

Then, for all $t \in (0, \lambda)$, $N(y + t[x - y]) = y$. The point $p(x) = y + \lambda(x - y)$ is called the *ridge point* of x in Ω and the *ridge* of Ω is defined by

$$R(\Omega) := \{p(x) \colon x \in \Omega\}. \tag{5.2.4}$$

Another important subset of Ω relevant to us is $\Sigma(\Omega) := \Omega \setminus G(\Omega)$, where $G(\Omega)$ is the *good set* defined by Li and Nirenberg in [129] as the largest open subset of Ω such that every point $x \in G(\Omega)$ has a unique near point.

The following connections between the sets $S(\Omega), R(\Omega), \Sigma(\Omega)$ are established in [15], Lemma 2.2.8:

$$\begin{aligned} S(\Omega) \subseteq R(\Omega) &\subsetneqq \overline{S(\Omega)}, \\ \Sigma(\Omega) = \overline{R(\Omega)} &= \overline{S(\Omega)}. \end{aligned} \tag{5.2.5}$$

In [85] Fremlin shows that $R(\Omega)$ coincides with the *central set* $R_C(\Omega)$ of centres of maximal open balls contained in Ω. It is also proved in [85] that for any proper open subset Ω of $\mathbb{R}^2$, $R(\Omega)$ has zero two-dimensional Lebesgue measure, but it does not appear to be known if this is the case for general open subsets of $\mathbb{R}^n$ for $n > 2$. An example is given in [131], page 10, of a convex open subset Ω of $\mathbb{R}^2$ with a $C^{1,1}$ boundary which is such that $\overline{S(\Omega)}$ has nonzero Lebesgue measure; hence $R(\Omega)$ is not closed in view of (5.2.4) and Fremlin's result. It is proved in [104] and [129] that $R(\Omega)$ is closed and is of zero measure if Ω is a domain in $\mathbb{R}^2$ with a $C^{2,1}$ boundary.

Bunt [39] and Motzkin [139] established independently the important result that

$$R(\Omega) = S(\Omega) = \emptyset, \tag{5.2.6}$$

if and only if $\mathbb{R}^n \backslash \Omega$ is convex. The following statements are therefore equivalent (see [15], Theorem 2.2.9):

1. $\mathbb{R}^n \setminus \Omega$ is convex;
2. δ is differentiable at every $x \in \Omega$;
3. for every $x \in \Omega$, there is a unique point $y \in \mathbb{R}^n \setminus \Omega$ at minimal distance from x; thus $N(x) = \{y\}$.

In Section 1.3.1, for an open subset Ω of $\mathbb{R}^n (n \geq 2)$ with non-empty boundary $\partial\Omega$, the smoothness class $C^{k,\alpha}, k \in \mathbb{N}_0,\ \alpha \in [0, 1]$, of the boundary was defined. The smoothness of the boundary of Ω is reflected in that of the distance function δ. For instance, if $\partial\Omega \in C^k = C^{k,0},\ k \geq 2$, then for some positive constant $\mu, \delta \in C^k(\Gamma_\mu)$, where $\Gamma_\mu = \{x \in \overline{\Omega} \colon \delta(x) < \mu\}$; see [89], Lemma 1 in the Appendix. Hence for every $x \in \Gamma_\mu$, there is a unique near point $y \in N(x)$ and consequently $\Gamma_\mu \subset G(\Omega)$, the good set. The same applies for the boundary smoothness condition $\partial\Omega \in C^{k,\alpha}, k \geq 1,\ 0 \leq \alpha \leq 1$. We refer to [89] for a full discussion of smoothness conditions on Ω and its boundary.

Let Ω be a domain in $\mathbb{R}^n (n \geq 2)$ with a C^2 boundary, thus locally, after a rotation of co-ordinates, $\partial\Omega$ is the graph of a C^2 function. To be specific, for any $y \in \partial\Omega$, let $\mathbf{n}(y), T(y)$ denote respectively the unit inward normal to $\partial\Omega$ at y and the tangent plane to Ω at y. The $\partial\Omega$ is of class C^2 if, given any $y_0 \in \partial\Omega$, there exists a neighbourhood $\mathcal{N}(y_0)$ in which $\partial\Omega$ is given in terms of local co-ordinates by $x_n = \phi(x_1, ..., x_{n-1}), \phi \in C^2\,(T(y_0) \cap \mathcal{N}\,(y_0))$, where x_n lies in the direction of $\mathbf{n}(\mathbf{y_0})$ and with $x' = (x_1, ..., x_{n-1})$, we have

$$\begin{aligned}\mathbf{D}\phi(y_0') &= (D_1, D_2, ..., D_{n-1})\,\phi(y_0')\\ &= \left[\left(\partial/\partial x_1, \partial/\partial x_2, ..., \partial/\partial x_{(n-1)}\right)\phi\right](y_0') = 0.\end{aligned}$$

The *principal curvatures* $\kappa_1, ..., \kappa_{n-1}$, of $\partial\Omega$ at y_0 are the eigenvalues of the Hessian matrix

$$\left[\mathbf{D}^2\phi(y_0')\right] = \left(D_i D_j \phi(y_0')\right)_{i,j=1,...,n-1}.$$

For a domain Ω in $\mathbb{R}^n$, $n \geq 2$, with C^2 boundary, it is proved in [15], Lemma 2.4.2 that $\delta \in C^2(G(\Omega))$, where by (5.2.5), $G(\Omega) = \Omega \setminus \overline{R(\Omega)} = \Omega \setminus \overline{S(\Omega)}$, and

$$\Delta\delta(x) = \Sigma_{i=1}^{n-1}\left(\frac{\kappa_i(y)}{1+\delta(x)\kappa_i(y)}\right), \quad x \in G(\Omega),\ N(x) = \{y\};$$

here $\kappa_i(y)$, $i = 1, 2, ..., n-1$ are the principal curvatures of $\partial\Omega$ at y with respect to the unit inward normal. Moreover, with *mean curvature* defined by

$$H(y) := \frac{1}{n-1}\Sigma_{j=1}^{n-1}\kappa_j(y), \quad y \in \partial\Omega,$$

it is proved in [15], Propositions 2.5.3 and 2.5.4 that for $x \in G(\Omega)$ and $N(x) = y$, we have

$$1 + \delta(x)H(y) > 0$$

and

$$\Delta\delta(x) \leq \frac{(n-1)H(y)}{1+\delta(x)H(y)}.$$

5.3 Convex Domains

Proofs of the following two important properties of δ for a convex domain Ω may be found in [15], Section 2.3:

1. δ is concave, i.e., for any $x, y \in \Omega$ and $z = \lambda x + (1-\lambda)y$, where $\lambda \in (0, 1)$,

$$\delta(z) \geq \lambda\delta(x) + (1-\lambda)\delta(y);$$

2. δ is superharmonic, i.e., $-\Delta\delta \geq 0$ in the distributional sense,

$$-\int_\Omega \delta(x)\Delta\phi(x)\,dx = -\int_\Omega \Delta\delta(x)\phi(x)\,dx \geq 0,\ \left(0 \leq \phi \in C_0^\infty(\Omega)\right). \tag{5.3.1}$$

For a domain Ω in $\mathbb{R}^n (n \geq 2)$ with a C^2 boundary, it is proved in [124] (see also [15], Proposition 2.5.4) that δ is superharmonic in the good subset $G(\Omega)$ of Ω if and only if Ω is weakly mean convex.

Since $|\delta(x)| = 1$ a.e. on Ω, we have for all non-negative $\phi \in C_0^\infty(\Omega)$ and $1 < p < \infty$,

$$\int_\Omega |\nabla\delta|^{p-2}\nabla\delta \cdot \nabla\phi dx = \int_\Omega \nabla\delta \cdot \nabla\phi dx = -\int_\Omega \delta\Delta\phi dx \geq 0, \tag{5.3.2}$$

and hence if δ is superharmonic, the p-Laplacian satisfies

$$-\Delta_p\delta = -\mathrm{div}\left(|\nabla\delta|^{p-2}\nabla\delta\right) \geq 0 \tag{5.3.3}$$

in the distributional sense; δ is then said to be *p-superharmonic* on Ω. It follows that

$$\int_\Omega |\nabla\delta|^{p-2}\nabla\delta \cdot \nabla\phi\, dx = \int_\Omega \nabla\delta \cdot \nabla\phi\, dx$$

for $0 \leq \phi \in C_0^1(\Omega)$.

The fact that δ is p-superharmonic on a convex domain Ω implies the validity of an inequality (5.1.10) on Ω with $C(p,\Omega) = (p/(p-1))^p$. To see this, we follow a trick of Moser in [138]. Let $\phi = |u|^p/\delta^{p-1}$ in (5.3.2), with $u \in C_0^\infty(G(\Omega))$, where $G(\Omega)$ is the *good set* in Ω. Since δ is differentiable on $G(\Omega)$, we have that $\phi \in C_0^1(G(\Omega))$ and

$$p\int_\Omega (\nabla\delta \cdot \nabla|u|)\frac{|u|^{p-1}}{\delta^{p-1}}\, dx - (p-1)\int_\Omega \frac{|u|^p}{\delta^p}|\nabla\delta|^2 dx \geq 0.$$

Hence, as $|\nabla|u|| \leq |\nabla u|$ a.e. and $|\nabla\delta| = 1$ on $G(\Omega)$, we have

$$\begin{aligned}(p-1)\int_\Omega \frac{|u|^p}{\delta^p}\, dx &\leq p\int_\Omega |\nabla u|\frac{|u|^{p-1}}{\delta^{p-1}}\, dx \\ &\leq p\left(\int_\Omega |\nabla u|^p dx\right)^{1/p}\left(\int_\Omega \frac{|u|^p}{\delta^p}\, dx\right)^{1-1/p},\end{aligned}$$

and so

$$\int_\Omega \left|\frac{u(x)}{\delta(x)}\right|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla u(x)|^p\, dx, \quad u \in \overset{0}{D}{}_p^1(G(\Omega)). \tag{5.3.4}$$

This is an example of a Hardy inequality which holds for functions defined on a subset $\Gamma_r := \{x \in \Omega\colon \delta(x) < r\}$ of Ω. In [155], Robinson investigates the general question of whether, for $s \geq 0$, $r_0 > 0$, $r \in (0, r_0)$ and $a > 0$, the weighted $L_2(\Omega)$ Hardy inequality

$$\int_\Omega \delta(x)^{s-2}\, |f(x)|^2\, dx \leq a^2 \int_\Omega \delta(x)^s\, |\nabla f(x)|^2\, dx \tag{5.3.5}$$

is valid for all $f \in C_0^1(\Gamma_r)$, where $\Gamma_r = \{x \in \Omega\colon \delta(x) < r\}$. The Hardy constant $a_s(\Gamma_r)$ is then defined to be the infimum of all the constants a for which (5.3.5) is satisfied. It clearly decreases as $r \to 0$ and, denoting the boundary of Ω by

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Γ, the *boundary constant* $a_s(\Gamma)$ is defined as the infimum of $a_s(\Gamma_r)$ over $r \in (0, r_0)$. The inequalities (5.3.4) and (5.3.5) are *boundary* Hardy inequalities, in the sense that they do not hold for functions on all of Ω. The example $\Omega = B_1(0)$ demonstrates that this is all that can be achieved in general. For (5.3.5) is then valid for $s \in [0, 1)$ with $a_s(\Omega) = 2/(1-s)$ whereas if $s > 1$, (5.3.5) holds on $C_0^1(\Gamma_r)$ for all $r \in (0, 1)$ but fails on $C_0^1(B_1(0))$; see [113].

In [155], Theorems 4.3 and 5.1, the precise value of the boundary Hardy constant $a_s(\Gamma)$ is determined under the assumption that Ω is either convex or a $C^{1,1}$ domain. The assumption that Ω is a $C^{1,1}$ domain implies that its boundary satisfies a uniform internal ball condition and a uniform external ball condition. The uniform internal ball condition requires that for each $y \in \partial\Omega$, there exists $x \in \Omega$ and $k > 0$ such that $\overline{B}(x; k) \cap \Omega^c = \{y\}$. Hence, for small enough r, $\Gamma_r \subset G(\Omega)$, the *good set* of Ω. The uniform exterior ball condition is similar with Ω and Ω^c interchanged.

Theorem 5.4 *Let Ω be either convex or a $C^{1,1}$ domain in $\mathbb{R}^n$. Then for all $r \in (0, r_0)$ with r_0 sufficiently small, and all s such that $0 \le s \ne 1$,*

$$\int_\Omega \delta(x)^{s-2} |f(x)|^2 \, dx \le a_s(\Gamma_r)^2 \int_\Omega \delta(x)^s \, |\nabla f(x)|^2 \, dx \tag{5.3.6}$$

for all $f \in C_0^1(\Gamma_r)$. Moreover, the Hardy boundary constant is $a_s(\Gamma) = \frac{2}{|s-1|}$.

The boundary constant $a_s(\Gamma)$ is characterised by local constants in the sense that

$$a_s(\Gamma) = \sup_{j\in\mathbb{N}} a_s(\Gamma \cap U_j),$$

where $\left(U_j\right)_{j\in\mathbb{N}}$ is a cover of Γ by bounded open subsets of $\mathbb{R}^n$.

In [155] the inequality (5.3.6) is shown to be equivalent to a weighted version of Davies' *weak Hardy inequality* in [48], with equality of the corresponding optimal constants. The weak Hardy inequality on $C_0^1(\Omega)$ is

$$\int_\Omega \delta(x)^{s-2} \, |\psi(x)|^2 \, dx \le b^2 \int_\Omega \delta(x)^s(x) \, |\nabla\psi(x)|^2 \, dx + c^2 \int_\Omega |\psi(x)|^2 \, dx \tag{5.3.7}$$

for all $\psi \in C_0^1(\Omega)$ and some finite constants b, c. The *weak* Hardy constant $b_s(\Omega)$ is defined to be the infimum of all the b for which there is a c such that (5.3.7) is valid. One can also define $b_s(\Gamma_r)$ and $b_s(\Gamma)$ by restriction to functions in $C_0^1(\Gamma_r)$, as was done for $a_s(\Gamma_r)$. The aforementioned equivalence is given in

Theorem 5.5 *Let $s \in [0, 2)$. Then the boundary Hardy inequality (5.3.6) is valid if and only if the weak Hardy inequality (5.3.7) on $C_0^1(\Gamma_r)$ is valid. Moreover, if the inequalities are valid then $a_s(\Gamma) = b_s(\Gamma) = b_s(\Omega)$.*

For a convex domain Ω with a C^1 boundary, (5.3.4) was proved in [133] to hold for all $u \in C_0^\infty(\Omega)$ and

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$$l(p,\Omega) := \inf_{u\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p\,dx}{\int_\Omega |u/\delta|^p\,dx} = \left(\frac{p-1}{p}\right)^p =: c_p. \tag{5.3.8}$$

This was also established in [132] assuming only that Ω is convex. The existence of a minimiser in the variational problem determined by (5.3.8) was also explored in [132] for a bounded Ω with a C^2 boundary. For $1 < p < \infty$, it was shown that $l(p,\Omega) \leq \left(\frac{p-1}{p}\right)^p$, with equality if there is no minimiser; if $p = 2$, there is equality if and only if there is no minimiser. The existence of minimisers of (5.3.8) for domains of class $C^{1,\gamma}$, $\gamma \in [0,1]$ is an important feature of [113]. It is proved in [113], Theorem 4.1, that if Ω is bounded and $l(p,\Omega) < c_p$, then there exists a positive minimiser $u \in \overset{0}{W}_{1,p}(\Omega)$ of (5.3.8). Also, if $\alpha \in ((p-1)/p, 1)$ is such that $l(p,\Omega) = \lambda_\alpha$, then $0 < u(x) < C\delta(x)^\alpha$ for all $x \in \Omega$. The identity (5.3.8) is proved in [124] for a domain Ω which is weakly mean convex in $\mathbb{R}^n (n \geq 2)$, as long as the set $\Sigma(\Omega) = \Omega \setminus G(\Omega)$ is assumed to have zero measure. The weak mean convexity condition is sharp in the sense that the equality fails if only the mean curvature $H \leq \varepsilon$ is assumed for $\varepsilon > 0$.

In [154], Proposition 2.5, the following L_p version of (5.3.6) is given. Let $s \geq 0$ and suppose that $p - 1 - s > 0$. Then

$$\int_\Omega \delta(x)^{s-p}\,|f(x)|^p\,dx \leq a_p^p \int_\Omega \delta(x)^s\,|\nabla f(x)|^p\,dx, \quad f \in C_0^1\left(G(\Omega)\right), \tag{5.3.9}$$

where $a_p = (p/[p-1-s])$.

5.3.1 Convex Complements

The conclusion of Theorem 5.4 continues to be correct if Ω is the complement of a convex set and $s > 1$, but if $s \in [0,1)$ the constant $a_s(\Omega)$ can be strictly larger than $2/|s-1|$. The following analogue of (5.3.6) is derived in [154] for $\Omega = \mathbb{R}^n \setminus K$, where K is a closed convex subset of $\mathbb{R}^n$. The existence of the inequality is given by Theorem 1.1 in [154] and the optimality of the derived constant in Theorem 4.2.

Theorem 5.6 *Let $\Omega = \mathbb{R}^n \setminus K$ $(n \geq 2)$, where K is a closed convex subset of $\mathbb{R}^n$, and denote the Hausdorff dimension of the boundary $\partial\Omega$ of Ω by d_H. Let $c_\Omega = c \circ d_H$, where $c(t) = t^s(1+t)^{s-s'}$ with $s, s' \geq 0$. If $n - d_H + (s \wedge s') - p > 0$, with $p \in [1,\infty)$ and $s \wedge s' := \max\{s, s'\}$, then for all $\phi \in C_0^1(\Omega)$,*

$$\int_\Omega c_\Omega |\nabla\phi(x)|^p dx \geq \int_\Omega c_\Omega\,|(\nabla\delta(x))\cdot(\nabla\phi(x))|^p\,dx \geq a_p^p \int_\Omega c_\Omega \frac{|\phi(x)|^p}{\delta(x)^p} dx, \tag{5.3.10}$$

where $a_p = \left(n - d_H + (s \wedge s') - p\right)^p$.

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Let $d_H \in \{1, ..., n-1\}$ *and define the optimal constant*

$$l(p,\Omega) = \inf\left\{ \frac{\int_\Omega c_\Omega |\nabla\phi(x)|^p dx}{\int_\Omega c_\Omega \frac{|\phi(x)|^p}{\delta(x)^p}\, dx} : \phi \in C_0^\infty(\Omega) \right\}.$$

Then

$$l(p,\Omega) \le ([n - d_H + s - p]/p)^p\,,$$

with equality if $s \le s'$.

Another weighted inequality on a domain with a convex complement is the following from [8], Theorem 3.

Theorem 5.7 *Let* $\Omega = \mathbb{R}^n \backslash K$ $(n \ge 2)$*, where K is a closed, non-empty, convex subset of* $\mathbb{R}^n$*. Then for any* $p \in [1,\infty),\ s \in \mathbb{R}$ *and real-valued* $u \in C_0^1(\Omega)$*,*

$$\int_\Omega \frac{|\nabla u(x)|^p}{\delta^{s-p}(x)}\, dx \ge c_{n,p,s} \int_\Omega \frac{|u(x)|^p}{\delta^s(x)}\, dx, \tag{5.3.11}$$

where

$$c_{n,p,s} = \min\{|s-k|^p/p^p : k = 1, 2, ..., n\} \tag{5.3.12}$$

is optimal. Hence with $s = p \in [n,\infty)$*,*

$$\inf_{u\in C_0^1(\Omega), u\neq 0} \frac{\int_\Omega |\nabla u(x)|^p dx}{\int_\Omega \frac{|u(x)|^p}{\delta^p(x)}\, dx} \ge (p-n)^p/p^p. \tag{5.3.13}$$

Note from Remark 5.3 that by Theorem 4 in [8], if K in Theorem 5.7 is assumed to be a non-empty compact subset of $\mathbb{R}^n$ rather than convex, then for any $p \in [1,\infty)$ and $s \in [n,\infty)$,

$$c_{n,p,s} = |s-n|^p/p^p. \tag{5.3.14}$$

5.3.2 Non-convex Domains

For any domain Ω in $\mathbb{R}^n (n \ge 1)$, it is proved in [12] that Hardy's inequality

$$\int_\Omega \frac{|f(x)|^2}{\delta(x)^2}\, dx \le C(\Omega) \int_\Omega |\nabla f(x)|^2 dx,\ \ f \in C_0^\infty(\Omega) \tag{5.3.15}$$

holds for a finite constant $C(\Omega)$ if and only if there exist a strictly positive superharmonic function g on Ω and a positive number ε such that

$$\Delta g + \frac{\varepsilon}{\delta^2} g \le 0 \tag{5.3.16}$$

in the distributional sense, i.e.,

$$\int_\Omega \left(\Delta g + (\varepsilon/\delta^2)g\right)\psi\, dx = \int_\omega \left(\Delta\psi + (\varepsilon/\delta^2)\psi\right) g\, dx \le 0,\ \ 0 \le \psi \in C_0^\infty(\Omega).$$

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The largest value of ε in (5.3.16) is $1/C(\Omega)$, where $C(\Omega)$ is the best possible constant in (5.3.15). The function g is a so-called 'strong barrier' on Ω. We refer to [12] for background information and the proof of this important result.

For non-convex domains (and ones not weakly mean convex), the best possible constant in the Hardy inequality is not known in general, but for arbitrary planar, simply connected domains Ω, there is the following celebrated result of Ancona in [12]:

Theorem 5.8 *Let $\Omega \subsetneq \mathbb{R}^2$ be a simply connected domain. Then (5.3.15) holds with $C(\Omega) \leq 16$.*

Since the optimal constant $C(\Omega)$ in (5.3.15) (which, for now, we call the *strong* Hardy constant to distinguish it from the *weak* Hardy constant of Section 5.3) is 4 for a convex planar domain Ω, it is natural to ask if $C(\Omega)$ can take values between 4 and 16 if Ω possesses some degree of convexity. This was answered by Laptev and Sobolev in [115]; they established a refinement of Kobe's '1/4' theorem and introduced two possible 'measures' of non-convexity in their solution. In particular they proved that if any $y \in \partial\Omega$ is the vertex of an infinite sector Λ of angle $\theta \in [\pi, 2\pi]$ independent of y such that $\Omega \subset \Lambda$, then $C(\Omega)$ in Theorem 5.8 can be replaced by $4\theta^2/\pi^2$. The convexity case corresponds to $\theta = \pi$ and then 4 is recovered for $C(\Omega)$.

In [48] Davies determined the value of $C(\Omega_\beta)$ for the plane sector

$$\Omega_\beta := \{re^{i\theta} : 0 < r < 1,\ 0 < \theta < \beta\},\quad 0 < \beta < 2\pi.$$

He proved that the strong and weak Hardy constants are equal whenever $0 < \beta < 2\pi$. Furthermore, denoting the common value by C_β, there exists a critical angle $\beta^c = 4.856$ such that $C_\beta = 4$ for all $\beta \leq \beta^c$, while for $\beta^c < \beta \leq 2\pi$, C_β is strictly increasing and $4 < C_\beta \leq C_{2\pi} = 4.869$.

The example of a quadrilateral Ω in $\mathbb{R}^2$ with exactly one non-convex angle β, $\pi < \beta < 2\pi$ is considered in [18]. The best possible constant C_β is shown to be the unique solution of

$$2\sqrt{C_\beta}\left(\frac{\Gamma\left(\frac{3+\sqrt{1-4C_\beta}}{4}\right)}{\Gamma\left(\frac{1+\sqrt{1-4C_\beta}}{4}\right)}\right)^2 = \tan\left(\sqrt{C_\beta}\left(\frac{\beta-\pi}{2}\right)\right)$$

when $\beta^c \leq \beta < 2\pi$ and $C_\beta = 1/4$ when $\pi < \beta \leq \beta^c$. The constant C_β is precisely that computed numerically in [48] for a sector of angle β. The critical angle β^c is the unique solution in $(\pi, 2\pi)$ of the equation

$$\tan\left(\frac{\beta^c - \pi}{4}\right) = 4\left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right)^2.$$

The Hardy constant for other non-convex planar domains is computed in [19].

Hardy inequalities on annular regions bounded by convex domains with smooth boundaries were investigated by Avkhadiev and Laptev in [5]. An analogue of Theorem 1 in [5] is used in [15], Section 3.8 to derive a Hardy inequality on a general doubly connected domain $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$. Such a domain has a boundary which is the disjoint union of 2 simple curves. If its boundary is smooth then it can be mapped conformally onto an annulus $\Omega_{\rho,R} = B_R \setminus B_\rho = \{z \in \mathbb{C} \colon \rho < |z| < R\}$ for some ρ, R. Let $\Omega := \Omega_2 \setminus \overline{\Omega_1} \subset \mathbb{C}$ and $B_\rho \subset B_R \subset \mathbb{C}, 0 < \rho < R$, where B_r is the disc of radius r centred at the origin. Let

$$F \colon \Omega_2 \setminus \overline{\Omega_1} \to B_R \setminus \overline{B_\rho}$$

be analytic and univalent. Then in [15], Lemma 3.8.3, it is shown that

$$\mathcal{F}(z) := -\frac{|F'(z)|^2}{|F(z)|^2} + |F'(z)|^2 \left\{ \frac{1}{|F(z)| - \rho} + \frac{1}{R - |F(z)|} \right\}^2$$

is invariant under scaling, rotation and inversion, which implies it does not depend on the mapping F but only on the geometry of $\Omega_2 \setminus \overline{\Omega_1}$. Theorem 1 in [5] can then be shown to yield

Theorem 5.9 *For* $\Omega := \Omega_2 \setminus \overline{\Omega_1} \subset \mathbb{R}^2$,

$$\int_\Omega |\nabla u(x)|^2 dx \geq \frac{1}{4} \int_\omega \mathcal{F}(x) |u(x)|^2 \, dx, \quad u \in \overset{0}{H}{}^1_2(\Omega). \tag{5.3.17}$$

Avkhadiev proves in [4] that for $\Omega := B_R \setminus \overline{B_\rho}$,

$$\int_\Omega |\nabla u(x)|^2 dx \geq \lambda(\Omega) \int_\Omega \frac{|u(x)|^2}{\delta(x)^2} \, dx, \quad u \in \overset{0}{H}{}^1_2(\Omega), \tag{5.3.18}$$

where

$$\frac{2}{\pi} \ln \frac{R}{\rho} \leq \frac{1}{\lambda(\Omega)} \leq \ln \frac{R}{\rho} + k_0,$$

and $k_0 = \Gamma(\frac{1}{4})^4 / 2\pi^2 = 8.75...$ This inequality is applied in [15], Example 3.8.7, to prove that for $n \geq 3$ and any $\varepsilon > 0$, there exist ellipsoids E_1, E_2 with $\overline{E_2} \subset E_1 \subset \mathbb{R}^n$ and a function $f \in C_0^1(E_1 \setminus \overline{E_2})$ such that

$$\int_{E_1 \setminus \overline{E_2}} |\nabla f(x)|^2 dx \leq \varepsilon \int_{E_1 \setminus \overline{E_2}} \frac{|f(x)|^2}{\delta(x)^2} \, dx,$$

where $\delta(x)$ is the distance from $x \in E_1 \setminus E_2$ to the boundary of $E_1 \setminus E_2$. Moreover, the mean curvature $H(N(x)) \leq \varepsilon$ for all $x \in E_1 \setminus \overline{E_2}$, $N(x)$ being the near point of x.

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5.4 The Mean Distance Function

Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$ with non-empty boundary, and for $x \in \Omega,\ \nu \in \mathbb{S}^{n-1}$, set

$$\tau_\nu(x) = \min\{t > 0\colon x + t\nu \notin \Omega\}, \quad \delta_\nu(x) = \min\{\tau_\nu(x), \tau_{-\nu}(x)\}.$$

The *mean distance function M_p* is defined by

$$\frac{1}{M_p(x)^p} := \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{S}^{n-1}} \frac{1}{\delta_\nu^p(x)}\, d\omega(\nu), \tag{5.4.1}$$

where the measure $d\omega(\nu)$ on $\mathbb{S}^{n-1}$ is assumed to be normalised, i.e., $\int_{\mathbb{R}^n} d\omega(\nu) = 1$. It was introduced by Davies in [47] for $p = 2$ and for any $p \in (0, \infty)$ in [167]. For background information we refer to [15], 3.3. It is an effective and much used tool for establishing Hardy and similar inequalities in two and higher dimensions by reduction to one-dimensional problems. If Ω has suitable geometric properties the mean distance function can be estimated in a useful way. For example, when $\partial\Omega$ satisfies a θ-cone condition (every $x \in \partial\Omega$ is the vertex of a circular cone of semi-angle θ that lies entirely in $\mathbb{R}^n\backslash\Omega$), the mean distance function is equivalent to the usual distance function (see [15], p. 85); when Ω is convex, it is shown below that it is bounded above by the ordinary distance function. The following theorem from [47] (for $p = 2$) and [167] demonstrates its use.

Theorem 5.10 *For all $f \in \overset{0}{\mathcal{D}}{}^1_p(\Omega)$, $1 < p < \infty$ and any domain Ω with non-empty boundary,*

$$\int_\Omega \left|\frac{f(x)}{M_p(x)}\right|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla f(x)|^p\, dx, \quad f \in \overset{0}{D}{}^1_p(\Omega). \tag{5.4.2}$$

Proof The starting point is the one-dimensional inequality

$$\int_a^b |\phi'(t)|^p dt \geq \left(\frac{p-1}{p}\right)^p \int_a^b \frac{|\phi(t)|^p}{\rho(t)^p}\, dt, \quad \phi \in C_0^\infty(a, b), \tag{5.4.3}$$

where $\rho(t) = min\{|t-a|, |t-b|\}$. Let ϕ be real and $c := (a+b)/2$. Then

$$\begin{aligned}
\int_a^c \frac{|\phi(t)|^p}{(t-a)^p}\, dt &= \int_a^c (t-a)^{-p} \left(\int_a^t [|\phi(x)|^p]' dx\right) dt \\
&= \int_a^c [|\phi(x)|^p]' \left(\int_x^c (t-a)^{-p} dt\right) dx \\
&\leq \frac{p}{p-1} \int_a^c \frac{|\phi(x)|^{p-1} |\phi'(x)|}{(x-a)^{p-1}}\, dx
\end{aligned}$$

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since $||\phi(x)|'| \leq |\phi'(x)|$ a.e. Similarly,

$$\int_c^b \frac{|\phi(t)|^p}{(t-a)^p}\,dt \leq \frac{p}{p-1}\int_c^b \frac{|\phi(x)|^{p-1}|\phi'(x)|}{(b-x)^{p-1}}\,dx.$$

The two inequalities combine to give

$$\begin{aligned}\int_a^b \frac{|\phi(t)|^p}{\rho(t)^p}\,dt &\leq \left(\frac{p}{p-1}\right)\int_a^b \frac{|\phi(x)|^{p-1}|\phi'(x)|}{\rho(x))^{p-1}}\,dx \\ &\leq \left(\int_a^b \frac{|\phi(t)|^p}{\rho(t)^p}\,dt\right)^{1-1/p}\left(\int_a^b |\phi'(x)|^p dx\right)^{1/p}\end{aligned}$$

whence (5.4.3).

For $\nu \in \mathbb{S}^{n-1}$ and $\partial_\nu := \nu \cdot \nabla$, let$(a_\nu, b_\nu)$ be the interval of intersection of Ω with the ray in direction ν, $\delta_\nu(t) := \min\{|t-a_\nu|, |b_\nu - t|\}$, and denote by $\langle \nu, \omega\rangle$ the angle between ν and $\omega \in \mathbb{R}^n$. Then from (5.4.3),

$$\int_{a_\nu}^{b_\nu} |\partial_\nu \phi(t)|^p dt \geq \left(\frac{p-1}{p}\right)^p \int_{a_\nu}^{b_\nu} \frac{|\phi(t)|^p}{\delta_\nu(t)^p}\,dt, \quad \phi \in C_0^\infty(a_\nu, b_\nu). \tag{5.4.4}$$

On integrating both sides with respect to the normalised measure $d\omega(\nu)$ and writing $\nu \cdot \nabla\phi = |\nabla\phi| \cos\langle \nu, \nabla\phi\rangle$, we obtain

$$\begin{aligned}&\int_\Omega \int_{\mathbb{S}^{n-1}} |\cos(\nu, \nabla\phi(x))|^p d\omega(\nu)|\nabla\phi(x)|^p dx \\ &\quad \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \int_{\mathbb{S}^{n-1}} \frac{1}{\delta_\nu(x)^p}\,d\omega(\nu)|\phi(x)|^p dx.\end{aligned} \tag{5.4.5}$$

For any fixed unit vector $\mathbf{e} \in \mathbb{R}^n$,

$$\int_{\mathbb{S}^{n-1}} |\cos\langle \nu, \nabla\phi(x)\rangle|^p d\omega(\nu) = \int_{\mathbb{S}^{n-1}} |\cos\langle \nu, \mathbf{e}\rangle|^p d\omega(\nu)$$

and a calculation gives

$$\int_{\mathbb{S}^{n-1}} |\cos\langle \nu, \mathbf{e}\rangle|^p d\omega(\nu) = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+p}{2}\right)}. \tag{5.4.6}$$

The inequality (5.4.2) follows from (5.4.5) for any real $\phi \in C_0^\infty(\Omega)$, and hence any real $\phi \in \overset{0}{\mathcal{D}}{}^1_p$. If ϕ is not real, then $|\phi| \in \overset{0}{\mathcal{D}}{}^1_p$, $|\nabla|\phi|| \leq |\nabla\phi|$ a.e., and (5.4.2) is a consequence of the inequality already established for real functions. □

Remark 5.11

In a case like $\Omega = \mathbb{R}^n \setminus \{0\}$, where $\rho_\nu(t) = \infty$ unless the ray ν passes through the origin, a co-ordinate change is necessary; see the proof of Theorem 2.3 in [143]. Let $\{u_1, u_2, ..., u_n\}$, $u_1 = \nu$ be an orthonormal basis of $\mathbb{R}^n$, let

$\mathbf{v} = (v_1, v_2, ..., v_n)$ denote co-ordinates with respect to that basis and let P be a co-ordinate transition matrix $x = \mathbf{v}P$ from $\mathbf{v}$ co-ordinates to standard co-ordinates. For fixed $\hat{\mathbf{v}} = (v_2, ..., v_n)$ let $\Omega_{\hat{\mathbf{v}}} = \{v_1 \in \mathbb{R} \colon \mathbf{v}P \in \Omega\}$, where $\mathbf{v} = (v_1, \hat{\mathbf{v}})$ and $\Omega_{v_1} = \{\hat{\mathbf{v}} \colon \mathbf{v}P \in \Omega\}$. Define $g_{\hat{\mathbf{v}}} \colon \Omega_{\hat{\mathbf{v}}} \to \mathbb{R}$ and $\delta_\nu \hat{\mathbf{v}} \to (0, \infty]$ by

$$g_{\hat{\mathbf{v}}}(v_1) := f(\mathbf{v}P), \quad \delta_{\hat{\mathbf{v}}}(v_1) = \delta_{v_1}(\hat{\mathbf{v}}P).$$

Then from (5.4.4),

$$\int_{\Omega_{\mathbf{v}}} |g'_{\hat{\mathbf{v}}}(v_1)|^p dv_1 \geq \left(\frac{p-1}{p}\right)^p \int_{\Omega_{\hat{\mathbf{v}}}} \frac{|g_{\hat{\mathbf{v}}}(v_1)|^p}{\delta_{\hat{\mathbf{v}}}(v_1)^p}\, dv_1$$

and hence

$$\begin{aligned}\left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|f(x)|^p}{M_p(x)^p}\, dx &= \left(\frac{p-1}{p}\right)^p \int_{\mathbb{S}^{n-1}} \int_{\Omega_{v_1}} \int_{\Omega_{\hat{\mathbf{v}}}} \frac{|g_{\hat{\mathbf{v}}}(v_1)|^p}{\delta_{\hat{\mathbf{v}}}(v_1)^p}\, dv_1\, d\hat{\mathbf{v}}\, d\nu \\ &\leq \int_{\mathbb{S}^{n-1}} \int_{\Omega_{v_1}} \int_{\Omega_{\hat{\mathbf{v}}}} |g'_{\hat{\mathbf{v}}}(v_1)|^p dv_1\, d\hat{\mathbf{v}}\, d\nu \\ &= \int_{\mathbb{S}^{n-1}} \int_\Omega |(\nu \cdot \nabla) f(x)|^p\, d\omega(\nu)\, dx,\end{aligned}$$

which corresponds to (5.4.3) and hence leads to (5.4.2).

Theorem 5.12 *If Ω is convex, then $M_p(x) \leq \delta(x)$ for all $x \in \Omega$ and hence*

$$\int_\Omega \left|\frac{f(x)}{\delta(x)}\right|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_\Omega |\nabla f(x)|^p\, dx, \quad f \in \overset{0}{D}{}^1_p(\Omega). \tag{5.4.7}$$

Proof Let $\mathbf{e}$ be a unit vector in $\mathbb{R}^n$ which is such that $\rho_{\mathbf{e}}(x) = \delta(x)$. Then as Ω is assumed to be convex,

$$\delta_\nu(x) \cos\langle \mathbf{e}, \nu\rangle \leq \delta(x).$$

Hence

$$\begin{aligned}\int_{\mathbb{S}^{n-1}} \frac{1}{\delta_\nu(x)^p}\, d\omega(\nu) &\geq \int_{\mathbb{S}^{n-1}} |\cos\langle \mathbf{e}, \nu\rangle|^p \frac{1}{\delta(x)^p}\, d\omega(\nu) \\ &= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n+p}{2}\right)} \frac{1}{\delta(x)^p}\end{aligned}$$

by (5.4.6) and so $M_p(x) \leq \delta(x)$ which yields (5.4.7) from (5.4.6). □

Let $r := \sup\{\delta(x) \colon x \in \Omega\}$ and $\mu := \sup\{M_2(x) \colon x \in \Omega\}$ denote respectively the *inradius* and *mean inradius* of Ω. From (5.4.7) with $p = 2$, we have that the least eigenvalue λ_Ω of the Dirichlet Laplacian $-\Delta_\Omega^D$ on Ω satisfies

$$\lambda_\Omega = \inf\{\int_\Omega |\nabla u(x)|^2\, dx\colon u \in \overset{0}{H}{}_2^1(\Omega),\ \int_\Omega |u(x)|^2\, dx = 1\} \geq \frac{1}{4\mu^2}.$$

A lower bound for λ_Ω can also be obtained in terms of r. For if $\rho < r$ then Ω contains a ball B_ρ of radius ρ and

$$\begin{aligned}\lambda_\Omega &\leq \inf\{\int_{B_\rho} |\nabla u(x)|^2 dx\colon u \in \overset{0}{H}{}_2^1(B_\rho),\ \int_{B_\rho} |u(x)|^2 dx = 1\}\\ &= \left(1/\rho^2\right)\inf\{\int_{B_1} |\nabla u(x)|^2 dx\colon u \in \overset{0}{H}{}_2^1(B_1),\ \int_{B_1} |u(x)|^2 dx = 1\}\\ &= \left(1/\rho^2\right)\lambda_1,\end{aligned}$$

where $\lambda_1 := \lambda_{B_1}$ denotes the smallest eigenvalue of $-\Delta_{B_1}^D$. Therefore

$$\frac{1}{4\mu^2} \leq \lambda_\Omega \leq \frac{\lambda_1}{r^2}. \tag{5.4.8}$$

For $n \geq 2$, the value of λ_Ω is unchanged when Ω is punctured by a finite number of points (see [64], Corollary VIII.6.4), which means that μ is unaffected while r is reduced. Thus the mean inradius μ is of greater significance than the inradius r in the determination of λ_Ω for $n \geq 2$.

If Ω is convex, then $\mu \leq r$ by Theorem 5.4.3 and hence $\lambda_\Omega \geq 1/4r^2$. If Ω is mean convex, it is known that $\lambda_\Omega \geq c/\rho^2$, for some constant c, where ρ is the radius of the largest ball contained in Ω. This is not true for a general Ω but in [125] Lieb proved it to be true if the largest ball B_ρ contained in Ω is replaced by a ball that *intersects* Ω *significantly*. We reproduce an alternative proof of Lieb's result from [80] which uses the Hardy inequality (5.4.7) in the case $p = 2$ and the inequality, for any $x \in \Omega$ and $\rho > 0$,

$$|\Omega \cap B_\rho(x)| \leq \left(1 - \frac{\rho^2}{nM_2(x)^2}\right)|B_\rho(x)|, \tag{5.4.9}$$

where $B_\rho(x) = \{y\colon |y - x| < \rho,\}$, and

$$\frac{1}{M_2(x)^2}\colon = \frac{n}{\omega_{n-1}}\left(\int_{\mathbb{S}^{n-1}} \frac{1}{\delta_\nu(x)^2}\, d\omega(\nu)\right),\quad \delta_\nu(x)\colon = \inf\{|t| : x + t\nu \notin \Omega\}; \tag{5.4.10}$$

note the inclusion of ω_{n-1} to normalise the surface measure, as required in our definition of the mean distance function M_2. The estimate (5.4.9) is proved as follows. Denoting the characteristic function of Ω by χ_Ω, we have

$$|\Omega \cap B_\rho(x)| = \int_{\mathbb{S}^{n-1}} \int_0^\rho \chi_\Omega(x + t\nu)t^{n-1} dt\, d\omega(\nu).$$

For any $\nu \in \mathbb{S}^{n-1}$ with $\delta_\nu(x) > \rho$, we have $x + t\nu \in \Omega$ for all $t \in (0, \rho)$ and thus

$$|\Omega \cap B_\rho(x)| \geq |\{\nu \in \mathbb{S}^{n-1}\colon \delta_\nu(x) \geq \rho\}| n^{-1}\rho^n. \tag{5.4.11}$$

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On the other hand,

$$\rho^{-2}|\{\nu \in \mathbb{S}^{n-1}: \delta_\nu(x) \le \rho\}| \le \int_{\mathbb{S}^{n-1}} \delta_\nu^{-2} d\omega(\nu) = \frac{\omega_{n-1}}{nM_2(x)^2},$$

and so

$$|\{\nu \in \mathbb{S}^{n-1}: \delta_\nu(x) \ge \rho\}| \ge \left(1 - \frac{\rho^2}{nM_2(x)^2}\right)\omega_{n-1}.$$

On inserting this in (5.4.11), (5.4.9) follows.

From (5.4.2),

$$\begin{aligned}\lambda_\Omega &\ge \frac{1}{4}\inf\left\{\int_\Omega \frac{|u(x)|^2}{M_2(x)^2}dx\colon u \in \overset{0}{H}{}^1_2(\Omega),\ \int_\Omega |u(x)|^2 dx = 1\right\}\\ &\ge \frac{1}{4}\inf\{M_2(x)^{-2}\colon x \in \Omega\}.\end{aligned}$$

The lower bound for λ_Ω in [80] is given by using (5.4.9) in the last estimate, to give

Theorem 5.13 *Let $\Omega \subset \mathbb{R}^n$ be open. Then for any $\rho > 0$,*

$$\lambda_\Omega \ge \frac{1}{4\rho^2}\left(1 - \sup_{x\in\Omega}\frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|}\right). \tag{5.4.12}$$

This implies for all $\theta \in (0, 1)$,

$$\lambda_\Omega \ge \frac{(1-\theta)}{4\rho_\theta^2},\ \textit{where}\ \ \rho_\theta := \inf\left\{\rho > 0\colon \sup_{x\in\Omega}\frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|} \le \theta\right\}. \tag{5.4.13}$$

As remarked in [80], Theorem 5.13 has a counterpart for the principal eigenvalue of the p-Laplacian $-\Delta_{p,\Omega}$. From (5.4.1) the mean distance function is now given by

$$\frac{1}{M_p(x)^p} = C(n,p)\int_{\mathbb{S}^{n-1}} \frac{1}{\delta_\nu^p(x)}\, d\omega(\nu),$$

where $d\omega(\nu)$ is normalised and

$$C(n,p) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}.$$

The first eigenvalue $\lambda_{p,\Omega}$ of the p-Laplacian satisfies

$$\begin{aligned}\lambda_{p,\Omega} &\ge \left(\frac{p-1}{p}\right)^p \inf\left\{\int_\Omega \frac{|u(x)|^p}{M_p(x)^p}\, dx\colon u \in \overset{0}{H}{}^1_p(\Omega),\ \int_\Omega |u(x)|^p\, dx = 1\right\}\\ &\ge \left(\frac{p-1}{p}\right)^p \inf\{M_p(x)^{-p}\colon x \in \Omega\}.\end{aligned}$$

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The analogue of Theorem 4.4.4 is that for any $\rho > 0$,

$$\lambda_{p,\Omega} \geq \left(\frac{p-1}{p}\right)^p \rho^{-p} \left(1 - \sup_{x\in\Omega} \frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|}\right), \tag{5.4.14}$$

which implies, for all $\theta \in (0, 1)$,

$$\lambda_{p,\Omega} \geq \left(\frac{p-1}{p}\right)^p \frac{1-\theta}{\rho_\theta^p}, \text{ where } \rho_\theta := \inf\left\{\rho > 0\colon \sup_{x\in\Omega} \frac{|\Omega \cap B_\rho(x)|}{|B_\rho(x)|} \leq \theta\right\}. \tag{5.4.15}$$

5.5 Extensions of Hardy's Inequality

The refinement

$$\int_0^\pi |u'(x)|^2 dx \geq \frac{1}{4}\int_0^\pi \frac{|u(x)|^2}{\sin^2 x} + \frac{1}{4}\int_0^\pi |u(x)|^2 dx, \quad u \in \overset{0}{H}{}^1(\Omega) \tag{5.5.1}$$

of the Hardy inequality

$$\int_0^\pi |u'(x)|^2 dx \geq \frac{1}{4}\int_0^\pi \frac{|u(x)|^2}{x^2} dx, \quad u \in \overset{0}{H}{}^1(\Omega)$$

is derived in [87] through knowing about the exact solvability of the differential equation

$$\tau_s y := -\frac{d^2 y}{dx^2} + \frac{s^2 - (1/4)}{\sin^2 x} y = zy, \ z \in \mathbb{C}.$$

Roughly speaking, the non-negativity of the Friedrichs extension associated with the differential expression $\tau_s - (1/4)$, implying the non-negativity of the underlying quadratic form defined on $\overset{0}{H}{}^1(0, \pi)$, yields the refinement. Both constants $1/4$ on the right-hand side of (5.5.1) are shown to be optimal and the inequality is strict in the sense that equality holds if and only if $u = 0$.

In [34] Brezis and Marcus investigated the quantity

$$\lambda^*(\Omega) := \inf_{u\in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega |u/\delta|^2 dx}{\int_\Omega |u|^2 dx}$$

for smooth bounded domains Ω. It was shown that the infimum is not achieved, that there are domains for which $\lambda^*(\Omega) < 0$, but for convex domains with C^2 boundary,

$$\lambda^*(\Omega) \geq \frac{1}{4\,\mathrm{diam}^2(\Omega)}.$$

Thus with $D(\Omega) := \mathrm{diam}\,(\Omega)$,

$$\int_\Omega |\nabla u|^2 dx - \frac{1}{4}\int_\Omega |u/\delta|^2 dx \geq \frac{1}{4D(\Omega)^2}\int_\Omega |u|^2 dx, \quad u \in \overset{0}{H}{}^1(\Omega). \tag{5.5.2}$$

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This result generated a great deal of research into obtaining estimates for $\lambda^*(\Omega)$. The problem posed in [34] of whether the diameter $D(\Omega)$ could be replaced in (5.5.2) by a constant multiple of the volume $|\Omega|$ of Ω, i.e.,

$$\lambda^*(\Omega) \geq \alpha\, |\Omega|^{-2/n},$$

was settled in the affirmative in [98] by a method which made use of the mean distance function M_2 and is valid for any domain with non-empty boundary. The approach in [97] was followed in [74] to give the following theorem. Theorem 2.1 in [167] is an L_p analogue for other values of $p > 1$.

Theorem 5.14 *For any $u \in C_0^1(\Omega)$,*

$$\int_\Omega |\nabla u(x)|^2 dx \geq \frac{1}{4}\int_\Omega \frac{|u(x)|^2}{M_2(x)^2} + \frac{3}{2}K(n)\int_\Omega \frac{|u(x)|^2}{|\Omega_x|^{2/n}}\, dx, \tag{5.5.3}$$

where M_2 is the mean distance function defined in (5.4.1) for $p = 2$, $K(n) := n\left[n^{-1}\omega_n\right]^{2/n}$ and

$$\Omega_x := \{y \in \Omega\colon x + t(y - x) \in \Omega,\ \forall t \in [0, 1]\},$$

i.e., Ω_x is the set of all $y \in \Omega$ which can be 'seen' from $x \in \Omega$.

If Ω is convex, $\Omega_x = \Omega$ and for any $u \in C_0^1(\Omega)$,

$$\int_\Omega |\nabla u(x)|^2 dx \geq \frac{1}{4}\int_\Omega \frac{|u(x)|^2}{\delta(x)^2} + \frac{3K(n)}{2|\Omega|^{2/n}}\int_\Omega |u(x)|^2 dx. \tag{5.5.4}$$

Also for Ω convex, Filippas et. al. [77] obtained an estimate

$$\lambda^*(\Omega) \geq \frac{3}{D_{\text{int}}(\Omega)}$$

in terms of the *interior diameter* $D_{\text{int}}(\Omega) := 2\sup_{x\in\Omega}\delta(x)$. Clearly $D_{\text{int}}(\Omega) \leq D(\Omega)$ and a significant fact is that Ω need not be assumed to be bounded or have finite volume. Following Theorem 3.1 in [77], we now give another form of extension, of the type discussed in [14]. It is assumed that δ is superharmonic in the distributional sense, which we recall from Section 5.3 is satisfied if Ω is convex; in fact the assumption is weaker for $n \geq 3$ but equivalent if $n = 2$. It was also noted in Section 5.2 that δ is superharmonic for a domain Ω with a C^2 boundary 1.1 if and only if Ω is weakly mean convex.

Theorem 5.15 *Let $\Omega \subset \mathbb{R}^n$ be such that $-\Delta\delta \geq 0$ in the distributional sense. Then for any $\alpha > -2$ and all $u \in \overset{0}{H}{}_2^1(\Omega)$,*

$$\int_\Omega |\nabla\delta(x)\cdot\nabla u(x)|^2\, dx - \frac{1}{4}\int_\Omega \frac{|u(x)|^2}{\delta(x)^2}\, dx \geq \frac{C_\alpha}{D_{\text{int}}(\Omega)^{\alpha+2}}\int_\Omega \delta(x)^\alpha |u(x)|^2\, dx \tag{5.5.5}$$

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with

$$D_\alpha = \begin{cases} 2^\alpha(\alpha+2)^2, & \alpha \in (-2,-1), \\ 2^\alpha(2\alpha+3), & \alpha \in [-1,\infty). \end{cases}$$

Under the same conditions, a Hardy–Sobolev–Maz'ya extension of the Hardy inequality was given in [77]:

$$\int_\Omega |\nabla u(x)|^2\, dx - \frac{1}{4}\int_\Omega \frac{|u(x)|^2}{\delta(x)^2}\, dx \geq C_\Omega \left(\int_\Omega |u(x)|^{2n/(n-2)} dx\right)^{\frac{n-2}{n}}$$

for $n \geq 3$ and all $u \in C_0^\infty(\Omega)$. The problem was posed: can the constant C_Ω be chosen to be independent of Ω? This was settled in [81], where it was also proved that if Ω is a convex domain in $\mathbb{R}^n (n \geq 3)$ and $p \in [2,n)$, there exists a constant $C_{n,p}$, depending only upon n and p, such that

$$\int_\Omega |\nabla u(x)|^p\, dx - \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u(x)|^p}{\delta(x)^p}\, dx \geq C_{n,p} \left(\int_\Omega |u(x)|^{pn/(n-p)} dx\right)^{\frac{n-p}{n}} \tag{5.5.6}$$

for all $u \in C_0^\infty(\Omega)$. For Ω the half-space $\mathbb{R}^n_+ := \{(x', x_n) : x' \in \mathbb{R}^{n-1}, x_n > 0\}$ (and so $\delta(x) = |x_n|$), the case $p = 2$ is proved in [135] and for $2 < p < n$ in [17]; also the sharp value of $C_{3,2}$ is given in [20].

Avkhadiev and Wirths also considered domains Ω which are convex and have finite inradius, and obtained in [6] the following generalisation of the Hardy inequality with weights and sharp constants. An L_p analogue for $p > 2$ is given in [140].

Theorem 5.16 *Let $\Omega \subset \mathbb{R}^n$ be convex with finite inradius $D_{\text{int}}(\Omega)$. Then for all $f \in C_0^1(\Omega)$,*

$$\int_\Omega \frac{\nabla f(x)|^2}{\delta(x)^{s-1}}\, dx \geq A\int_\Omega \frac{|f(x)|^2}{\delta(x)^{s+1}}\, dx + \frac{\lambda^2}{D_{\text{int}}(\Omega)^q}\int_\Omega \frac{|f(x)|^2}{\delta(x)^{s-q+1}}\, dx, \tag{5.5.7}$$

where A and λ are sharp constants given by

$$A = \frac{s^2 - \nu^2 q^2}{4} \geq 0, \lambda = \frac{q}{2}\lambda_\nu(2s/q) > 0,$$

and s, q are positive numbers, $\nu \in [0, s/q]$ and $z = \lambda_\nu(s)$ is the Lamb constant defined as the positive root of the Bessel equation $sJ_\nu(z) + 2J'_\nu(z) = 0$. In particular, with $s = 1$ and $\nu = 0$,

$$\int_\Omega |\nabla f(x)|^2 dx \geq \frac{1}{4}\int_\Omega \frac{|f(x)|^2}{\delta(x)^2}\, dx + \frac{\lambda_0^2}{D_{\text{int}}(\Omega)^2}\int_\Omega |f(x)|^2\, dx, \tag{5.5.8}$$

and $\lambda_0 = 0.940...$ is the first zero of $J_0(t) - 2J_1(t)$. The inequality is sharp for $n \geq 1$.

The proof of (5.5.7) is based on the one-dimensional inequality

$$\int_0^1 f'(x)^2 dx > \frac{1}{4}\int_0^1 \frac{f^2(x)}{x^2}\,dx + \lambda_0^2 \int_0^1 f(x)^2 dx \tag{5.5.9}$$

for real functions f which are absolutely continuous on $[0, 1]$ and such that $f(0) = 0, f' \in L_2(0, 1)$. The constant λ_0 is shown to be sharp by exhibiting, for each $\varepsilon > 0$, a real function $f \in C_0^1(0, 2)$ which is such that $f'(1) = 0$ and

$$\int_0^1 f'(x)^2 dx < \frac{1}{4}\int_0^1 \frac{f^2(x)}{x^2}\,dx + \left(\lambda_0^2 + \varepsilon\right) \int_0^1 f(x)^2 dx.$$

Theorem 5.16 is established in [6] by means of inequalities derived from (5.5.9) and the application of an approximation technique of Hadwiger [92] for domains; the same applies to the L_p analogue in [140]. This technique implies, in particular, that for a convex domain $\Omega \subset \mathbb{R}^n$ and any compact set $K \subset \Omega$, there exists a convex n-dimensional polytope Q such that $K \subset \operatorname{int} Q \subset \Omega$. Thus for any $f \in C_0^\infty(\Omega)$, there is a convex n-dimensional polytope Q such that $\operatorname{supp} f \subset \operatorname{int} Q \subset \Omega$. This provides an effective method for establishing many-dimensional inequalities from ones of one dimension, as demonstrated by the many significant contributions made by Avkhadiev and his collaborators.

Brezis–Marcus type inequalities are considered in Section 3.7 of [15] and [124], where assumptions are made on the curvature of the boundary of Ω and δ, rather than the convexity of Ω. Applications to some non-convex domains such as the torus and 1-sheeted hyperboloid follow. For instance, for a ring torus $\Omega \subset \mathbb{R}^3$ with minor ring r and major ring $R \geq 2r$, the ridge $\mathcal{R}(\Omega)$ is closed and of measure zero, $-\Delta\delta > 0$ in $G(\Omega) = \Omega \setminus \mathcal{R}(\Omega)$ and

$$\begin{aligned}\int_\Omega |\nabla\delta(x)\cdot\nabla f(x)|^p &\geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|f(x)|^p}{\delta(x)^p}\,dx \\ &\quad + \left(\frac{p-1}{p}\right)^{p-1} \frac{R-2r}{r(R-r)} \int_\Omega \frac{|f(x)|^p}{\delta(x)^{p-1}}\,dx\end{aligned} \tag{5.5.10}$$

for all $f \in C_0^\infty(\Omega)$.

5.6 Discrete Laptev–Weidl Type Inequalities

We present two discrete versions of the Laptev–Weidl inequality (5.1.9); the first is that in [75] on a discretised cylinder $\mathbb{R} \times \mathbb{S}^1$ obtained from the punctured plane $\mathbb{R}^2 \setminus \{0\}$; the second version is the one derived in [91] on the standard lattice $\mathbb{Z}^2$. The following two proofs of (5.1.9) will provide background for the discrete versions considered.

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In the Laptev–Weidl inequality

$$\int_{\mathbb{R}^2} |(\nabla + i\mathbf{A})f(x)|^2\,dx > C\int_{\mathbb{R}^{\not{\in}}} \frac{|f(x)|^2}{|x|^2}\,dx,\ C = \min_{k\in\mathbb{Z}} |k-\Psi|^2,$$

given in (5.1.9), the left-hand side is conveniently written in polar co-ordinates as

$$h_A := \int_0^\infty \int_0^\infty \left(\left|\frac{\partial f}{\partial r}\right|^2 + r^{-2}|K_\theta f|^2 \right) r\,dr\,d\theta$$

where $K_\theta := i\frac{\partial}{\partial\theta} + \Psi$ is a self-adjoint operator with domain $H^1(\mathbb{S}^1)$ in $L_1(\mathbb{S}^1)$; it has eigenvalues $\lambda_k = k + \Psi$, $k \in \mathbb{Z}$ and corresponding eigenvectors

$$\phi_k(\theta) = \frac{1}{\sqrt{2\pi}} \exp(-ik\theta).$$

The sequence $\{\phi_k\}$ is an orthonormal basis of $L_2(\mathbb{S}^1)$ and hence any $f \in L_2(\mathbb{S}^1)$ has the representation

$$f(r,\theta) = \sum_{k\in\mathbb{Z}} f_k(r)\phi_k(\theta),$$

where

$$f_k(r) = \int_0^{2\pi} f(r,\theta)\overline{\phi_k(\theta)}\,d\theta.$$

For any $f \in H^1(\mathbb{S}^1)$,

$$h_A[f] = \sum_{k\in\mathbb{Z}} \int_0^\infty \left(|f_k'(r|^2 + \frac{\lambda_r^2}{r^2}|f_k(r)|^2 \right) r\,dr$$

and so

$$\begin{aligned}
\int_{\mathbb{R}^2} \frac{|f(x)|^2}{|x|^2}\,dx &= \sum_{k\in\mathbb{Z}} \int_0^\infty \frac{|f_k(r)|^2}{r^2}\,r\,dr \\
&\leq \sum_{k\in\mathbb{Z}} \frac{1}{\min_{m\in\mathbb{Z}} \lambda_m^2} \int_0^\infty \lambda_k^2 \frac{|f_k(r)|^2}{r^2}\,r\,dr \\
&\leq \frac{1}{(\min_{k\in\mathbb{Z}} |k+\Psi|)^2} h_A[f],
\end{aligned}$$

which proves (5.1.9). We refer to [116], Theorem 3, for a proof that the constant in (5.1.9) is sharp; see also [15], Chapter 5, which also contains background information.

The inequality (5.1.9) can also be proved simply by observing that the punctured plane $\mathbb{R}^2 \setminus \{0\}$ is topologically equivalent to the cylinder $\mathbb{R} \times \mathbb{S}^1$; see [75]. On setting

$$g(u,v) = e^{i\int_0^v \psi(\theta)d\theta} f\left(e^u \cos v, e^u \sin v\right),$$

(5.1.9) with $f \in C_0^\infty(\mathbb{R} \setminus \{0\})$ becomes

$$\int_{-\infty}^{\infty}\int_0^{2\pi} |g(u,v)|^2\, dv\, du \le C^{-1}\int_{-\infty}^{\infty}\int_0^{2\pi}\left\{\left|\frac{\partial g}{\partial u}\right|^2 + \left|\frac{\partial g}{\partial v}\right|^2\right\} dv\, du, \quad (5.6.1)$$

where $g \in C_0^\infty(\mathbb{R} \times [0, 2\pi])$ and

$$g(u, 2\pi) = e^{2\pi\Psi} g(u,0), \quad \frac{\partial g}{\partial v}(u, 2\pi) = e^{2\pi\Psi}\frac{\partial g}{\partial v}(u,0). \quad (5.6.2)$$

On integration by parts, the integral on the right-hand side of (5.6.1) is then seen to equal

$$\int_{-\infty}^{\infty}\int_0^{2\pi} \overline{g}\left(-\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}\right) g\, dv\, du.$$

The symmetric operator defined by

$$-\frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2}$$

for functions in $C_0^\infty(\mathbb{R} \times [0, 2\pi])$ satisfying (5.6.2), has a self-adjoint extension (its Friedrichs extension) with spectrum $[C, \infty)$; this follows by separation of variables, for it decouples into the non-negative self-adjoint operator $-\frac{d^2}{du^2}$ in $L_2(\mathbb{R})$ with spectrum $[0, \infty)$ and the self-adjoint operator $\frac{d^2}{dv^2}$ in $L_2([0, 2\pi])$ with boundary conditions (5.6.2) which has spectrum $\{\lambda \ge 0 \colon \sqrt{\lambda} = \pm\Psi \mod \mathbb{Z}\}$. Hence $C = \min_{k\in\mathbb{Z}} |k - \Psi|^2$ and the inequality (5.1.9) follows.

The form of the Laptev–Weidl inequality in (5.6.1) has a discrete variant which was studied in [75]. This was based on discretising the cylinder $\mathbb{R} \times [0, 2\pi]$ to $\{j/m \colon j \in \mathbb{Z}\} \times \{2\pi j/n \colon j \in \{1, 2, ..., n\}\}$; the corresponding point set in the physical plane accumulates at the origin and is exponentially sparse at infinity. On replacing $\partial g/\partial u$ and $\partial g/\partial v$ by $m\{g(j/k) - g(j-1,k)\}$ and $(n/2k)\{g(j,k) - g(j,k-1)\}$ respectively, the discrete analogue of (6.1.6) is

$$\frac{2\pi}{mn}\sum_{j\in\mathbb{Z}}\sum_{k=1}^{n} |g(j,k)|^2$$
$$\le C^{-1}\frac{2\pi}{mn}\sum_{j\in\mathbb{Z}}\sum_{k=1}^{n}\left\{m^2\,|g(j,k) - g(j-1,k)|^2 + \left(\frac{n}{2\pi}\right)^2 |g(j,k) - g(j,k-1)|^2\right\}. \quad (5.6.3)$$

Denoting by Δ the left-difference operator $\Delta u(l) = u(l) - u(l-1)$ and $\Delta^2 u(l) = \Delta^*\Delta u(l) = u(l+1) + u(l-1) - 2u(l)$, and using the notation

$$\Delta_1 g(j,k) := g(j,k) - g(j-1,k), \quad \Delta_2 g(j,k) := g(j,k) - g(j,k-1),$$

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the double sum on the right-hand side of (5.6.3) becomes

$$-\sum_{j\in\mathbb{Z}}\sum_{k\in\Lambda_n}\left\{m^2\overline{g(j,k)}\Delta_1^2 g(j,k)+\left(\frac{n}{2\pi}\right)^2\overline{g(j,k)}\Delta_2^2 g(j,k)\right\}, \tag{5.6.4}$$

where $\Lambda_n = \{1, 2, ..., n\}$. The following discrete analogue of the Laptev–Weidl inequality is established in [75], Theorem 1.4:

Theorem 5.17 *For all $g \in l_2(\mathbb{Z}\times\Lambda_n)$ and $\Psi \in (0, 1/2]$,*

$$\begin{aligned}\left\{\frac{n}{\pi}\sin\left(\frac{\pi}{n}\Psi\right)\right\}^2\sum_{j\in\mathbb{Z}}\sum_{k\in\Lambda_n}|g(j,k)|^2 &\le D_{mn}[g]\\ &\le \left(4m^2\left\{\frac{n}{\pi}\sin\left[\frac{\pi}{n}\left(\Psi+\left[\frac{n}{2}\right]\right)\right]\right\}^2\right)\sum_{j\in\mathbb{Z}}\sum_{k\in\Lambda_n}|g(j,k)|^2\end{aligned} \tag{5.6.5}$$

where

$$D_{mn}[g] := \sum_{j\in\mathbb{Z}}\sum_{k\in\Lambda_n}\left\{m^2|\Delta_1 g(j,k)|^2+\left(\frac{n}{2\pi}\right)^2|\Delta_2 g(j,k)|^2\right\}.$$

Another discrete analogue of a Sobolev-type inequality for Schrödinger operators with Aharonov–Bohm magnetic potential is also derived in [75], Theorem 1.5:

Theorem 5.18 *For all $g \in l_2(\mathbb{Z}\times\Lambda_n)$,*

$$\sup_{j\in\mathbb{Z}}\sum_{k\in\Lambda_n}|g(j,k)|^2 \le \frac{\pi}{2mn\sin(\Psi\pi/n)}D_{mn}[g].$$

The first step in the proof of the result in [91] is the definition of the discrete Aharonov–Bohm potential. For $k = (k_1, k_2) \in \mathbb{Z}^2$, let $|k|_\infty$ denote the norm $\max\{|k_1|, |k_2|\}$. The circles centred at the origin in this norm are the squares $\mathbb{S}(n) = \{k \in \mathbb{Z}^2 : |k|_\infty = n\}$, with n serving as the radial variable. Denote the phases $\phi_1(k)$ and $\phi_2(k)$ by

$$\begin{aligned}\phi_1(k) &= -\frac{k_2}{8|k|_\infty^2} = +(8n)^{-1}, \quad k_2 = n, \quad -n \le k_1 \le n,\\ \phi_2(k) &= -\frac{k_1}{8|k|_\infty^2} = +(8n)^{-1}, \quad k_1 = -n, \quad -n < k_2 \le n,\\ \phi_1(k) &= \frac{k_2}{8|k|_\infty^2} = -(8n)^{-1}, \quad k_2 = -n, \quad -n \le k_1 \le n,\\ \phi_2(k) &= \frac{k_1}{8|k|_\infty^2} = -(8n)^{-1}, \quad k_1 = n, \quad -n \le k_2 < n.\end{aligned}$$

The discrete version of the Aharonov–Bohm potential corresponding to the magnetic flux Ψ is defined as

$$\mathcal{A}_\Psi(k) = -i\,(A_1(k), A_2(k)) = -i\left(1 - e^{2\pi i\Psi\phi_1(k)}, 1 - e^{2\pi i\Psi\phi_2(k)}\right).$$

Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The main result is Theorem 1.1 of [91]:

Theorem 5.19 *For all functions $u\colon \mathbb{Z}^2 \to \mathbb{C}$ decaying sufficiently fast,*

$$\sum_{k\in\mathbb{Z}^2}\sum_{j=1,2}\left|u(k+e_j) - u(k) + iA_j(k)u(k)\right|^2$$
$$\geq 4\sin^2\left(\pi\,\frac{dist(\Psi,\mathbb{Z})}{8}\right)\sum_{k\in\mathbb{Z}^2\setminus\{0\}}\frac{|u(k)|^2}{|k|_\infty^2}. \tag{5.6.6}$$

Since $\mathrm{dist}(\Psi, \mathbb{Z}) \leq 1/2$, we have

$$4\sin^2\left(\pi\,\frac{\mathrm{dist}(\Psi,\mathbb{Z})}{8}\right) \geq 4\left[\pi\,\frac{\mathrm{dist}(\Psi,\mathbb{Z})}{8}\right]^2\frac{\sin^2(\frac{\pi}{16})}{\left(\frac{\pi}{16}\right)^2}$$
$$= 16\sin^2\left(\frac{\pi}{16}\right)\min_{l\in\mathbb{Z}}|l-\Psi|^2,$$

and (5.6.6) implies

Corollary 5.20 *For all functions $u\colon \mathbb{Z}^2 \to \mathbb{C}$ decaying sufficiently fast,*

$$\sum_{k\in\mathbb{Z}^2}\sum_{j=1,2}\left|u(k+e_j) - u(k) + iA_j(k)u(k)\right|^2$$
$$\geq 16\sin^2\left(\frac{\pi}{16}\right)\min_{k\in\mathbb{Z}^2}|l-\Psi|^2\sum_{k\in\mathbb{Z}^2\setminus\{0\}}\frac{|u(k)|^2}{|k|_\infty^2}. \tag{5.6.7}$$

Note that $16\sin^2\left(\frac{\pi}{16}\right) = 4\left(2 - \sqrt{2+\sqrt{2}}\right) \sim 0.50896....$