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L^p BOUNDS FOR MARCINKIEWICZ INTEGRALS

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Abstract In this paper the authors establish the L^p boundedness for several classes of Marcinkiewicz integral operators with kernels satisfying a condition introduced by Grafakos and Stefanov in *Indiana Univ. Math. J.* **47** (1998), 455–469.

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1. Introduction and results

Let $n \geqslant 2$ and S^{n-1} be the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $\mathrm{d}\sigma$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ (which is then naturally identified with a function on S^{n-1}) satisfying $\Omega \in L^1(S^{n-1})$ and

$$\int_{S^{n-1}} \Omega(y)\,\mathrm{d}\sigma(y) = 0. \tag{1.1}$$

For a suitable mapping $\Phi : \mathbb{R}^n \to \mathbb{R}^d$ we define the Marcinkiewicz integral operator $\mu_{\Phi,\Omega}$ along a mapping Φ on $\mathbb{R}^d$ by

$$\mu_{\Phi,\Omega}(f)(x) = \left(\int_0^\infty |F_{\Phi,t}(x)|^2 \frac{\mathrm{d}t}{t^3}\right)^{1/2},$$

where

$$F_{\Phi,t}(x) = \int_{|y|\leqslant t} \frac{\Omega(y)}{|y|^{n-1}} f(x - \Phi(y))\,\mathrm{d}y.$$

If $d = n$ and $\Phi(y) = (y_1, y_2, \ldots, y_n)$, we shall simply denote the operator $\mu_{\Phi,\Omega}$ by μ_Ω.

The study of the Marcinkiewicz integral operator μ_Ω began in Stein [**13**], where Ω was assumed to be in a certain Lipschitz class (see also [**2**]). In two recent papers [**5**,**6**], the L^p boundedness of the operators $\mu_{\Phi,\Omega}$ was established for Ω in the Hardy space $H^1(S^{n-1})$ and Φ in several classes of mappings.

The purpose of this paper is to investigate the L^p boundedness of the operators $\mu_{\Phi,\Omega}$ when $\Omega \in F_\alpha(S^{n-1})$, where for an $\alpha > 0$, $F_\alpha(S^{n-1})$ denotes the set of all Ω which are integrable over S^{n-1} and satisfy

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y)| \left(\log \frac{1}{|\langle \xi, y\rangle|}\right)^{1+\alpha} \mathrm{d}\sigma(y) < \infty. \tag{1.2}$$

Condition (1.2) was introduced by Grafakos and Stefanov in [**9**]. The examples in [**9**] show that there is the following relationship between $F_\alpha(S^{n-1})$ and $H^1(S^{n-1})$:

$$\bigcap_{\alpha>0} F_\alpha(S^{n-1}) \not\subset H^1(S^{n-1}) \not\subset \bigcup_{\alpha>0} F_\alpha(S^{n-1}).$$

It was proved in [**9**] that, under condition (1.2), the usual singular integral operator with the kernel $\Omega(y)|y|^{-n}$ is bounded on $L^p(\mathbb{R}^n)$ for

$$p \in \left(\frac{2+\alpha}{1+\alpha}, 2+\alpha\right).$$

The range of p was later enlarged to

$$\left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$$

in [**8**].

We shall state our main results as follows.

Theorem 1.1. *Let $d \in \mathbb{N}$ and $\mathcal{P}(y) = (P_1(y), \dots, P_d(y))$, where P_j is a real-valued polynomial on $\mathbb{R}^2$ for $1 \leqslant j \leqslant d$. If $\Omega \in F_\alpha(S^1)$ for some $\alpha > 0$, then $\mu_{\mathcal{P},\Omega}$ is bounded on $L^p(\mathbb{R}^d)$ for*

$$p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right).$$

Moreover, the bound on the operator norm is independent of the coefficients of the polynomials $\{P_j\}_{1 \leqslant j \leqslant d}$.

There is a similar result for $n \geqslant 3$ when the condition $\Omega \in F_\alpha$ is properly modified (see Theorem 4.1).

Singular integrals along surfaces of revolution have been studied quite extensively (see, for example, [**4**, **10**–**12**]). Theorems 1.2 and 1.3 deal with L^p bounds for corresponding Marcinkiewicz integrals.

Theorem 1.2. *Let $d = n + 1$ and $\Phi(y) = (y, \phi(|y|))$ be the surface of revolution generated by a function $\phi : [0, \infty) \to \mathbb{R}$. Suppose that $\phi \in C^1([0, \infty))$, ϕ' is convex and increasing, and $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 0$.*

(i) *If $n = 2$, then $\mu_{\Phi,\Omega}$ is bounded on $L^p(\mathbb{R}^3)$ for*

$$p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right).$$

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(ii) *If $n \geqslant 3$ and $\phi'(0) = 0$, then $\mu_{\Phi,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for*

$$p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right).$$

Theorem 1.3. *Let $d = n+1$ and $\Phi(y) = (y, \phi(|y|))$, where ϕ is a polynomial. In addition, let $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 0$.*

(i) *If $n = 2$, then $\mu_{\Phi,\Omega}$ is bounded on $L^p(\mathbb{R}^3)$ for*

$$p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right).$$

(ii) *If $n \geqslant 3$ and $\phi'(0) = 0$, then $\mu_{\Phi,\Omega}$ is bounded on $L^p(\mathbb{R}^{n+1})$ for*

$$p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right).$$

Moreover, in both (i) and (ii), the bounds on the operator norm are independent of the coefficients of ϕ.

Our method is based on a lemma presented in § 2. The proofs of our results can be found in §§ 3 and 4.

2. Main lemma

We shall begin by establishing some notation. For a family of measures $\tau = \{\tau_{k,t} : k \in \mathbb{N},\ t \in \mathbb{R}\}$ on $\mathbb{R}^d$, we define the operators Δ_τ and τ_k^* by

$$\Delta_\tau(f)(x) = \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}} |(\tau_{k,t} * f)(x)|^2 \,\mathrm{d}t\right)^{1/2} \quad \text{and} \quad \tau_k^*(f)(x) = \sup_{t\in\mathbb{R}}(|\tau_{k,t}| * |f|)(x).$$

Lemma 2.1. *Let $m \in \mathbb{N}$ and $L : \mathbb{R}^d \to \mathbb{R}^m$ be a linear transformation. Suppose that there are constants $C_0, C_p, \alpha, \gamma > 0$ such that the following hold for $k \in \mathbb{N}$, $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$:*

$$\|\tau_{k,t}\| \leqslant C_0 2^{-k}; \tag{2.1}$$

$$|\hat{\tau}_{k,t}(\xi)| \leqslant C_0 2^{-k} |2^{\gamma(t-k)} L\xi|; \tag{2.2}$$

$$|\hat{\tau}_{k,t}(\xi)| \leqslant C_0 2^{-k} (\log|2^{\gamma(t-k)} L\xi|)^{-(1+\alpha)}, \quad \text{if } |2^{\gamma(t-k)} L\xi| > 2; \tag{2.3}$$

$$\|\tau_k^*(f)\|_{L^p(\mathbb{R}^d)} \leqslant C_p 2^{-k} \|f\|_{L^p(\mathbb{R}^d)}, \quad \text{for } 1 < p < \infty. \tag{2.4}$$

Then, for

$$p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right),$$

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there exists a constant $A_p > 0$ such that

$$\|\Delta_\tau(f)\|_{L^p(\mathbb{R}^d)} \leqslant A_p\|f\|_{L^p(\mathbb{R}^d)} \tag{2.5}$$

for all $f \in L^p(\mathbb{R}^d)$. The constant A_p may depend on C_0, C_p, α, γ, d and m, but it is independent of the linear transformation L.

Proof. By an argument in [**7**] we may assume that $m \leqslant d$ and $L\xi = (\xi_1, \dots, \xi_m) = \xi'$ for $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. Choose a C^∞ function $\psi : \mathbb{R} \to [0, 1]$ such that $\operatorname{supp}(\psi) \subset [\frac{1}{4}, 4]$ and

$$\int_0^\infty \frac{\psi(r)}{r}\,\mathrm{d}r = 2. \tag{2.6}$$

Define the Schwartz functions $\Psi, \Psi_t : \mathbb{R}^m \to \mathbb{C}$ by

$$\hat{\Psi}(\xi_1, \dots, \xi_m) = \psi(\xi_1^2 + \cdots + \xi_m^2)$$

and $\Psi_t(u) = t^{-m}\Psi(u/t)$ for $t > 0$ and $u \in \mathbb{R}^m$. If we let δ_{d-m} represent the Dirac delta on $\mathbb{R}^{d-m}$, then by (2.6), for any Schwartz function f,

$$f(x) = \int_0^\infty (\Psi_t \otimes \delta_{d-m}) * f(x)\frac{\mathrm{d}t}{t} = (\gamma \log 2)\int_{\mathbb{R}} (\Psi_{2^{\gamma s}} \otimes \delta_{d-m}) * f(x)\,\mathrm{d}s. \tag{2.7}$$

Define the g-function $g(f)$ by

$$g(f)(x) = \left(\int_{\mathbb{R}} |(\Psi_{2^{\gamma s}} \otimes \delta_{d-m}) * f(x)|^2\,\mathrm{d}s\right)^{1/2}.$$

By $\int_{\mathbb{R}^m} \Psi_t(z)\,\mathrm{d}z = \psi(0) = 0$ and Littlewood–Paley theory, we have

$$\|g(f)\|_{L^p(\mathbb{R}^d)} \leqslant C\|f\|_{L^p(\mathbb{R}^d)}, \quad \text{for } 1 < p < \infty. \tag{2.8}$$

For $s \in \mathbb{R}$, $k \in \mathbb{N}$ and Schwartz function f, let

$$H_{s,k}(f)(x) = \left(\int_{\mathbb{R}} |(\Psi_{2^{\gamma(s+t)}} \otimes \delta_{d-m}) * \tau_{k,t} * f(x)|^2\,\mathrm{d}t\right)^{1/2} \tag{2.9}$$

and

$$H_s(f) = \sum_{k=1}^\infty H_{s,k}(f).$$

It follows from (2.7) and Minkowski's inequality that

$$\Delta_\tau(f)(x) \leqslant (\gamma \log 2)\int_{\mathbb{R}} H_s(f)(x)\,\mathrm{d}s. \tag{2.10}$$

Hence, if we can prove that, for

$$p \in \left(\frac{2 + 2\alpha}{1 + 2\alpha}, 2 + 2\alpha\right),$$

there exist $\theta_p > 0$ and $\theta_p' > 1$ such that

$$\|H_s\|_{p,p} \leqslant \begin{cases} C_p 2^{-s\theta_p}, & \text{for } s > 0, \\ C_p |s|^{-\theta_p'}, & \text{for } s < -N, \\ C_p, & \text{for } -N \leqslant s \leqslant 0, \end{cases} \tag{2.11}$$

where $N > 0$ depended only α and γ, then (2.5) follows from (2.10) and (2.11).

We shall first establish (2.11) for $p = 2$. When $s > 0$, by (2.2) we have

$$\begin{aligned} \int_{\mathbb{R}} |\psi(|2^{\gamma(s+t)}\xi'|^2)\hat{\tau}_{k,t}(\xi)|^2 \,\mathrm{d}t &\leqslant C 2^{-2k} \int_{(2^{\gamma s+1}|\xi'|)^{-1} \leqslant 2^{\gamma t} \leqslant 2(2^{\gamma s}|\xi'|)^{-1}} (2^{\gamma(t-k)}|\xi'|)^2 \,\mathrm{d}t \\ &\leqslant C(2^{k(\gamma+1)+\gamma s})^{-2}. \end{aligned} \tag{2.12}$$

It then follows from Plancherel's Theorem and (2.12) that

$$\|H_s\|_{2,2} \leqslant C 2^{-\gamma s}. \tag{2.13}$$

Now let us consider the case of $s < 0$. For given $\alpha > 0$ and $\gamma > 0$, take

$$-s > \max\left\{1 + \frac{8}{\gamma}, \frac{\gamma(1+\alpha)}{\log 2}\right\}.$$

Then for $1 \leqslant k < -s - (4/\gamma)$, by (2.3) we have

$$\begin{aligned} &\int_{\mathbb{R}} |\psi(|2^{\gamma(s+t)}\xi'|^2)\hat{\tau}_{k,t}(\xi)|^2 \,\mathrm{d}t \\ &\qquad \leqslant C 2^{-2k} \int_{(2^{\gamma s+1}|\xi'|)^{-1} \leqslant 2^{\gamma t} \leqslant 2(2^{\gamma s}|\xi'|)^{-1}} (\log |2^{\gamma(t-k)}\xi'|)^{-2(1+\alpha)} \,\mathrm{d}t \\ &\qquad \leqslant C 2^{-2k}(1 + \gamma|s+k|)^{-2(1+\alpha)}. \end{aligned} \tag{2.14}$$

On the other hand, for s chosen above and $k \geqslant -s - (4/\gamma)$, by (2.2) we have

$$\int_{\mathbb{R}} |\psi(|2^{\gamma(s+t)}\xi'|^2)\hat{\tau}_{k,t}(\xi)|^2 \,\mathrm{d}t \leqslant C 2^{-2k} 2^{-2\gamma(s+k)}. \tag{2.15}$$

Apply Plancherel's Theorem again, by (2.14) and (2.15), for s chosen above we have

$$\|H_{s,k}(f)\|_{L^2(\mathbb{R}^d)} \leqslant \begin{cases} C 2^{-k}(1 + \gamma|s+k|)^{-(1+\alpha)} \|f\|_{L^2(\mathbb{R}^d)}, & \text{for } 1 \leqslant k < -s - (4/\gamma), \\ C 2^{-k} 2^{-\gamma(s+k)} \|f\|_{L^2(\mathbb{R}^d)}, & \text{for } k \geqslant -s - (4/\gamma). \end{cases} \tag{2.16}$$

Thus, by (2.16) we get

$$\|H_s\|_{2,2} \leqslant C\left\{ \sum_{1 \leqslant k < -s-(4/\gamma)} 2^{-k}(1 + \gamma|s+k|)^{-(1+\alpha)} + \sum_{k \geqslant -s-(4/\gamma)} 2^{-k} 2^{-\gamma(s+k)} \right\}. \tag{2.17}$$

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We have

$$\begin{aligned}
&\sum_{1\leqslant k<-s-(4/\gamma)} 2^{-k}(1+\gamma|s+k|)^{-(1+\alpha)} \\
&\quad = 2^{s} \sum_{(4/\gamma)<j\leqslant-(s+1)} 2^{j}(1+\gamma j)^{-(1+\alpha)} \\
&\quad \leqslant 2^{s}\biggl(\sum_{(4/\gamma)<j\leqslant-(s+1)/2} 2^{j}(1+\gamma j)^{-(1+\alpha)} + \sum_{-(s+1)/2<j\leqslant-(s+1)} 2^{j}(1+\gamma j)^{-(1+\alpha)}\biggr) \\
&\quad \leqslant 2^{s}\biggl[2^{-(s+1)/2}\sum_{4<j<\infty}(1+j)^{-(1+\alpha)} + \biggl(1-\frac{\gamma(s+1)}{2}\biggr)^{-(1+\alpha)} \sum_{-(s+1)/2<j\leqslant-(s+1)} 2^{j}\biggr] \\
&\quad \leqslant C(2^{s/2}+|s|^{-(1+\alpha)})
\end{aligned} \tag{2.18}$$

and

$$\sum_{k\geqslant-s-(4/\gamma)} 2^{-k}2^{-\gamma(s+k)} \leqslant 2^{s} \sum_{j\geqslant-[4/\gamma]-1} 2^{-j(1+\gamma)} \leqslant C2^{s}. \tag{2.19}$$

It is easy to see that, for given $\alpha>0$ and $\gamma>0$, there exists an

$$N>\max\biggl\{1+\frac{8}{\gamma},\frac{\gamma(1+\alpha)}{\log 2}\biggr\}$$

such that, for all $s<-N$, $2^{s}<2^{s/2}<|s|^{-(1+\alpha)}$. Hence, by (2.17) and (2.18), (2.19), we see that

$$\|H_s\|_{2,2}\leqslant C|s|^{-(1+\alpha)}, \quad \text{for } s<-N. \tag{2.20}$$

Next we shall prove that, for every $p\in(1,\infty)$, there exists a $C_p>0$ such that for any $s\in\mathbb{R}$

$$\|H_s\|_{p,p}\leqslant C_p. \tag{2.21}$$

Let $G_u(x)=(\Psi_{2^{\gamma u}}\otimes\delta_{d-m})*f(x)$. Then by (2.1),

$$\biggl\|\biggl|\int_{\mathbb{R}}\tau_{k,t}*G_{s+t}(\cdot)\,\mathrm{d}t\biggr|\biggr\|_{L^1(\mathbb{R}^d)} \leqslant C2^{-k}\biggl\|\int_{\mathbb{R}}|G_t(\cdot)|\,\mathrm{d}t\biggr\|_{L^1(\mathbb{R}^d)}. \tag{2.22}$$

On the other hand, by (2.4), for $1<q<\infty$ we get

$$\Bigl\|\sup_{t\in\mathbb{R}}|\tau_{k,t}*G_{s+t}|\Bigr\|_{L^q(\mathbb{R}^d)} \leqslant \Bigl\|\tau_k^*\Bigl(\sup_{t\in\mathbb{R}}|G_t|\Bigr)\Bigr\|_{L^q(\mathbb{R}^d)} \leqslant C2^{-k}\Bigl\|\sup_{t\in\mathbb{R}}|G_t|\Bigr\|_{L^q(\mathbb{R}^d)}. \tag{2.23}$$

Hence, (2.22) and (2.23) show that the linear mapping $T:G_t\to\tau_{k,t}*G_{s+t}$ is bounded from $L^1(L^1(\mathbb{R}),\mathbb{R}^d)$ to itself and from $L^q(L^\infty(\mathbb{R}),\mathbb{R}^d)$ to itself, respectively. If $q>1$ satisfies $1/q=2/p-1$, then by using the operator interpolation theorem between (2.22) and (2.23), it can be concluded that for $1<p<2$ the mapping T is bounded from $L^p(L^2(\mathbb{R}),\mathbb{R}^d)$ to itself. By using an appropriate duality argument, we know that T is also bounded from $L^p(L^2(\mathbb{R}),\mathbb{R}^d)$ to itself for $2<p<\infty$. Thus, for $1<p<\infty$,

$$\biggl\|\biggl(\int_{\mathbb{R}}|\tau_{k,t}*G_{s+t}(\cdot)|^2\,\mathrm{d}t\biggr)^{1/2}\biggr\|_{L^p(\mathbb{R}^d)} \leqslant C_p2^{-k}\biggl\|\biggl(\int_{\mathbb{R}}|G_t(\cdot)|^2\,\mathrm{d}t\biggr)^{1/2}\biggr\|_{L^p(\mathbb{R}^d)}.$$

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From this and (2.8), we get that

$$\|H_{s,k}(f)\|_{L^p(\mathbb{R}^d)} \leqslant C_p 2^{-k} \|f\|_{L^p(\mathbb{R}^d)} \quad \text{for } 1 < p < \infty, \tag{2.24}$$

holds for $s \in \mathbb{R}$ and $k \in \mathbb{N}$, which implies that (2.21) holds for $s \in \mathbb{R}$ and $1 < p < \infty$.

Finally, by interpolating between (2.13) and (2.21), (2.20) and (2.21), respectively, we obtain (2.11) for every p in

$$\left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right)$$

with $\theta_p > 0$ and $\theta'_p > 1$. Lemma 2.1 is proved. □

3. Theorems 1.2 and 1.3

Proof of Theorem 1.3. Let $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 0$ and let Ω satisfy (1.1). Let $\Phi(y) = (y, \phi(|y|))$, where ϕ is a real-valued polynomial. In addition, we assume that $\phi'(0) = 0$ when $n \geqslant 3$.

Let $D_s = \{y \in \mathbb{R}^n : 2^s < |y| \leqslant 2^{s+1}\}$ and define the family of measures $\tau = \{\tau_{k,t} : t \in \mathbb{R},\ k \in \mathbb{N}\}$ on $\mathbb{R}^{n+1}$ by

$$\int_{\mathbb{R}^{n+1}} f(y, y_{n+1})\,\mathrm{d}\tau_{k,t} = 2^{-t} \int_{D_{t-k}} f(y, \phi(|y|)) \frac{\Omega(y)}{|y|^{n-1}}\,\mathrm{d}y. \tag{3.1}$$

Then

$$\mu_{\Phi,\Omega}(f) \leqslant \Delta_\tau(f). \tag{3.2}$$

It is easy to see that (2.1) follows from the integrability of Ω on S^{n-1}. In light of (3.2) and Lemma 2.1, it suffices to show that (2.2) and (2.3) also hold when we choose $\gamma = 1$ and $L(\xi, \xi_{n+1}) = \xi$.

For $\lambda \in \mathbb{R}$, let

$$I_\lambda(\xi, \xi_{n+1}, y) = \int_1^2 \mathrm{e}^{\mathrm{i}[\lambda(\xi\cdot y)u + \xi_{n+1}\phi(\lambda u)]}\,\mathrm{d}u. \tag{3.3}$$

By using a van der Corput type estimate in [**3**, Corollary 7.3] and (1.2) we obtain

$$\int_{S^{n-1}} |I_\lambda(\xi, \xi_{n+1}, y)\Omega(y)|\,\mathrm{d}\sigma(y) \leqslant C(\log^+ |\lambda\xi|)^{-(1+\alpha)} \tag{3.4}$$

for $\lambda \in \mathbb{R}$ and $(\xi, \xi_{n+1}) \in \mathbb{R}^{n+1}$. The distinction between the cases $n = 2$ and $n \geqslant 3$ was made clear in [**4**] (see also the example given at the end of § 3 in [**4**]). Thus

$$\begin{aligned} |\hat{\tau}_{k,t}(\xi, \xi_{n+1})| &\leqslant 2^{-k} \int_{S^{n-1}} |I_{2^{t-k}}(\xi, \xi_{n+1}, y)\Omega(y)|\,\mathrm{d}\sigma(y) \\ &\leqslant C 2^{-k} (\log^+ |2^{t-k}\xi|)^{-(1+\alpha)}. \end{aligned} \tag{3.5}$$

On the other hand, by (1.1),

$$\begin{aligned} |\hat{\tau}_{k,t}(\xi, \xi_{n+1})| &\leqslant 2^{-t} \int_{D_{t-k}} |\mathrm{e}^{\mathrm{i}[\xi\cdot y + \xi_{n+1}\phi(|y|)]} - \mathrm{e}^{\mathrm{i}\xi_{n+1}\phi(|y|)}| \frac{|\Omega(y)|}{|y|^{n-1}}\,\mathrm{d}y \\ &\leqslant C 2^{-k} |2^{t-k}\xi|. \end{aligned} \tag{3.6}$$

Clearly, (3.5) and (3.6) imply (2.2).

Finally, one may apply a theorem of Stein and Wainger on maximal operators along curves in [**14**] to obtain (2.4). This completes the proof of Theorem 1.3. □

The proof of Theorem 1.2 is similar. Details are omitted.

4. Proof of Theorem 1.1 and additional results

For $n, m \in \mathbb{N}$ we let $A(n,m)$ denote the set of polynomials on $\mathbb{R}^n$ which have real coefficients and degrees not exceeding m. Let

$$U(n,m) = \bigg\{ \sum_{|\beta|=m} a_\beta y^\beta \in A(n,m) \setminus A(n,m-1) : \sum_{|\beta|=m} |a_\beta|^2 = 1 \bigg\}.$$

Based on the work in [**1**] regarding singular integrals, we have the following theorem.

Theorem 4.1. *Let $\alpha > 0$, $n \geqslant 2$, $m, d \in \mathbb{N}$ and $\mathcal{P}(y) = (P_1(y), \dots, P_d(y)) \in (A(n,m))^d$. If $\Omega \in L^1(S^{n-1})$ and Ω satisfies*

$$\sup_{P \in \bigcup_{l=1}^m U(n,l)} \int_{S^{n-1}} |\Omega(y)| \bigg(\log \frac{1}{|P(y)|} \bigg)^{1+\alpha} \mathrm{d}\sigma(y) < \infty, \tag{4.1}$$

then $\mu_{\mathcal{P},\Omega}$ is bounded on $L^p(\mathbb{R}^d)$ for

$$p \in \bigg(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha \bigg).$$

Moreover, the bound on the operator norm is independent of the coefficients of the polynomials $\{P_j\}_{1 \leqslant j \leqslant d}$.

Proof. Define the family of measures $\sigma = \{\sigma_{k,t} \mid k \in \mathbb{N},\ t \in \mathbb{R}\}$ on $\mathbb{R}^d$ by

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}\sigma_{k,t}(x) = 2^{-t} \int_{D_{t-k}} f(x - \mathcal{P}(y)) \frac{\Omega(y)}{|y|^{n-1}}\,\mathrm{d}y.$$

Then

$$\mu_{\mathcal{P},\Omega}(f) \leqslant \Delta_\sigma(f). \tag{4.2}$$

By the arguments in [**7**] and [**1**], there are families of measures

$$\tau^{(1)} = \{\tau^{(1)}_{k,t} : k \in \mathbb{N},\ t \in \mathbb{R}\}, \dots, \tau^{(m)} = \{\tau^{(m)}_{k,t} : k \in \mathbb{N},\ t \in \mathbb{R}\},$$

each of which satisfies (2.1)–(2.4) with appropriate choices of $\gamma_1, \dots, \gamma_m$ and linear transformations $L^{(1)}, \dots, L^{(m)}$, such that

$$\sigma_{k,t} = \sum_{l=1}^m \tau^{(l)}_{k,t} \tag{4.3}$$

for $k \in \mathbb{N}$, $t \in \mathbb{R}$. It then follows from Lemma 2.1 and Minkowski's inequality that

$$\|\mu_{\mathcal{P},\Omega}(f)\|_{L^p(\mathbb{R}^d)} \leqslant \sum_{l=1}^{m} \|\Delta_{\tau^{(l)}}(f)\|_{L^p(\mathbb{R}^d)} \leqslant C_p \|f\|_{L^p(\mathbb{R}^d)}$$

for $f \in L^p(\mathbb{R}^d)$ and

$$p \in \left(\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha\right).$$

Theorem 4.1 is proved. □

It was shown in [**1**] that, when $n = 2$ and $\Omega \in F_\alpha(S^1)$, (4.1) holds for all $m \in \mathbb{N}$. Therefore, one obtains Theorem 1.1 as a corollary of Theorem 4.1.

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