A Multiplier Theorem on Anisotropic Hardy Spaces

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Abstract. We present a multiplier theorem on anisotropic Hardy spaces. When m satisfies the anisotropic, pointwise Mihlin condition, we obtain boundedness of the multiplier operator $T_m \colon H_A^p(\mathbb{R}^n) \to H_A^p(\mathbb{R}^n)$, for the range of p that depends on the eccentricities of the dilation A and the level of regularity of a multiplier symbol m. This extends the classical multiplier theorem of Taibleson and Weiss.

1 Introduction

We present a multiplier theorem (Theorem 1.2) on anisotropic Hardy space $H_A^p(\mathbb{R}^n)$. This space was first studied by Bownik [4] and generalizes the classical Hardy space of Fefferman and Stein [13] as well as the parabolic Hardy spaces of Calderón and Torchinsky [10] with a geometry and quasinorm induced by an expansive matrix A. Since the introduction of H_A^p, the anisotropic structure has been extended to a number of settings: Besov [6] and Triebel–Lizorkin spaces [7], weighted anisotropic Hardy spaces [8], variable Hardy–Lorentz spaces [1], and pointwise variable anisotropy [11], to name just a few. However, the study of the Fourier transform on these further generalizations is still incomplete, given that analysis of the Fourier transform becomes substantially harder.

To state our multiplier theorem, we require a few definitions; more details are in Section 2. Let A be an $n \times n$ matrix, and $|\det A| = b$. We say that A is a *dilation matrix* if all eigenvalues λ of A satisfy $|\lambda| > 1$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, ordered by their norm from smallest to largest, then define λ_- and λ_+ to satisfy $1 < \lambda_- < |\lambda_1|$ and $|\lambda_n| < \lambda_+$. Associated with A is a sequence of nested ellipsoids $\{B_j\}_{j \in \mathbb{Z}}$ such that $B_{j+1} = A(B_j)$ and $|B_0| = 1$. If A^* is the adjoint of A, then A^* is also a dilation matrix with the same determinant b and eigenvalues, with its own nested ellipsoids $\{B_k^*\}_{k \in \mathbb{Z}}$.

We use $\hat{f}$ and $\check{f}$ to denote the Fourier and inverse Fourier transforms of f, respectively. We say a measurable function $m \in L^\infty$ is a multiplier on H_A^p if its associated multiplier operator, initially defined by $T_m f = (\hat{f} m)^{\vee}$ for $f \in L^2 \cap H_A^p$, is bounded $H_A^p \to H_A^p$. We reserve ξ for the independent variable in the frequency domain, and ∂_ξ denotes differentiation with respect to ξ. For a dilation matrix A, we define the dilation operator by $D_A f(x) = f(Ax)$. Henceforth, C will denote a general constant

that may depend on the dilation matrix A and any scalar parameters n, p, q, and may change from line to line, but is independent of $f \in H_A^p$. The regularity requirement of a multiplier m will be given by the following Mihlin condition.

Definition 1.1 Let A be a dilation matrix. Let $N \in \mathbb{N} \cup \{0\}$ and let $m \in C^N(\mathbb{R}^n \backslash \{0\})$. We say m satisfies the anisotropic Mihlin condition of order N if there exists a constant $C = C_N$ such that for all multi-indices β, $|\beta| \leq N$, all $j \in \mathbb{Z}$, and all $\xi \in B_{j+1}^* \backslash B_j^*$,

$$(1.1) \qquad |D_{A^*}^{-j} \partial_\xi^\beta D_{A^*}^j m(\xi)| \leq C.$$

We can now state our main result. For $r \in \mathbb{R}$, the (integer) floor of r is given by $\lfloor r \rfloor$.

Theorem 1.2 *Let A be a dilation matrix, $N \in \mathbb{N}$, and denote*

$$L = \Big(N \frac{\log \lambda_-}{\log b} - 1 \Big) \frac{\log b}{\log \lambda_+}.$$

If m satisfies the Mihlin condition of order N and T_m is the corresponding multiplier operator, then $T_m \colon H_A^p \to H_A^p$ is bounded provided p satisfies

$$(1.2) \qquad 0 \leq \frac{1}{p} - 1 < \lfloor L \rfloor \frac{(\log \lambda_-)^2}{\log b \log \lambda_+}.$$

Remark 1.3 We have implicitly fixed λ_- and λ_+, determining the eccentricities of our dilation matrix. However, we can always "tighten" the eccentricities by defining $\widetilde{\lambda}_-$ and $\widetilde{\lambda}_+$ so that $1 < \lambda_- < \widetilde{\lambda}_- < |\lambda_1| \leq \cdots \leq |\lambda_n| < \widetilde{\lambda}_+ < \lambda_+$. In the proof of Theorem 1.2, we will exploit this simple fact.

Remark 1.4 An instructive example for the dilation matrix is by setting $A = 2I_n$, so $\lambda_- = \lambda_+ = 2$ and $b = 2^n$. Then (1.2) is equivalent to $\frac{n}{N-1} < p \leq 1$, thus recovering the classical case.

As an essential class of singular integral operators, multiplier operators have been well studied for the classical Hardy space H^p and its various extensions. We briefly discuss four classical multiplier theorems that are related to Theorem 1.2.

First, our proof of Theorem 1.2 most closely resembles that of Peetre [16] in that if m satisfies a classical pointwise Mihlin condition with respect to the Euclidean norm (which condition (1.1) generalizes), then it is a multiplier on Triebel–Lizorkin and Besov–Lipschitz spaces. Second, this pointwise Mihlin condition is stronger than an integral Hörmander condition on m, used in Taibleson and Weiss [17] and paired with molecular decomposition of H^p to prove the boundedness of T_m. Third, this Hörmander condition is equivalent to a Herz norm condition on the inverse Fourier transforms of smooth truncations of the multiplier m, which Calderón and Torchinsky [10] used to prove the multiplier theorem in the parabolic setting. Lastly, Baernstein and Sawyer [2] further generalized this with a weaker Herz norm condition, generalizing the previous three multiplier theorems.

For our multiplier theorem, we will assume the (strongest) Mihlin condition on m in order to overcome the issues native to the anisotropic setting. This approach was first considered by Benyi and Bownik [3] in the study of symbols associated

with pseudo-differential operators. Our Theorem 1.2 is closely related to their result, though we require minimal regularity requirement on m, and we obtain a more precise range of exponents p for which multiplier operators are bounded in terms of eccentricity of the dilation A, as measured by $\log \lambda_- / \log \lambda_+$ and $\log_{\lambda_-} / \log b$. Ding and Lan [12] extended the multiplier theorems of [2] to the spaces $T : H_A^p \to L^p$, though with an additional requirement that the dilation A is symmetric.

The rest of the paper will be organized as follows. In Section 2, we give the background information on anisotropic Hardy spaces H_A^p. In Section 3, we give the lemmas needed for the proof of Theorem 1.2, from which the theorem follows immediately. In Section 4, we provide the proofs of the lemmas as well as the molecular decomposition of H_A^p.

2 Anisotropic Hardy Spaces and Multiplier Operators

We now introduce the anisotropic structure and the associated Hardy spaces. Given a dilation matrix A, we can find a (non-unique) homogeneous quasi-norm, that is, a measurable mapping $\rho_A : \mathbb{R}^n \to [0, \infty)$ with a doubling constant c satisfying:

$$
\begin{array}{lll}
\rho_A(x) = 0 & \text{exactly when} & x = 0, \\
\rho_A(Ax) = b\rho(x) & \text{for all} & x \in \mathbb{R}^n, \\
\rho_A(x+y) \le c(\rho_A(x) + \rho_A(y)) & \text{for all} & x, y \in \mathbb{R}^n.
\end{array}
$$

Note that $(\mathbb{R}^n, dx, \rho_A)$ is a space of homogeneous type (dx denotes the Lebesgue measure), and any two quasi-norms associated with A will give the same anisotropic structure. In the isotropic setting, the "basic" geometric object is the Euclidean ball $B(x, r)$, centered at $x \in \mathbb{R}^n$ with radius r. This has the nice property that whenever $r_1 < r_2$, we have $B(x, r_1) \subset B(x, r_2)$. But for a dilation matrix A, we do not expect $B(x, r) \subset A(B(x, r))$. Instead, one can construct "canonical" ellipsoids $\{B_k\}_{k \in \mathbb{Z}}$, associated with A such that for all k, $B_{k+1} = A(B_k)$, $B_k \subseteq B_{k+1}$, and $|B_k| = b^k$. These nested ellipsoids will serve as the basic geometric object in the anisotropic setting. Moreover, we can use the ellipsoids to define the canonical quasinorm associated with A as follows:

$$
\rho_A(x) = \begin{cases} b^j & \text{if } x \in B_{j+1} \backslash B_j, \\ 0 & \text{if } x = 0. \end{cases}
$$

By setting ω to be the smallest integer so that $2B_0 \subset A^\omega B_0 = B_\omega$, ρ_A is a quasi-norm with the doubling constant $c = b^\omega$. Once A is fixed, we will drop the subscript and ρ will always denote the step norm. The anisotropic quasi-norm is related to the Euclidean structure by the following lemma of Lemarie–Rieusset [14].

Lemma 2.1 *Suppose ρ_A is a homogeneous quasi-norm associated with dilation A. Then there is a constant c_A such that:*

$$
\begin{array}{ll}
(2.1) & \frac{1}{c_A} \rho_A(x)^{\zeta_-} \le |x| \le c_A \rho_A(x)^{\zeta_+} & \text{if } \rho_A(x) \ge 1, \\
& \frac{1}{c_A} \rho_A(x)^{\zeta_+} \le |x| \le c_A \rho_A(x)^{\zeta_-} & \text{if } \rho_A(x) < 1.
\end{array}
$$

where c_A depends only on the eccentricities of $A : \zeta_{\pm} = \frac{\ln \lambda_{\pm}}{\ln b}$.

Lastly, we observe that if A^* is the adjoint of A, then A^* is also a dilation matrix with its own (canonical) norm ρ_*, though A^* and A have the same eigenvalues and eccentricities.

We let $\mathcal{S}$ be the Schwartz space, and let $\mathcal{S}'$ be the space of tempered distributions. Suppose we fix $\varphi \in \mathcal{S}$ such that $\int \varphi \, dx \neq 0$. If $k \in \mathbb{Z}$, we denote the anisotropic dilation by $\varphi_k(x) = b^k \varphi(A^k x)$. Then the radial maximal function on $f \in \mathcal{S}'$ is given by

$$M_\varphi^0 f(x) = \sup_{k \in \mathbb{Z}} |f * \varphi_k(x)|.$$

The anisotropic Hardy space H_A^p consists of all tempered distributions $f \in \mathcal{S}'$ so that $M_\varphi f \in L^p$, with $\|f\|_{H_A^p} \simeq \|M_\varphi f\|_{L^p}$. Analogous to the isotropic setting, this definition is independent of the choice of φ and is equivalent to the grand maximal function formulation (see [4, Theorem 7.1]).

We now present the atomic and molecular decompositions of H_A^p, which greatly simplify the analysis of Hardy spaces. For a fixed dilation A, we say that (p, q, s) is an admissible triple if $p \in (0, 1]$, $1 \leq q \leq \infty$ with $p < q$, and $s \in \mathbb{N}$ satisfies

$$s \geq \left\lfloor \left(\frac{1}{p} - 1 \right) \frac{\ln b}{\ln \lambda_-} \right\rfloor.$$

For the rest of this article, (p, q, s) will always denote an admissible triple. A (p, q, s) atom is a function a supported on $x_0 + B_j$ for some $x_0 \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, satisfying size condition $\|a\|_q \leq |B_j|^{1/q - 1/p}$, and vanishing moments condition: For $|\alpha| \leq s$,

$$\int_{\mathbb{R}^n} a(x) x^\alpha dx = 0.$$

The following theorem is the atomic decomposition of H_A^p; see [4, Theorem 6.5].

Theorem 2.2   *Suppose $p \in (0, 1]$ and (p, q, s) is admissible. Then $f \in H_A^p(\mathbb{R}^n)$ if and only if*

$$f = \sum_i \lambda_i a_i,$$

for some sequence $(\lambda_i)_i \in \ell^p$ and (a_i) a sequence of (p, q, s) atoms. Moreover,

$$\|f\|_{H_A^p} \simeq \inf \left\{ \|(\lambda_i)\|_{\ell^p} : f = \sum_i \lambda_i a_i \right\},$$

where the infimum is taken over all possible atomic decompositions.

We can also decompose $f \in H_A^p$ with molecules, which generalize the notion of atoms.

Definition 2.3   Let (p, q, s) be admissible, and fix d satisfying

$$(2.2) \qquad\qquad d > s \frac{\ln \lambda_+}{\ln b} + 1 - \frac{1}{q},$$

and define $\theta = (\frac{1}{p} - \frac{1}{q})/d$. Then we say a function M is a (p, q, s, d) molecule centered at $x_0 \in \mathbb{R}^n$ if it satisfies the following size and vanishing moments conditions:

(i) $N(M) = \|M\|_q^{1-\theta} \|\rho(x - x_0)^d M\|_q^\theta < \infty,$

(ii) $\displaystyle\int x^\beta M(x)dx = 0$ for all $|\beta| \le s$.

The quantity $N(M)$ is the molecular norm of M. We say the quadruple (p, q, s, d) is admissible if the triple (p, q, s) is an admissible triple and d satisfies (2.2). If we say that M is a molecule, then it implicitly has an admissible quadruple. A straightforward computation shows that if a is an atom, then $N(a) \le C$, where C is a uniform constant.

The following theorem gives the molecular decomposition of Hardy spaces. It is not new, since the crucial ideas are implicit in [4, Lemma 9.3], though our definition of molecules is more general than what is used there. For completeness, we will include the proof in the last section.

Theorem 2.4   *Every molecule M is in H_A^p and satisfies*

$$(2.3) \qquad\qquad \|M\|_{H_A^p} \le CN(M),$$

where $C = C(A, p, q, s, d)$. Moreover, $f \in H_A^p(\mathbb{R}^n)$ if and only if there exist (p, q, s, d) molecules $\{M_j\}_j$ such that $f = \sum_j M_j$ in $\mathcal{S}'$, and $\sum_j N(M_j)^p < \infty$. In this case, we have

$$\|f\|_{H_A^p}^p \le C \sum_j N(M_j)^p.$$

3   Proof of the Multiplier Theorem 1.2

In proving that the multiplier operator is bounded on H_A^p, we will follow this outline.

(a) Show that our multiplier operator is a convolution operator of a certain regularity. This is the key result of this paper, given by Lemma 3.2.
(b) As is often the case with Hardy spaces, we show it suffices to verify the action of operators on atoms. As we will see in Lemma 3.3, we only need to consider (p, ∞, s) atoms.
(c) Lastly, by Lemma 3.4, we show that the action of this operator on atoms will produce molecules whose (molecular) norms are uniformly bounded. By Theorem 2.4, this completes the proof of Theorem 1.2.

In this section, we state these lemmas, and provide a proof of Theorem 1.2 (which follows immediately). The proofs of these lemmas are in the next section.

We start by generalizing the notion of regularity to the anisotropic setting, taken from [4].

Definition 3.1   Let (p, q, s) be admissible, let $R \in \mathbb{N}$ satisfy

$$(3.1) \qquad\qquad R > \max\left\{ \left(\frac{1}{p} - 1\right)\frac{\log b}{\log \lambda_-}, s\frac{\log \lambda_+}{\log \lambda_-} \right\},$$

and let $K \in C^R(\mathbb{R}^n \backslash \{0\})$. We say K is a *Calderón–Zygmund convolution kernel of order R* if there exists a constant C such that for all multi-indices α with $|\alpha| \le R$, and

all $k \in Z$, $x \in B_{k+1} \backslash B_k$,

$$(3.2) \qquad \left| D_A^{-k} \partial_x^\alpha D_A^k K(x) \right| \le \frac{C}{\rho(x)}.$$

If K is such a kernel, we say K satisfies CZC-R and its associated singular integral operator T is defined by $Tf = K * f$.

The following lemma is our key result.

Lemma 3.2   *Let $N \in \mathbb{N}$ and $m \in L_{loc}^1(\mathbb{R}^n \backslash \{0\})$. Suppose m satisfies the Mihlin condition* (1.1) *of order N, and define K by $K = \check{m}$. Then K is a Calderón–Zygmund convolution kernel of order R provided $R \in \mathbb{N}$ and satisfies*

$$(3.3) \qquad 0 \le R < \Big(N \frac{\ln \lambda_-}{\ln b} - 1 \Big) \frac{\ln b}{\ln \lambda_+}.$$

The general method in proving that an operator $T: H_A^p \to H_A^p$ is bounded is to show that T is uniformly bounded on all (p, q, s) atoms, that is, $\|Ta\|_{H_A^p} \le C$ where a is a (p, q, s) atom. However, as we see in [5], in general it is not sufficient to deal with (p, ∞, s) atoms, though by the work of Meda et al [15], it suffices if $q < \infty$. This suggests that we simply need to show that our operator satisfies $\|Ta\|_{H_A^p} < \infty$ for $(p, 2, s)$ atoms.

However, this approach will not work for us, because of the following complication. Observe that we have the inclusions of the subspaces

$$H_{fin}^{p,2} \subseteq L^2 \cap H_A^p \subseteq H_A^p.$$

Suppose we use the approach outlined above, and after verifying $\|Ta\|_{H_A^p} \le C$ for all $(p, 2, s)$ atoms, we can then extend $T: H_{fin}^{p,2} \to H_A^p$ to the unique bounded extension $\widetilde{T}: H_A^p \to H_A^p$. Next, consider the operator T_m on the (middle) subspace $L^2 \cap H_A^p$, which we initially defined by $T_m f = (m\hat{f})\check{\,}$. It is not clear that the extension $\widetilde{T}$ will agree with T_m on $(L^2 \cap H_A^p) \backslash H_{fin}^{p,2}$. Because of this uncertainty, we cannot conclude that $\widetilde{T}$ is indeed the extension of T_m on H_A^p.

Fortunately, for multiplier operators, we have another approach, aided by a regularity result of [9, Theorem 1]. This approach also shows that it suffices, at least in our case, to verify uniform boundedness of (p, ∞, s) atoms.

Lemma 3.3   *Suppose that (p, ∞, s) is an admissible triple, $m \in L^\infty$ and T_m is the associated multiplier operator initially defined on $L^2 \cap H_A^p$. Then T_m has a a unique, bounded extension $\widetilde{T}_m: H_A^p \to H_A^p$ if for all (p, ∞, s) atoms, $\|T_m a\|_{H_A^p} \le C$, where C is independent of the atom a.*

This last lemma (and the regularity condition (3.2)) first appeared in [4, Theorem 9.8] for the more general Calderón–Zygmund operators. We give an alternate proof using Theorem 2.4.

Lemma 3.4   *Let $R \in \mathbb{N}$. Suppose T is a singular integral operator whose kernel K is a Calderón–Zygmund convolution kernel of order R. Then $T: H_A^p(\mathbb{R}^n) \to H_A^p(\mathbb{R}^n)$ is*

bounded provided p satisfies

$$(3.4) \qquad\qquad 0 < \frac{1}{p} - 1 < R\Big(\frac{(\log \lambda_-)^2}{\log b \log \lambda_+} \Big).$$

Now that all the pieces are here, we can prove Theorem 1.2.

Proof of Theorem 1.2 Suppose m is a multiplier satisfying the Mihlin condition (1.1) of order N, and $L \notin \mathbb{N}$. Then by Lemma 3.2, we have a kernel K of order R, satisfying (3.3) such that $\hat{K} = m$. Then by Lemma 3.4, the operator $Tf = K * f$ satisfies the bound $\|Ta\|_{H^p} \leq C$ for all (p, ∞, s) atoms, which by Lemma 3.3, gives a unique extension $\widetilde{T} \colon H_A^{p_A} \to H_A^p$, provided p is in the range (3.4), which implies the range given in Theorem 1.2.

However, if $L \in \mathbb{N}$, then $\lceil L - 1 \rceil \leq \lfloor L \rfloor$. To make the above argument hold, recall Remark 1.3, and let $\widetilde{\lambda}_-$ and $\widetilde{\lambda}_+$ be defined such that

$$1 < \lambda_- < \widetilde{\lambda}_- < |\lambda_1| \leq \cdots \leq |\lambda_n| < \widetilde{\lambda}_+ < \lambda_+,$$

so that the new $\widetilde{L}$, defined in terms of the new eccentricities, is slightly larger and no longer an integer. However, $\lfloor \widetilde{L} \rfloor = \lfloor L \rfloor$, and we can repeat the above argument and obtain the bound (1.2). ∎

4 Proofs of Lemmas and the Molecular Decomposition

In this section, we give the proofs of Lemma 3.2–3.4, as well as the proof of Theorem 2.4. Lemma 3.2 is the key result of this paper. Lemma 3.4 and Theorem 2.4 originally appear in [4], and we reprove them here with our notion of molecules.

4.1 Proof of Lemmas

Proof of Lemma 3.2 Let m satisfy the Mihlin condition of order N and let R satisfy (3.3). Fix $\Psi \in S(\mathbb{R}^n)$ such that $\hat{\Psi}$ is supported on $B_1^* \backslash B_{-1}^*$, and for all $\xi \neq 0$,

$$\sum_{j \in \mathbb{Z}} \hat{\Psi}(A^{-j}\xi) = 1.$$

By setting $\Psi_j(x) = b^j \Psi(A^j x)$, we have the identity

$$\widehat{\Psi}_j(\xi) = D_{A^*}^{-j} \hat{\Psi}(\xi) = \hat{\Psi}\big((A^*)^{-j})\xi \big),$$

and $\widehat{\Psi}_j$ is supported on $B_{j+1}^* \backslash B_{j-1}^*$. We define $m_j = m\widehat{\Psi}_j$, which is supported on $B_{j+1}^* \backslash B_{j-1}^*$, and define $K_j = (m_j)^\vee$. Then we have

$$m = \sum_{j \in \mathbb{Z}} m_j \qquad \text{holds pointwise and in } S', \text{ and} \qquad K = \sum_{j \in \mathbb{Z}} K_j \text{ in } S'.$$

We will see that the equality for K also holds pointwise. We make the following reductions to prove the CZC-R condition (3.2). First, it suffices to show that for all multi-index β such that $|\beta| \leq R$, $k \in \mathbb{Z}$, and $x \in B_1 \backslash B_0$, $|\partial_x^\alpha D_A^k K(x)| \leq C/b^k$, which follows from the absolute convergence

$$\sum_{j \in \mathbb{Z}} \big| \partial_x^\beta D_A^k K_j(x) \big| \leq \frac{C}{b^k}.$$

To prove this, it suffices to prove the above convergence for $k = 0$:

$$\text{(4.1)} \qquad \sum_{j \in \mathbb{Z}} \left| \partial_x^\beta K_j(x) \right| \le C.$$

Indeed, suppose (4.1) holds. Then if $k \in \mathbb{Z}$, and m has the Mihlin property, then so does $D_{A^*}^k m$, with the same constant C. Therefore, if $\xi \in B_{j+1}^* \backslash B_j^*$, then $(A^*)^k \xi \in B_{j+k+1}^* \backslash B_{j+k}^*$, so

$$\left| (D_{A^*}^{-j} \partial_\xi^\beta D_{A^*}^j)(D_{A^*}^k m)(\xi) \right| = \left| (D_{A^*}^{-j-k} \partial_\xi^\beta D_{A^*}^{j+k} m((A^*)^k \xi) \right| \le C_\beta.$$

To prove (4.1), we decompose the sum using a well-chosen integer M. Denote $\lambda_{\max}^*$ as the eigenvalue of A^* with the largest norm and $\| \cdot \|_{\text{op}}$ as the operator norm on $\mathbb{R}^n \to \mathbb{R}^n$. By the spectral theorem,

$$\lambda_{\max}^* = \limsup_{j \to \infty} \| A^{*j} \|_{\text{op}}^{1/j}.$$

Let $\epsilon > 0$. Then there exists an integer $M > 0$ such that for all $j > M$,

$$\| A^{*j} \|_{\text{op}}^{1/j} \le (1 + \epsilon) \lambda_{\max}^* \le (1 + \epsilon) \lambda_+.$$

With this M, we write

$$\sum_{j \in \mathbb{Z}} \left| \partial_x^\beta K_j(x) \right| = \sum_{j \le M} \left| \partial_x^\beta K_j(x) \right| + \sum_{j > M} \left| \partial_x^\beta K_j(x) \right| = S_L + S_H.$$

We call S_L and S_H the low and high spatial terms, respectively. Starting with the high spatial terms, we fix $j > M$ and $x \in B_{-1} \backslash B_0$. Then we can fix another multi-index α satisfying $|\alpha| = N$ such that there exists a constant c depending only on n such that $|(A^j x)^\alpha| \ge c |A^j x|^N$. This can be done by picking $\alpha = N e_i$ where e_i is the i-th unit vector in the canonical basis of $\mathbb{R}^n$ and the direction i is where $A^j x$ has the largest value in norm. Define $w(u) = (A^{*j} u)^\beta m(A^{*j} u) \hat{\Psi}(u)$. Using Parseval's identity, integration by parts, and a change of variables, we have

$$\partial_x^\beta K_j(x) = c_\beta b^j \int_{B_1^* \backslash B_{-1}^*} (\partial_u^\alpha w)(u) \frac{e^{2\pi i \langle A^j x, u \rangle}}{(2\pi i A^j x)^\alpha} \, du,$$

which we estimate using the bound from the spectral theorem.

Then the product rule gives

$$\text{(4.2)} \qquad (\partial^\alpha w)(u) = \sum_{\gamma \le \alpha} \binom{\alpha}{\gamma} \underbrace{\partial^{\alpha - \gamma} (D_{A^*}^j m \cdot \hat{\Psi})(u)}_{I_1} \cdot \underbrace{\partial^\gamma ((A^{*j} u)^\gamma)}_{I_2}.$$

By another application of the product rule, we have a uniform constant c', independent of m, j, u such that

$$I_1 \le c \sup_{\delta \le \gamma} \left| \partial^\delta D_{A^*}^j m)(u) \right| = c \sup_{\delta \le \gamma} \left| (D_{A^*}^{-j} \partial^\delta D_{A^*}^j m)(A^{*j} u) \right| \le c'.$$

We now bound I_2. With $u \in B_1^* \backslash B_{-1}^*$, elementary considerations from expressing $(A^{*j} u)^\beta$ as a sum of monomials show that there exists c depending only on N, such that by our choice of M and $j > M$,

$$I_2 = \left| \partial^\gamma (A^{*j} u)^\beta \right| \le c \| A^{*j} \|_{\text{op}}^{|\beta|} \le c \left(\lambda_+^* (1 + \epsilon) \right)^{j|\beta|},$$

Combining our estimates of I_1 and I_2 in (4.2), we have a constant C, depending on the past constants, such that

$$\left| (\partial^\alpha w)(u) \right| \le C \left(\lambda_+^*(1+\epsilon) \right)^{j|\beta|}.$$

Then we have

$$\left| \partial^\beta K_j(x) \right| \le b^j \int_{B_1^* \setminus B_{-1}^*} \left| \frac{(\partial^\alpha w)(u)}{(2\pi i A^j x)^\alpha} \right| du \le C \left(\frac{b^j(\lambda_+^*(1+\epsilon))^{j|\beta|}}{|A^j x|^{|\alpha|}} \right)$$
$$\le C \left(\frac{b^j(\lambda_+^*(1+\epsilon))^{j|\beta|}}{b^{j|\alpha|\zeta_-}} \right).$$

Note that with our choice of α and (2.1), we can sum $|\partial^\beta K_j(x)|$ for $j > M$ if

$$\frac{b(\lambda_+^*(1+\epsilon))^{|\beta|}}{b^{|\alpha|\zeta_-}} < 1, \qquad \text{that is,} \qquad |\beta| < \left(\frac{N \log \lambda_-}{\log b} - 1 \right) \frac{\log b}{\log(\lambda_+^*(1+\epsilon))}.$$

Indeed, for $|\beta| \le R$, there exists $\epsilon > 0$ such that the series below converges. For C_1 depending only on A, n, Ψ, β, M, we have

$$\sum_{j=M+1}^\infty |\partial^\beta K_j(x)| \le C \sum_{j=M+1}^\infty \left(\frac{b(\lambda_+^*(1+\epsilon))^{|\beta|}}{b^{|\alpha|\zeta_-}} \right)^j \le C_1.$$

Turning our attention to S_L, we start with Parseval's identity and a change of variables. With C a dimensional constant, we have

$$|\partial_x^\beta K_j(x)| = \left| \int_{B_{j+1}^* \setminus B_{j-1}^*} (2\pi i \xi)^\beta m_j(\xi) e^{2\pi i \langle x, \xi \rangle} d\xi \right|$$
$$\le cb^j \int_{B_1^* \setminus B_{-1}^*} \underbrace{|(A^{*j}u)^\beta|}_{J_1} \cdot \underbrace{|m_j(A^{*j}u)|}_{J_2} du \le \begin{cases} Cb^{j(1+|\beta|\zeta_+)} & \text{if } j \ge 0, \\ Cb^{j(1+|\beta|\zeta_-)} & \text{if } j < 0. \end{cases}$$

Indeed, for u in the unit annulus $B_1^* \setminus B_{-1}^*$, we have $A^{*j}u \in B_{j+1}^* \setminus B_{j-1}^*$, $J_1 \le c|A^{*j}u|^{|\beta|} \le Cb^{j\zeta_\pm}\rho_*(u)^{\zeta_\pm}$, with the eccentricity $\zeta_\pm$ depending on the sign of j. Since $m \in L^\infty$ and $J_2 \le C(m, \Psi)$, we obtain the above estimate. Returning to S_L, we have a constant C, depending only on n, A, N, Ψ, M such that

$$S_L \le \sum_{j=-\infty}^{-1} |\partial_x^\beta K_j(x)| + \sum_{j=0}^M |\partial_x^\beta K_j(x)| \le C \sum_{j=-\infty}^{-1} b^{j(1+|\beta|\zeta_-)} + C \sum_{j=0}^M b^{j(1+|\beta|\zeta_+)} \le C_2,$$

with $C_2 = C_2(n, A, N, \Psi, M)$. This completes the estimate (4.1) and this proof. ∎

For the proof of Lemma 3.3, we need the following result from [9].

Theorem 4.1 ([9, Theorem 1]) *Let $p \in (0,1]$. If $f \in H_A^p$, then $\hat{f}$ is a continuous function and satisfies*

$$|\hat{f}(\xi)| \le C\|f\|_{H_A^p} \rho_*(\xi)^{\frac{1}{p}-1}.$$

In particular, if $f = \sum_j \lambda_j a_j$, then $\hat{f}(\xi) = \sum_j \lambda_j \hat{a}_j(\xi)$ almost everywhere and in $\mathcal{S}'$.

Proof of Lemma 3.3 If a is a (p, ∞, s) atom, it is compactly supported, so that it is in $L^2 \cap H_A^p$, and $T_m a = (m\hat{a})^{\vee}$ is well defined. Now let $f \in L^2 \cap H_A^p$, with an infinite atomic decomposition $f = \sum_j \lambda_j a_j$ using (p, ∞, s) atoms. We first establish T_m can pass through the infinite sum:

$$(4.3) \qquad T_m f = T_m \Big(\sum_j \lambda_j a_j \Big) = \sum_j \lambda_j T_m a_j.$$

Observe that passing the operator through the infinite sum is the main issue raised by [5] and the rationale as to why the result of [15] is needed for a general sublinear operator. In our case, where our operator is a multiplier, we show that we can do this directly. If we denote the right-hand term above by $g = \sum_j \lambda_j T_m a_j$, then (4.3) holds if we can show $(T_m f)^{\wedge} = m\hat{f} = \hat{g}$ in $\mathcal{S}'$. To show (4.3), we note that by Lemma 4.1, for $\xi \in \mathbb{R}^n$ almost everywhere, we have $(m\hat{f})(\xi) = \sum_j \lambda_j m(\xi)\hat{a}_j(\xi)$. Then

$$\hat{g} = \Big(\sum_j \lambda_j T_m a_j \Big)^{\wedge} = \sum_j \lambda_j (T_m a_j)^{\wedge} = \sum_j \lambda_j (m\hat{a}_j) = m \sum_j \lambda_j \hat{a}_j = m\hat{f}.$$

Since $\hat{g} = m\hat{f} = (T_m f)^{\wedge}$ in $\mathcal{S}'$ and pointwise, we must also have that $g = T_m f$, thus establishing equality (4.3). The boundedness of $T_m \colon L^2 \cap H_A^p \to H_A^p$ follows immediately:

$$\|T_m f\|_{H_A^p}^p = \Big\| \sum_j \lambda_j T(a_j) \Big\|_{H_A^p}^p \le \sum_j |\lambda_j|^p \|T(a_j)\|_{H_A^p}^p \le C \sum_j |\lambda_j|^p.$$

Taking the infimum over all possible atomic decompositions, we have $\|T_m f\|_{H_A^p} \le C\|f\|_{H_A^p}$. Lastly, with $L^2 \cap H_A^p$ a dense subset of H_A^p, there exists a unique bounded extension $\widetilde{T}_m \colon H_A^p \to H_A^p$ such that $\widetilde{T}_m = T_m$ on $L^2 \cap H_A^p$. ∎

Proof of Lemma 3.4 Let p satisfy (3.4) and let (p, q, s) be an admissible triple. Let a be a (p, q, s) atom supported on the ellipsoid $x_0 + B_r$ for some $x_0 \in \mathbb{R}^n$ and $r \in \mathbb{Z}$. The boundedness of T follows once we establish the uniform bound of the molecular norm $N(Ta) = \|Ta\|_q^{1-\theta} \|\rho(x-x_0)^d Ta(x)\|_q^{\theta} \le C$. Note that since $(\mathbb{R}^n, dx, \rho)$ is a space of homogeneous type, T is bounded from L^q to L^q for $q > 1$. There is a C, depending only on T, q, and θ, such that

$$\|Ta\|_q^{1-\theta} \le C\|a\|_q^{1-\theta} \le Cb^{r(\frac{1}{q}-\frac{1}{p})(1-\theta)}.$$

By Minkowski's inequality,

$$\|\rho(x-x_0)^d Ta(x)\|_q \le \Big(\int_{x_0 + B_{r+2\omega}} |\rho(x-x_0)^{dq} Ta(x)|^q dx \Big)^{1/q}$$
$$+ \Big(\int_{(x_0 + B_{r+2\omega})^c} |\rho(x-x_0)^{dq} Ta(x)|^q dx \Big)^{1/q} = I_1 + I_2.$$

The estimate for I_1 is immediate:

$$I_1 \le b^{d(r+2\omega)} \Big(\int_{x_0 + B_{r+2\omega}} |Ta(x)|^q dx \Big)^{1/q} \le b^{d(r+2\omega)} \|Ta\|_q$$
$$\le Cb^{dr} b^{r(\frac{1}{q}-\frac{1}{p})} = Cb^{r(d+\frac{1}{q}-\frac{1}{p})}.$$

To estimate I_2, we require the following pointwise estimate from [4, Lemma 9.5]. Suppose T is a singular integral operator whose kernel k is CZC-R, with R satisfying (3.1).

Then there exists a constant C such that for every (p, q, s) atom a with support $x_0 + B_r$, all $l \geq 0$ and $x \in x_0 + (B_{r+l+2\omega+1} \setminus B_{r+l+2\omega})$,

$$|Ta(x)| \leq Cb^{-lR\zeta_- - l}|B_r|^{-1/p}.$$

With this estimate, we have

$$I_2 = \sum_{j=0}^{\infty} \int_{x_o + (B_{r+2\omega+j+1} \setminus B_{r+2\omega+j})} \rho(x - x_o)^{dq}|Ta(x)|^q dx$$

$$\leq Cb^{-\frac{rq}{p}} \sum_{j=0}^{\infty} b^{-jq(1+R\zeta_-)} \Big(\int_{x_o + B_{r+2\omega+j+1} \setminus B_{r+2\omega+j}} \rho(x - x_o)^{dq} dx \Big)$$

$$= Cb^{-\frac{rq}{p}} \sum_{j=0}^{\infty} b^{-jq(1+R\zeta_-)} b^{(dq+1)(r+2\omega+j)} = Cb^{r(dq+1-\frac{q}{p})} \sum_{j=0}^{\infty} b^{j(dq+1-q(1+R\zeta_-))}.$$

The geometric series converges exactly when R satisfies (3.4). Taking the power θ/q on both sides, we have

$$\big\| \rho(x - x_o)^d Ta(x) \big\|_q^\theta \leq Cb^{r\theta(d+\frac{1}{q}-\frac{1}{p})}.$$

All together, we have $N(Ta) \leq Cb^{k(\frac{1}{q}-\frac{1}{p})(1-\theta)} b^{k\theta(d+\frac{1}{q}-\frac{1}{p})} = C$, as the exponent is exactly 0. ∎

4.2   Proof of Theorem 2.4

We need a few preliminary results on projections and molecules, which we state without proof as they are implicit in the proof of [4, Lemma 9.3]. To define the projections needed, recall that given a dilation A, $\{B_j\}_{j \in \mathbb{Z}}$ denotes the "canonical" ellipsoids so that for all $j \in \mathbb{Z}$, $A(B_j) = B_{j+1}$. We also define $\|f\|_{L^1(B)} = \int_B |f(x)| dx$.

Definition 4.2   Let $s \in \mathbb{N}$ and $\mathcal{B} = \{x + B_j : x \in \mathbb{R}^n, j \in \mathbb{Z}\}$. Define P_s to be the space of polynomials on $\mathbb{R}^n$ of degree at most s. If $B \in \mathcal{B}$, we define π_B as the natural projection defined by the Riesz Lemma:

$$\int_B \big(\pi_B f(x) \big) Q(x) dx = \int_B f(x) Q(x) dx, \qquad \text{for all } f \in L^1(B) \text{ and } Q \in P_s.$$

With these projections, we make some elementary observations. Let $Q = \{Q_\alpha\}_{|\alpha| \leq s}$ be an orthonormal basis of P_s in $L^2(B_0)$-norm, that is,

$$\langle Q_\alpha, Q_\beta \rangle = \int_{B_0} Q_\alpha(x) \overline{Q_\beta(x)} dx = \delta_{\alpha, \beta}.$$

Then the projection $\pi_{B_0} : L^1(B_0) \to P_s$ is given by

$$\pi_{B_0} f = \sum_{|\alpha| \leq s} \Big(\int_{B_0} f(x) \overline{Q_\alpha(x)} dx \Big) Q_\alpha.$$

Generally, if $j \in \mathbb{Z}$, then $\pi_{B_j} : L^1(B_j) \to P_s$ is given by

$$\pi_{B_j} f = (D_A^{-j} \pi_{B_0} D_A^j) f.$$

If $B = y + B_j$, then $\pi_B : L^1(B) \to P_s$ is given by $\pi_B f = (T_y \pi_{B_j} T_{-y}) f$, where $T_y f = f(x - y)$ is the translation operator, and there exists C_0, depending only on s and Q, such that for all $B \in \mathcal{B}$, given $x \in B$,

$$|\pi_B f(x)| \leq C_0 \int_B |f| \frac{dx}{|B|}.$$

Let $\widetilde{\pi}_B = \mathrm{Id} - \pi_B$ be the complementary projection. Then for all $B \in \mathcal{B}$, $\widetilde{\pi}_B : L^q(B) \to L^q(B)$ is bounded, with

$$\|\widetilde{\pi}_B(f)\|_{L^q(B)} \leq (1 + C_0) \|f\|_{L^q(B)}.$$

Furthermore, for all α with $|\alpha| \leq s$, we have $\int_B x^\alpha \cdot (\widetilde{\pi}_B f)(x) \, dx = 0$.

Lemma 4.3 *Let M be a (p, q, d) molecule centered at x_0.*
(i) *Then $\|\pi_j M\|_{L^1(B_j)} \to 0$ as $j \to \infty$.*
(ii) *Define $g_j = (\widetilde{\pi}_{B_j} M) \mathbf{1}_{B_j} = (M - \pi_{B_j} M) \mathbf{1}_{B_j}$. Then $g_j \to M$ in L^1 as $j \to \infty$.*

Proof of Theorem 2.4 We first prove estimate (2.3). Let M be a (p, q, d) molecule. Without loss of generality, we assume that $N(M) = \|M\|_q^{1-\theta} \|M(u)\rho(u)^d\|_q^\theta = 1$. Define the quantity σ by $\|M\|_q = \sigma^{1/q - 1/p}$ and choose $k \in \mathbb{Z}$ such that $b^k \leq \sigma < b^{k+1}$. From Lemma 4.3, we have the following expression for M, with convergence in L^1:

$$M = g_k + \sum_{j=k}^{\infty} (g_{j+1} - g_j).$$

Note that for each j, g_j has vanishing moments of order up to s, and has compact support. We will decompose M by setting $g_k = \mu_k a_k$ and $g_{j+1} - g_j = \mu_j a_j$, where $(\mu_j)_{j=k}^{\infty} \in \ell^p$ has a uniform norm independent of M and $(a_j)_{j=k}^{\infty}$ is a sequence of (p, q, s) atoms. We start with $g_k = (M - \pi_k M) \mathbf{1}_{B_k}$. With C_0 as in Definition 4.2, we have

$$\|g_k\|_{L^q(B_k)} \leq \|M\|_{L^q(B_k)} + \|\pi_k M\|_{L^q(B_k)} \leq (1 + C_0) \|M\|_{L^q(B_k)}.$$

Scaling the measure, we obtain

$$\|g_k\|_{L^q\left(\frac{x_{B_k}}{|B_k|} dx\right)} = \left(\int_{B_k} |g_k(x)|^q \frac{dx}{|B_k|} \right)^{1/q} \leq (1 + C_0) \|M\|_{L^q\left(\frac{dx}{|B_k|}\right)}.$$

Note that because $\frac{1}{q} - \frac{1}{p} < 0$, we have $\sigma \geq b^k \Rightarrow \sigma^{\frac{1}{q} - \frac{1}{p}} \leq b^{k\left(\frac{1}{q} - \frac{1}{p}\right)}$. Continuing our estimate using the definition of σ, we have

$$\|M\|_{L^q\left(\frac{dx}{|B_k|}\right)} = |B_k|^{-1/q} \|M\|_q = |B_k|^{-1/q} \sigma^{\frac{1}{q} - \frac{1}{p}} \leq |B_k|^{-\frac{1}{q}} |B_k|^{\frac{1}{q} - \frac{1}{p}} = |B_k|^{\frac{1}{p}}.$$

Therefore, we have $\|g_k\|_q \leq (1 + C_0) |B_k|^{\frac{1}{q} - \frac{1}{p}}$, which gives $g_k = \mu_k a_k$ where a_k is a (p, q, s) atom and $\mu_k = 1 + C_0$. For $j > k$, we have

$$g_{j+1} - g_j = \mathbf{1}_{B_{j+1} \backslash B_j} M - \pi_{B_{j+1}} M + \pi_{B_j} M.$$

Estimating the first term, we have

$$\|M\mathbf{1}_{B_{j+1}\setminus B_j}\|_{L^q\left(\frac{dx}{|B_{j+1}|}\right)} = |B_{j+1}|^{-\frac{1}{q}} b^{-jd} \Big(\int_{B_{j+1}\setminus B_j} |M(x)|^q \rho(x)^{dq} dx \Big)^{1/q}$$

$$\leq b^d |B_{j+1}|^{-\frac{1}{q}} |B_{j+1}|^{-d} \|M(x)\rho(x)^d\|_q = b^d |B_{j+1}|^{-\frac{1}{q}-d} \|M\|_q^{\frac{\theta-1}{\theta}}$$

$$\leq b^d |B_{j+1}|^{-\frac{1}{p}} b^{(k-j)d(1-\theta)}.$$

Setting $r = d(1-\theta) > 0$, we obtain the estimate $\|M\mathbf{1}_{B_{j+1}\setminus B_j}\|_q \leq b^d |B_{j+1}|^{\frac{1}{q}-\frac{1}{p}} b^{(j-k)(-r)}$. Next, we estimate $\pi_{B_j}M$ with Minkowski's inequality:

$$\|\pi_{B_j}M\|_{L^q(B_j)} = \Big\| \sum_{|\alpha|\leq s} \Big(\int_{B_j} M(u)\overline{Q_\alpha(A^{-j}u)}\frac{du}{b^j} \Big) Q_\alpha(A^{-j}x) \Big\|_{L^q(B_j)}$$

$$\leq \sum_{|\alpha|\leq s} b^{-j} \Big| \int_{B_j} M(u)\overline{Q_\alpha(A^{-j}u)}du \Big| \|D_A^{-j}Q_\alpha\|_{L^q(B_j)}.$$

Let $C(Q)$ a uniform bound for $\|Q_\alpha\|_{L^q(B_0)}$. By a change of variables, we have

$$\|D_A^{-j}Q_\alpha\|_{L^q(B_j)} = b^{\frac{j}{q}} \|Q_\alpha\|_{L^q(B_0)} \leq C(Q) b^{\frac{j}{q}}.$$

Next, since M has vanishing moments and $\frac{1}{q} + \frac{1}{q'} = 1$,

$$\Big| \int_{B_j} M(u)\overline{Q_\alpha(A^{-j}u)}du \Big|$$

$$= \Big| \int_{B_j^c} M(u)\overline{Q_\alpha(A^{-j}u)}du \Big| \leq \int_{B_j^c} |M(u)||Q_\alpha(A^{-j}u)|du$$

$$\leq C(Q) \int_{B_j^c} |M(u)||A^{-j}u|^s du \leq C(Q)c_A \int_{B_j^c} |M(u)|\rho(A^{-j}u)^{s\zeta_+} du$$

$$\leq C(Q)c_A b^{-js\zeta_+} \Big(\int_{B_j^c} |M(u)|^q \rho(u)^{dq} du \Big)^{1/q} \Big(\int_{B_j^c} \rho(u)^{q'(s\zeta_+ - d)} du \Big)^{1/q'}.$$

The first integral in the last expression can be computed as follows:

$$\Big(\int_{B_j^c} |M(u)|^q \rho(u)^{dq} du \Big)^{1/q} \leq \|M(x)\rho(x)^d\|_q = \|M\|_q^{\frac{\theta-1}{\theta}} = \sigma^{(\frac{1}{q}-\frac{1}{p})(\frac{\theta-1}{\theta})}$$

$$= \sigma^{d(1-\theta)} \leq b^{kr}.$$

The second integral from Holder's inequality can be computed directly as a geometric series. With C a constant depending only on A, q, s, and d, and $d > s\zeta_+1 - \frac{1}{q}$, we have

$$\int_{B_j^c} \rho(u)^{q'(s\zeta_+ - d)} du = \sum_{m=j}^{\infty} \int_{B_{m+1}\setminus B_m} \rho(u)^{q'(s\zeta_+ - d)} du = C b^{j(1+q'(s\zeta_+ - d))}.$$

This gives

$$\Big| \int_{B_j} M(u)\overline{Q_\alpha(A^{-j}u)}du \Big| \leq C b^{-js\zeta_+} b^{kr} b^{j(1-\frac{1}{q}+s\zeta_+ - d)} = C b^{kr} b^{j(1-\frac{1}{q}-d)}.$$

Then we have the following estimate on $\|\pi_{B_j}M\|_{L^q}$:

$$\|\pi_{B_j}M\|_{L^q} \leq C b^{-j} b^{kr} b^{j(1-\frac{1}{q}-d)} b^{\frac{j}{q}} = C b^{-jd} b^{-k(-r)} = C b^{j(\frac{1}{q}-\frac{1}{p})} b^{(j-k)(-r)}.$$

Finally, returning to the estimate on $g_{j+1} - g_j$, we have

$$\|g_{j+1} - g_j\|_q \le \|M\mathbf{1}_{B_{j+1}\setminus B_j}\|_q + \|\pi_{B_{j+1}}M\|_q + \|\pi_{B_j}M\|_q \le C|B_{j+1}|^{\frac{1}{q}-\frac{1}{p}}b^{(j-k)(-r)}.$$

Therefore, if $j > k$, $g_{j+1} - g_j = \mu_j a_j$, with $\mu_j = Cb^{(j-k)(-a)}$ and where a_j is a (p, q, s) atom supported on B_{j+1}. Summing the coefficients, we have

$$\sum_{j=k}^{\infty} |\mu_j|^p = \mu_k + \sum_{j=1}^{\infty} C^p b^{-jrp} = (1 + C_0) + \frac{C}{1 - b^{-rp}}.$$

This establishes (2.3) with C depending only on A, p, q, s, d and the cube Q, and is independent of M.

Lastly, we prove the molecular decomposition. If $f \in H_A^p$, then its atomic decomposition $\sum_j \lambda_j a_j$ can be seen as a molecular decomposition with $M_j = \lambda_j a_j$. Then by (2.3), we have

$$\|f\|_{H_A^p}^p \le \sum_j \|\lambda_j a_j\|_{H_A^p}^p \le C \sum_j N(\lambda_j a_j)^p \le C' \sum_j \lambda_j^p < \infty,$$

where in the penultimate inequality, we used the fact that the molecular norm of atoms are uniformly bounded.

As for the converse, suppose $f \in S'$ has the molecular decomposition $f = \sum_j M_j$ with $\sum_j N(M_j)^p < \infty$. Then again by (2.3), we have

$$\|f\|_{H_A^p} = \|\sum_j M_j\|_{H_A^p}^p \le \sum_j \|M_j\|_{H_A^p}^p \le C \sum_j N(M_j)^p < \infty.$$

So $f \in H_A^p$, and this completes our proof. ∎

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