INTERSECTION IRREDUCIBLE IDEALS OF A NON-COMMUTATIVE PRINCIPAL IDEAL DOMAIN

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Introduction. Let R always denote a fixed non-commutative principal ideal domain. A right (left) ideal aR (Ra) is termed *right* (*left*) $\cap$ *irreducible* provided it is not the intersection of two right (left) ideals that properly include it. In this case, the element a is called *right* (*left*) $\cap$ *irreducible.*

Since R satisfies the $A.C.C.$ for right ideals every right ideal aR can be written in the form $aR = a_1R \cap a_2R \cap \ldots \cap a_nR$, where a_iR properly include aR and is right $\cap$ irreducible, $i = 1, 2, \ldots, n$. We shall investigate properties (including primary properties) of right $\cap$ irreducible one-sided and two-sided ideals of R. These properties will depend on the results given in **(1)** and **(2**, chapter III**)**.

An element a is *irreducible* if it is not zero or a unit and has no proper factors. In this case aR (Ra) is a maximal right (left) ideal.

1. Right $\cap$ irreducible one-sided ideals.

From Theorem 2 of **(2**, p. 31**)** it follows that the zero ideal is right and left $\cap$ irreducible. This is also trivially true for R. The special case where a is irreducible will be discussed briefly in §3. Hence in this paper, unless otherwise stated, the ideals aR mentioned will not be 0, R or maximal.

THEOREM 1.1. *For $aR \subset R$ the following statements are equivalent*:

1. a is right $\cap$ irreducible.
2. aR is contained in a unique minimal right overideal not equal to R.
3. Ra is contained in a unique maximal left ideal.
4. If $a = bc = b'c'$ where c' and c are irreducible then $Rc = Rc'$.
5. If $a = bc = b'c'$ where c' and c are irreducible then $bR = b'R$.

Proof. $1 \leftrightarrow 2$. Let a be right $\cap$ irreducible. From Theorem 1 of **(2**, p. 31**)** we have that every right ideal has at least one minimal overideal. If bR and dR are distinct minimal overideals of aR then $aR = bR \cap dR$—a contradiction. Conversely, let bR be the unique minimal overideal of aR. Suppose $aR = dR \cap eR$ where dR and eR properly include aR. Hence $bR \subseteq dR$ and $bR \subseteq eR$ and $aR \subset bR \subseteq dR \cap eR$—a contradiction.

In order, we now show that $2 \to 3 \to 4 \to 5 \to 2$. To show that $2 \to 3$ let bR be the unique minimal overideal of aR. Then $a = bc$ where c is irreducible **(2**, p. 34**)**. Hence $Ra \subset Rc$. Suppose $Ra \subset Rc'$ where Rc' is a maximal left ideal. Then $a = dc'$ and $aR \subset dR$. Therefore, $bR \subseteq dR$, that is, $b = dk$. Hence $a = dc' = dkc$ which implies $c' = kc$. Thus k is a unit and $Rc' = Rc$.

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$3 \to 4$. Let Rc be the unique maximal left ideal containing Ra. If $a = b'c' = b''c''$ where c' and c'' are irreducible, then $Ra \subset Rc'$ and $Ra \subset Rc''$. Hence $Rc = Rc' = Rc''$.

$4 \to 5$. Let $a = b'c' = b''c''$ where c' and c'' are irreducible. By 4 we have $Rc' = Rc''$. Hence there exists a unit u such that $c' = uc''$. Therefore, $a = b'uc'' = b''c''$. Thus $b'u = b''$ and $b'R = b''R$.

$5 \to 2$. Suppose bR and $b'R$ are minimal overideals of aR. Then $a = bc = b'c'$ where c and c' are irreducible. By 5 we can conclude that $bR = b'R$.

From 3 of this theorem the following corollary is immediate.

COROLLARY 1.1. *Let aR be right $\cap$ irreducible. If $Ra \subset Rd_1 \subset Rd_2 \subset \ldots \subset Rd_n \subset R$ is a composition series where Rd_n is the unique maximal left ideal containing Ra, then d_i, $i = 1, 2, \ldots, n$, is right $\cap$ irreducible.*

From the corollary we have

COROLLARY 1.2. *Let a be right $\cap$ irreducible. If $a = kd = k'd'$ where $Rd = Rd'$ then $a = kbc = k'b'c'$ where $bR = b'R$.*

STATEMENT 1.1. *Let a be right $\cap$ irreducible. If bR is the unique minimal overideal of aR and $a = bc$ then*

$$cR = \{aR|b\}_r = \{x|bx \in aR, x \in R\}.$$

Proof. Certainly $cR \subseteq \{aR|b\}_r$. If $d \in \{aR|b\}_r$ then $bd = am = bcm$. Hence $d = cm$ and $d \in cR$.

From (**2**, p. 34) we know that every element may admit several factorizations as a product of irreducible elements. Part 4 of Theorem 1.1 tells us that a is right $\cap$ irreducible if and only if for any two factorizations $a = a_1a_2 \ldots a_n = b_1b_2 \ldots b_n$ where the a_i's and b_i's are irreducible we have $Ra_n = Rb_n$.

2. Primary properties. The ring R can be considered as an $A - K$ module as defined in (**1**) by taking as A the ring of left multiplications and as K the ring of right multiplications. By Theorems 2.1. and 2.2. of (**1**) we have the following.

Let aR be right $\cap$ irreducible. The normalizer (centralizer) B of aR is $\{b|ba \in aR\}$. Then $P = \{x|x^n \in aR, x \in B\}$ is a completely prime two-sided ideal of B and $cx \in aR$ for $c \in B$, $x \notin aR$ implies $c \in P$.

Since aR is a right ideal we may consider $R - aR$ as an R module. Let us now apply Theorem 1 of (**3**, p. 25) to this R module. Since $R \ni 1$, the module $R - aR$ is cyclic. Then $I = \{x|\bar{1}x = \bar{0}, x \in R\}$ is a right ideal of R where $\bar{1}$ and $\bar{0}$ denotes the cosets of 1 and 0 in $R - aR$. Certainly $I = aR$. Since $R \ni 1$ the ideal $K = \{k|kR \subseteq aR, k \in B\} = aR$. Applying the theorem above we have the ring of endomorphisms C of $R - aR$ is B/aR.

Since $a \neq 0$ the module $R - aR$ satisfies both chain conditions for submodules. Since aR is right $\cap$ irreducible certainly $R - aR$ is indecomposable. Applying Theorem 3 of (**2**, p. 57) then the ring of endomorphisms of $R - aR$,

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which is B/aR, is completely primary, that is, if $P = \{x | x^n \in aR, x \in B\}$ then $(B/aR)/(P/aR) \cong B/P$ is a division ring. Hence P is a maximal right ideal of B and we have proved

THEOREM 2.1. *If aR is right $\cap$ irreducible and B the normalizer of aR in R, then the radical P of aR in B is a maximal right ideal and B/P is a division ring.*

LEMMA 2.1. *Let aR be right $\cap$ irreducible and P the radical of aR in B. If bR is the unique minimal overideal of aR then $xbR \subseteq aR$ for $x \in P$.*

Proof. If $x \in P$ then $x^n \in aR$ for some integer n. If $n = 1$ the statement is obviously true. Assume $n > 1$ and $x^n \in aR$, $x^{n-1} \notin aR$. Since $x^{n-1} \notin aR$ then $bR \subseteq x^{n-1}R + aR$. Hence $b = x^{n-1}g_1 + ag_2$, $xb = x^n g_1 + xag_2$. Since $x \in B$ and $x^n \in aR$ we have $xb \in aR$. Hence $xbR \subseteq aR$.

STATEMENT 2.1. *Let aR be right $\cap$ irreducible and P be the radical of aR in its normalizer B. If bR is the unique minimal overideal of aR then $P = \{aR|b\}_l = \{x | xb \in aR, x \in R\}$.*

Proof. By Lemma 2.1 certainly $P \subseteq \{aR|b\}_l$. If $x \in \{aR|b\}_l$ then $xb \in aR$ and $xbR \subseteq aR$. Since $aR \subseteq bR$ certainly $x \in B$. From the discussion at the beginning of this section and since $xb \in aR$, $x \in B$, $b \notin aR$ then $x \in P$.

3. Summary. In the commutative case for a principal ideal domain an ideal aR is $\cap$ irreducible if and only if $aR = c^n R$ where c is irreducible. Here $aR \subseteq cR$, c is irreducible, cR is maximal and cR is called the radical $(aR)^{\frac{1}{2}}$ of aR. (Read **(4**, chapter I**)**.)

In the non-commutative case the radical P of aR in B has many of the properties of the radical in the commutative case. These are: (1) if $x \subseteq P$ then $x^n \in aR$. (2) P is maximal in B. (3) $P = \{aR|b\}_l$ by Statement 2.1. (4) If $xd \in aR$, $x \in B$, $d \notin aR$ then $x \in P$.

If Rc is the unique maximal left ideal containing Ra then c has many of the properties of the element which generates the radical in the commutative case. These are: (1) Rc and cR are maximal. (2) c is irreducible. (3) c is right factor of a. (4) $cR = \{aR|b\}_r$ by Statement 1.1.

We shall now consider the case where a is irreducible. Since aR is a maximal right ideal certainly aR is $\cap$ irreducible. By the corollary of **(3**, p. 26**)** we have that B/aR is a division ring where B is the normalizer of aR. Hence in this case $aR = P$, the radical of aR in B.

4. Intersection irreducible two-sided ideals. Suppose $a^*R = Ra^*$ is a two-sided ideal of R which is right $\cap$ irreducible. The normalizer B of a^*R is R. Hence the radical P of a^*R in R is a two-sided ideal of R and R/P is a division ring. Therefore, the unique maximal left ideal Rc^* which contains a^*R is equal to P and is a two-sided ideal. Thus $Rc^* = c^*R = P$. Certainly

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c^*R is a unique maximal right ideal containing a^*R. Hence by Theorem 1.1 a^*R is also left $\cap$ irreducible. We have proved

THEOREM 4.1. *If $a^*R = Ra^*$ is right $\cap$ irreducible then a^*R is left irreducible. In addition if P is the radical of a^*R in R then R/P is a division ring and $P = c^*R = Rc^*$ where c^* is irreducible.*

STATEMENT 4.1. *If $p^*R = Rp^*$ where p^* is irreducible then $(p^*)^nR = R(p^*)^n$ is right $\cap$ irreducible for all integers n.*

Proof. If $n = 1$ this is obvious. For $n > 1$ then $(p^*)^nR \subset p^*R$ and assume $(p^*)^nR \subset Rc$ where c is irreducible and $p^*R \neq Rc$. Then $(p^*)^k \in Rc$ for some integer k. If $k = 1$ then $p^*R = Rc$ — contradiction. If $k > 1$ and $(p^*)^{k-1} \notin Rc$, then $R = Rc + R(p^*)^{k-1}$ and thus $1 = d_1c + d_2(p^*)^{k-1}$. Multiplying on the left by p^* we have $p^* = p^*\, d_1c + p^*d_2(p^*)^{k-1}$. But $p^*d_2 = d_3p^*$ and $(p^*)^k = d_4c$. Then $p^* = p^*d_1c + d_3d_4c$. Hence $p^* \in Rc$ and $p^*R = Rc$—a contradiction. Since the assumption that $p^*R \neq Rc$ always leads to a contradiction, we can only conclude that $p^*R = Rc$. Thus $(p^*)^nR$ is contained in a unique maximal left ideal and is, therefore, right $\cap$ irreducible.

From Theorem 2.1 of **(1)** it follows that $a^*R = Ra^*$ is primary in the sense that $cd \in a^*R$ implies $c^n \in a^*R$ if $d \notin a^*R$ and $d^n \in a^*R$ if $c \notin a^*R$.

THEOREM 4.2. *A two-sided ideal $a^*R = Ra^*$ is right $\cap$ irreducible if and only if $a^* = u(p^*)^n$ for some integer n, where u is a unit of R, $p^*R = Rp^*$, and p^* is irreducible.*

Proof. Suppose a^*R is right $\cap$ irreducible. Then $a^*R \subset c^*R$ where c^* is irreducible and $(c^*)^n \in a^*R$ for some integer n. If $a^* = bc^*$ and if $c^* \notin a^*R$ then $b \in c^*R$ and $b = dc^*$, $a^* = d(c^*)^2$. This process continues until $a = k(c^*)^n = ea$. Then $a = kea$ and k is a unit. The converse follows from Statement 4.1.

5. An application. If R is real closed field then a polynomial $f(x)$ in $R(\sqrt{-1})[x]$ is $\cap$ irreducible if and only if $f(x) = u(x - r)^n$ where u is a unit and $r \in R(\sqrt{1-})$. Thus in this case all the roots of $f(x)$ are equal.

Let Q be the ring of quaternions over a real closed field. From the discussion given in **(2**, p. 36**)** a polynomial $f(x)$ in$Q[x]$, where $ax = xa$ for $a \in Q$, is irreducible if and only if it is linear. In addition r is a right-hand root of $f(x)$ if and only if $f(x) = q(x)\ (x - r)$.

THEOREM 5.1. A polynomial $f(x) \in Q[x]$, where Q denotes the ring of quaternions over a real closed field, is right $\cap$ irreducible if and only if all its right-hand roots are equal.

Proof. Let r_1 and r_2 be right-hand roots of $f(x)$ which is right $\cap$ irreducible. Then $f(x) = q_1(x)\ (x - r_1) = q_2(x)\ (x - r_2)$. From Theorem 1.1 then $u(x - r_1) = (x - r_2)$ where u is a unit. Hence $r_1 = r_2$.

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Conversely suppose all the right-hand roots of $f(x)$ are equal and let $f(x) = q_1(x)\ (ax + b) = q_2(x)\ (cx + d)$. Then $a^{-1}b = c^{-1}d$. Hence $ca^{-1}(ax + b) = (cx + d)$ and by Theorem 1.1 we have $f(x)$ is right $\cap$ irreducible.

If the coefficients of $f(x)$ are in the centre of Q and $f(x)$ is right $\cap$ irreducible we conclude from this section and §4 that $f(x) = u(x - r)^n$ where u is a unit and $r \in Q$.

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