THE NUMBER OF PAIRS OF GENERALIZED INTEGERS WITH L.C.M. $\leqq x$

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1. Introduction

Generalized integers are defined in [2] as follows: Suppose there is given a finite or infinite sequence $\{p\}$ of real numbers which are called generalized primes such that $1 < p_1 < p_2 < \cdots$. Form the set $\{l\}$ of all possible p-products, i.e. the products of the form $p_1^{\alpha_1} p_2^{\alpha_2} \cdots$, where $\alpha_1, \alpha_2, \cdots$ are integers $\geqq 0$ of which all but a finite number are 0. Call these numbers generalized integers and suppose that no two generalized integers are equal if their α's are different. Then arrange $\{l\}$ in an increasing sequence $1 = l_1 < l_2 < l_3 < \cdots < l_n < \cdots$.

Let x be any real number $\geqq 1$ and let $[x]$ denote the number of generalized integers $\leqq x$. We assume throughout the paper that

$$[x] = x + 0(x^{\alpha}), \text{ where } 0 \leqq \alpha < 1. \tag{1.1}$$

This assumption is fundamental and under this assumption it has been shown by Horadam ([4], theorem 1) that the number of generalized primes is infinite. Generalized primes were first introduced by Beurling [1], who proved using an assumption equivalent to (1.1) that

$$\zeta(s) = \prod_{r=1}^{\infty} \left(\frac{1}{1 - p_r^{-s}} \right), \text{ where } \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{l_n^s}, \ (s > 1). \tag{1.2}$$

A generalized integer d is called a divisor of l_n if there exists a $\delta \in \{l\}$ such that $d\delta = l_n$. Let (l_r, l_s) and $[l_r, l_s]$ respectively denote the greatest common divisor and the least common multiple of l_r and l_s. A divisor d of l_n is called unitary if $d\delta = l_n$ and $(d, \delta) = 1$. Let $\tau^*(l_n)$ and $t(l_n)$ denote respectively the number of unitary divisors of l_n and number of ordered pairs of generalized integers l_r and l_s with $[l_r, l_s] = l_n$. It is clear that $\tau^*(l_n) = 2^r$, where r is the number of distinct generalized prime divisors of l_n. Also, $t(l_n) = \sum_{d | l_n} \tau^*(d)$ as indicated in [6], so that $t(l_n) = \tau(l_n^2)$, where $\tau(l_n)$ is the number of divisors of l_n. Let $\theta(x)$ denote the number of ordered pairs of generalized integers with l.c.m. $\leqq x$. Clearly $\theta(x) = \sum_{l_n \leqq x} t(l_n)$. It has been recently shown by Horadam [6] using an estimate for $\sum_{l_n \leqq x} \tau^*(l_n)$ obtained by her in [5] that

$$\theta(x) = \frac{x \log^2 x}{2\zeta(2)} + \left(\frac{3\gamma_1 - 1}{\zeta(2)} - \frac{2\zeta'(2)}{\zeta^2(2)}\right) x \log x + O(x), \tag{1.3}$$

where γ_1 is the constant given in (2.1) below and $\zeta'(s)$ is the derivative of $\zeta(s)$.

The object of the present paper is to give a more exact estimation of $\theta(x)$ (see theorem 3.2 below) with an error term equal to $0(x^{(2+\alpha)/3} \log x)$.

2. Auxiliary results

The following elementary estimates given by Horadam in [3] and [6] are needed in our present discussion. These estimates can be proved by using Abel's transformation as described in [3] and (1.1).

$$\sum_{l_n \leqq x} \frac{1}{l_n} = \log x + \gamma_1 + O(x^{\alpha - 1}), \tag{2.1}$$

where γ_1 is a constant.

$$\sum_{l_n \leqq x} \frac{1}{l_n^{\beta}} = O(x^{1-\beta}), \text{ if } \beta < 1 \tag{2.2}$$

$$\sum_{l_n > x} \frac{1}{l_n^2} = O\left(\frac{1}{x}\right) \tag{2.3}$$

$$\sum_{l_n > x} \frac{\log l_n}{l_n^2} = O\left(\frac{\log x}{x}\right) \tag{2.4}$$

$$\sum_{l_n > x} \frac{\log^2 l_n}{l_n^2} = O\left(\frac{\log^2 x}{x}\right) \tag{2.5}$$

$$\sum_{l_n \leqq x} \frac{\log l_n}{l_n} = \tfrac{1}{2} \log^2 x + \delta_1 + O(x^{\alpha - 1} \log x), \tag{2.6}$$

where δ_1 is a constant.

Further, we need the following:

LEMMA 2.1. ([3], lemma 2.2). *If* $f(l_n) = \sum_{d\delta = l_n} g(d)h(\delta)$ *and* $G(x) = \sum_{l_n \leqq x} g(l_n)$, $H(x) = \sum_{l_n \leqq x} h(l_n)$, *then for all* x_1, x_2 *satisfying* $x_1 x_2 = x$,

$$\sum_{l_n \leqq x} f(l_n) = \sum_{l_n \leqq x_1} g(l_n) H\left(\frac{x}{l_n}\right) + \sum_{l_n \leqq x_2} h(l_n) G\left(\frac{x}{l_n}\right) - G(x_1)H(x_2).$$

LEMMA 2.2.

$$T(x) = \sum_{l_n \leqq x} \tau(l_n) = x(\log x + 2r_1 - 1) + O(x^{(1+\alpha)/2}). \tag{2.7}$$

PROOF. Taking $g(l_n) = h(l_n) = 1$, $x_1 = x_2 = \sqrt{x}$ in lemma 2.1, we get (2.7) by making use of (1.1), (2.1) and (2.2).

LEMMA 2.3.

$$\sum_{l_n \leqq x} \frac{\tau(l_n)}{l_n} = \tfrac{1}{2}\log^2 x + 2\gamma_1 \log x + \gamma_1^2 - 2\delta_1 + O(x^{(\alpha-1)/2}\log x). \tag{2.8}$$

PROOF. Taking $g(l_n) = h(l_n) = 1/l_n$, $x_1 = x_2 = \sqrt{x}$ is lemma 2.1, we get (2.8) by making use of (2.1), (2.2) and (2.6).

LEMMA 2.4. *If $\tau_3(l_n)$ denotes the number of ordered triads (l_r, l_s, l_t) of generalized integers such that $l_r l_s l_t = l_n$, then*

$$\sum_{l_n \leqq x} \tau_3(l_n) = \frac{x}{2}\log^2 x + (3\gamma_1 - 1)x\log x + (3\gamma_1^2 - 3\gamma_1 - 3\delta_1 + 1)x + O(x^{(2+\alpha)/3}\log x). \tag{2.9}$$

PROOF. We have $\tau_3(l_n) = \sum_{l_r l_s l_t = l_n} = \sum_{d\delta = l_n} \tau(d)$. Taking $g(l_n) = \tau(l_n)$, $h(l_n) = 1$, $x_1 = x^{\frac{2}{3}}$, $x_2 = x^{\frac{1}{3}}$ in lemma 2.1 we get by lemma 2.2, (2.1) and (1.1),

$$\begin{aligned}
\sum_{l_n \leqq x} \tau_3(l_n) &= \sum_{l_n \leqq x^{\frac{2}{3}}} \tau(l_n)\left[\frac{x}{l_n}\right] + \sum_{l_n \leqq x^{\frac{1}{3}}} T\left(\frac{x}{l_n}\right) - T(x^{\frac{2}{3}})[x^{\frac{1}{3}}] \\
&= \sum_{l_n \leqq x^{\frac{2}{3}}} \tau(l_n)\left\{\frac{x}{l_n} + O\left(\frac{x^\alpha}{l_n^\alpha}\right)\right\} + \sum_{l_n \leqq x^{\frac{1}{3}}} \left\{\frac{x}{l_n}\left(\log\frac{x}{l_n} + 2\gamma_1 - 1\right) + O\left(\frac{x^{(1+\alpha)/2}}{l_n^{(1+\alpha)/2}}\right)\right\} \\
&\quad - \{x^{\frac{2}{3}}(\tfrac{2}{3}\log x + 2\gamma_1 - 1) + O(x^{(1+\alpha)/3})\}\{x^{\frac{1}{3}} + O(x^{\alpha/3})\} \\
&= x\sum_{l_n \leqq x^{\frac{2}{3}}} \frac{\tau(l_n)}{l_n} + O\left(x^\alpha \sum_{l_n \leqq x^{\frac{2}{3}}} \frac{\tau(l_n)}{l_n^\alpha}\right) + x(\log x + 2\gamma_1 - 1)\sum_{l_n \leqq x^{\frac{1}{3}}} \frac{1}{l_n} \\
&\quad - x\sum_{l_n \leqq x^{\frac{1}{3}}} \frac{\log l_n}{l_n} + O\left(x^{(1+\alpha)/2} \sum_{l_n \leqq x^{\frac{1}{3}}} \frac{1}{l_n^{(1+\alpha)/2}}\right) \\
&\quad - x(\tfrac{2}{3}\log x + 2\gamma_1 - 1) + O(x^{(2+\alpha)/3}\log x).
\end{aligned}$$

We have by (2.2) and (2.1),

$$\begin{aligned}
\sum_{l_n \leqq x^{\frac{2}{3}}} \frac{\tau(l_n)}{l_n^\alpha} &= \sum_{l_n \leqq x^{\frac{2}{3}}} \sum_{d\delta = l_n} \frac{1}{d^\alpha \delta^\alpha} = \sum_{d\delta \leqq x^{\frac{2}{3}}} \frac{1}{d^\alpha \delta^\alpha} = \sum_{d \leqq x^{\frac{2}{3}}} \frac{1}{d^\alpha} \sum_{\delta \leqq x^{\frac{2}{3}}/d} \frac{1}{\delta^\alpha} \\
&= O\left(\sum_{d \leqq x^{\frac{2}{3}}} \frac{1}{d^\alpha}\left(\frac{x^{\frac{2}{3}}}{d}\right)^{1-\alpha}\right) = O\left(x^{(2-2\alpha)/3} \sum_{d \leqq x^{\frac{2}{3}}} \frac{1}{d}\right) \\
&= O(x^{(2-2\alpha)/3}\log x).
\end{aligned}$$

Also, by (2.2)

$$\sum_{l_n \leqq x^{\frac{1}{3}}} \frac{1}{l_n^{(1+\alpha)/2}} = O(x^{(1-\alpha)/6}).$$

Hence by lemma 2.3 and (2.6),

$$\sum_{l_n \leq x} \tau_3(l_n) = x\left\{\tfrac{1}{2}(\tfrac{2}{3})^2 \log^2 x + \frac{4\gamma_1}{3}\log x + \gamma_1^2 - 2\delta_1 + O(x^{(\alpha-1)/3}\log x)\right\}$$
$$+x(\log x + 2\gamma_1 - 1)\{\tfrac{1}{3}\log x + \gamma_1 + O(x^{(\alpha-1)/3})\}$$
$$-x\{\tfrac{1}{2}(\tfrac{1}{3})^2\log^2 x + \delta_1 + O(x^{(\alpha-1)/3}\log x)\}$$
$$-x(\tfrac{2}{3}\log x + 2\gamma_1 - 1) + O(x^{(2+\alpha)/3}\log x)$$
$$= \frac{x}{2}\log^2 x + (3\gamma_1 - 1)x\log x + (3\gamma_1^2 - 3\gamma_1 - 3\delta_1 + 1)x + O(x^{(2+\alpha)/3}\log x).$$

Hence lemma 2.4 follows.

LEMMA 2.5. If $g(l_n)$ and $h(l_n)$ are multiplicative functions, then

$$f(l_n) = \sum_{d^2 | l_n} g(d) h\left(\frac{l_n}{d^2}\right)$$

is also multiplicative.

PROOF. This can be proved exactly in the same way as the corresponding result for natural numbers proved in ([7], lemma 2.4 for $k = 2$).

3. Asymptotic formula for $\theta(x)$

Let $\mu(l_n)$ be the Möbius function for generalized integers defined by Horadam [2] as follows: $\mu(l_n) = 0$ if l_n has a square factor; $\mu(l_n) = (-1)^r$, where r is the number of distinct generalized prime factors of l_n and l_n has no square factor; $\mu(1) = 1$. It is clear that $\mu(l_n)$ is multiplicative.

LEMMA 3.1. ([3], (2.1)). *If* $s > 1$,

$$\sum_{n=1}^{\infty} \frac{\mu(l_n)}{l_n^s} = \frac{1}{\zeta(s)}. \tag{3.1}$$

LEMMA 3.2. *If* $s > 1$,

$$\sum_{n=1}^{\infty} \frac{\mu(l_n)\log l_n}{l_n^s} = -\eta'(s), \tag{3.2}$$

where $\eta'(s)$ *is the derivative of* $\eta(s) = 1/\zeta(s)$.

PROOF. Since the series in (3.2) is uniformly convergent for $s \geqq 1+\varepsilon > 1$, we obtain (3.2) by term-wise differentiation of the series in (3.1) with respect to s.

LEMMA 3.3. *If* $s > 1$,

$$\sum_{n=1}^{\infty} \frac{\mu(l_n)\log^2 l_n}{l_n^s} = \eta''(s), \tag{3.3}$$

where $\eta''(s)$ *is the second derivative of* $\eta(s)$.

PROOF. This follows by term-wise differentiation of the series in (3.2). We now prove the following:

THEOREM 3.1.

$$t(l_n) = \sum_{d^2 | l_n} \mu(d)\tau_3\left(\frac{l_n}{d^2}\right).$$

PROOF. Since $\mu(l_n)$ is multiplicative and $\tau_3(l_n) = \sum_{d|l_n} \tau(d)$ is multiplicative, it follows by lemma 2.5 that the function on the right side of the theorem is multiplicative. Also, $t(l_n) = \tau(l_n^2)$ is multiplicative. Hence, it is enough, if we prove the theorem for $l_n = p^\nu$, where p is a generalized prime and this can be done by making use of

$$\tau_3(p^\nu) = \sum_{d|p^\nu} \tau(d) = \frac{(\nu+1)(\nu+2)}{2}$$

and $t(p^\nu) = 2\nu+1$.

THEOREM 3.2.

$$\theta(x) = \frac{x\log^2 x}{2\zeta(2)} + \left(\frac{3\gamma_1 - 1}{\zeta(2)} + 2\eta'(2)\right) x\log x + \left\{\frac{3\gamma_1^2 - 3\gamma_1 - 3\delta_1 + 1}{\zeta(2)} + 2(3\gamma_1 - 1)\eta'(2) + 2\eta''(2)\right\} x + O(x^{(2+\alpha)/3}\log x), \tag{3.4}$$

where γ_1 and δ_1 are constants given in (2.1) *and* (2.6), *$\eta'(2)$ and $\eta''(2)$ are the values of the first and second derivatives of $\eta(s) = 1/\zeta(s)$ at $s = 2$.*

PROOF. We have by theorem 3.1 and lemma 2.4,

$$\begin{aligned}
\theta(x) &= \sum_{l_n \leqq x} t(l_n) = \sum_{l_n \leqq x} \sum_{d^2\delta = l_n} \mu(d)\tau_3(\delta) = \sum_{d^2\delta \leqq x} \mu(d)\tau_3(\delta) \\
&= \sum_{d \leqq \sqrt{x}} \mu(d) \sum_{\delta \leqq x/d^2} \tau_3(\delta) \\
&= \sum_{d \leqq \sqrt{x}} \mu(d) \left\{\frac{x}{2d^2}\log^2\left(\frac{x}{d^2}\right) + (3\gamma_1 - 1)\frac{x}{d^2}\log\frac{x}{d^2} + (3\gamma_1^2 - 3\gamma_1 - 3\delta_1 + 1)\frac{x}{d^2} \right. \\
&\qquad \left. + O\left(\frac{x^{(2+\alpha)/3}}{d^{(4+2\alpha)/3}}\log\frac{x}{d^2}\right)\right\} \\
&= \left\{\frac{x}{2}\log^2 x + (3\gamma_1 - 1)x\log x + (3\gamma_1^2 - 3\gamma_1 - 3\delta_1 + 1)x\right\} \sum_{l_n \leqq \sqrt{x}} \frac{\mu(l_n)}{l_n^2} \\
&\qquad - 2x(\log x + 3\gamma_1 - 1) \sum_{l_n \leqq \sqrt{x}} \frac{\mu(l_n)\log l_n}{l_n^2} \\
&\qquad + 2x \sum_{l_n \leqq \sqrt{x}} \frac{\mu(l_n)\log^2 l_n}{l_n^2} + O(x^{(2+\alpha)/3}\log x).
\end{aligned}$$

Hence, by lemmas 3.1, 3.2, 3.3 for $s = 2$ and (2.3), (2.4), (2.5), we have

$$\begin{aligned}\theta(x) = &\left\{\frac{x}{2}\log^2 x + (3\gamma_1 - 1)x\log x + (3\gamma_1^2 - 3\gamma_1 - 3\delta_1 + 1)x\right\}\left\{\frac{1}{\zeta(2)} + O\left(\frac{1}{\sqrt{x}}\right)\right\}\\ &-2x(\log x + 3\gamma_1 - 1)\left\{-\eta'(2) + O\left(\frac{\log x}{\sqrt{x}}\right)\right\}\\ &+2x\left\{\eta''(2) + O\left(\frac{\log^2 x}{\sqrt{x}}\right)\right\} + O(x^{(2+\alpha)/3}\log x)\\ = &\frac{x\log^2 x}{2\zeta(2)} + \left(\frac{3\gamma_1 - 1}{\zeta(2)} + 2\eta'(2)\right)x\log x\\ &+\left\{\frac{3\gamma_1^2 - 3\gamma_1 - 3\delta_1 + 1}{\zeta(2)} + 2(3\gamma_1 - 1)\eta'(2) + 2\eta''(2)\right\}x\\ &+O(x^{(2+\alpha)/3}\log x).\end{aligned}$$

Thus theorem 3.2 is proved.

References

[1] A. Beurling, 'Analyse de la loé asymptotique de la distribution des nombres premiers généralisés', *Acta Mathematica*, 68 (1937), 255—291.
[2] E. M. Horadam, 'Arithmetical functions of generalized primes', *Amer. Math. Monthly*, 68 (1961), 626—629.
[3] E. M. Horadam, 'The Euler φ-function for generalized integers', *Proc. Amer. Math. Soc.*, 14 (1963), 754—762.
[4] E. M. Horadam, 'The order of Arithmetical functions of generalized integers', *Amer. Math. Monthly*, 70 (1963), 506—512.
[5] E. M. Horadam, The number of unitary divisors of a generalized integer', *Amer. Math. Monthly*, 71 (1964), 893—895.
[6] E. M. Horadam, 'On the number of pairs of generalized integers with least common multiple not exceeding x', *Amer. Math. Monthly*, 74 (1967), 811—812.
[7] D. Suryanarayana, 'The number of k-ary divisors of an integer', *Monatsh. Math.*, 72 (1968), 445—450.

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