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# ON THE EQUIVALENCE OF BROWDER'S AND GENERALIZED BROWDER'S THEOREM

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**Abstract.** In this note we answer two question posed by Berkani and Koliha [*Acta Sci. Math.* **69** (2003), 359–376]. We show that generalized Browder's (resp. generalized $a$-Browder's) theorem holds for a Banach space operator if and only if Browder's (resp. $a$-Browder's) theorem does. We also give condition under which generalized Weyl's (resp. generalized $a$-Weyl's) theorem is equivalent to Weyl's (resp. $a$-Weyl's) theorem.

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**1. Introduction.** Throughout this paper, $\mathcal{B}(X)$ denote the algebra of all bounded linear operators acting on a Banach space $X$. For $T \in \mathcal{B}(X)$, let $T^*$, $N(T)$, $R(T)$, $\sigma(T)$ and $\sigma_{ap}(T)$ denote the adjoint, the null space, the range, the spectrum and the approximate point spectrum of $T$ respectively. Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of $T$ defined by

$$\alpha(T) = \dim N(T), \text{ and } \beta(T) = \operatorname{codim} R(T).$$

If the range $R(T)$ of $T$ is closed and complemented and $\alpha(T) < \infty$ (resp. $N(T)$ is complemented and $\beta(T) < \infty$), then $T$ is called an *upper* (resp. a *lower*) *semi-Fredholm* operator. In the sequel $SF_+(X)$ (resp. $SF_-(X)$) will denote the set of all upper (resp. lower ) semi-Fredholm operators. If $T \in \mathcal{B}(X)$ is either upper or lower semi-Fredholm, then $T$ is called a *semi-Fredholm* operator, and the *index* of $T$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is a *Fredholm* operator. An operator $T$ is called *Weyl* if it is Fredholm of index zero. The *descent* $q(T)$ and the *ascent* $p(T)$ are given by

$$q(T) = \inf\{n : R(T^n) = R(T^{n+1})\},$$

$$p(T) = \inf\{n : N(T^n) = N(T^{n+1})\}.$$

A bounded linear operator $T$ is called *Browder* if it is Fredholm of finite ascent and descent. The essential spectrum $\sigma_e(T)$, Weyl spectrum $\sigma_w(T)$, and Browder spectrum

$\sigma_b(T)$ of $T \in \mathcal{B}(X)$ are defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Fredholm}\},$$
$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\},$$
$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.$$

Evidently

$$\sigma_e(T) \subseteq \sigma_w(T) \subseteq \sigma_b(T).$$

For $T \in \mathcal{B}(X)$, define the set $LD(X)$ by

$$LD(X) = \{T \in \mathcal{B}(X) : p(T) < \infty \text{ and } R(T^{p(T)+1}) \text{ is closed}\}.$$

An operator $T \in \mathcal{B}(X)$ is said to be *left Drazin invertible* if $T \in LD(X)$. We say that $\lambda \in \sigma_{ap}(T)$ is a *left pole* of $T$ if $T - \lambda \in LD(X)$, and that $\lambda \in \sigma_{ap}(T)$ is a left pole of $T$ of finite rank if $\lambda$ is a left pole of $T$ and $\alpha(T - \lambda) < \infty$. We denote by $\pi^a(T)$ the set of all left poles of $T$, and by $\pi_0^a(T)$ the set of all left poles of finite rank.

We say that *Weyl's theorem* holds for $T \in \mathcal{B}(X)$ if

$$\sigma(T) \setminus \sigma_w(T) = E_0(T);$$

where $E_0(T)$ is the set of isolated point of $\sigma(T)$ which are eigenvalues of finite multiplicity, and that *Browder's theorem* holds for $T \in \mathcal{B}(X)$ if

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T),$$

where $\pi_0(T)$ is the set of all poles of $T$ of finite rank.

For $T \in \mathcal{B}(X)$, let $SF_+^-(X)$ the class of all $T \in SF_+(X)$ with ind $T \leq 0$. The *essential approximate point spectrum* $\sigma_{SF_+^-}(T)$ is defined by

$$\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not in } SF_+^-(X)\}.$$

We say that *a-Weyl's theorem* holds for $T \in \mathcal{B}(X)$ if

$$\sigma_{ap}(T) \setminus \sigma_{SF_+^-}(T) = E_0^a(T),$$

where $E_0^a(T)$ is the set of isolated points of $\sigma_{ap}(T)$ which are eigenvalues of finite multiplicity, and that *a-Browder's theorem* holds for $T \in \mathcal{B}(X)$ if

$$\sigma_{ap}(T) \setminus \sigma_{SF_+^-}(T) = \pi_0^a(T).$$

In [**10**, **20**], it is shown that :

$$a\text{-Weyl's theorem} \Rightarrow \text{Weyl's theorem} \Rightarrow \text{Browder's theorem},$$

$$a\text{-Weyl's theorem} \Rightarrow a\text{-Browder's theorem} \Rightarrow \text{Browder's theorem}.$$

The investigation of operators obeying Weyl's theorem, $a$-Weyl's theorem, Browder's theorem or $a$-Browder's theorem was studied by many mathematicians [**7**, **8**, **9**, **10**, **11**, **12**, **14**, **15**, **18**, **20**, **22**] and the references cited therein.

For a bounded linear operator $T$ and a nonnegative integer $n$ define $T_n$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular

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$T_0 = T$). If for some integer $n$ the range space $R(T^n)$ is closed and $T_n$ is an upper (resp. a lower) semi-Fredholm operator, then $T$ is called an *upper* (resp. *lower*) *semi-B-Fredholm* operator. In this case the *index* of $T$ is defined as the index of the semi-B-Fredholm operator $T_n$, see [**4**]. Moreover if $T_n$ is a Fredholm operator, then $T$ is called a *B-Fredholm* operator. A *semi-B-Fredholm* operator is an upper or a lower semi-B-Fredholm operator. An operator $T \in \mathcal{L}(X)$ is said to be a *B-Weyl operator* if it is a B-Fredholm operator of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of $T$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a B-Weyl operator}\}.$$

We say that *generalized Weyl's theorem* holds for $T$ if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T),$$

where $E(T)$ is the set of all isolated eigenvalues of $T$, and *generalized Browder's theorem* holds for $T$ if

$$\sigma(T) \setminus \sigma_{BW}(T) = \pi(T),$$

where $\pi(T)$ is the set of all poles of $T$. Similarly, let $SBF_+(X)$ be the class of all upper semi-B-Fredholm operators, and $SBF_+^-(X)$ the class of all $T \in SBF_+(X)$ such that $\text{ind}(T) \leq 0$. Also let

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not in } SBF_+^-(X)\},$$

called the *semi-essential approximate point spectrum*, see [**4**]. We say that $T$ obeys *generalized a-Weyl's theorem* if

$$\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus E^a(T),$$

where $E^a(T)$ is the set of all eigenvalues of $T$ which are isolated in $\sigma_{ap}(T)$ ([**4** Definition 2.13]). From [**4**], we know that

generalized $a$-Weyl's theorem $\Rightarrow$ generalized Weyl's theorem $\Rightarrow$ Weyl's theorem,
generalized $a$-Weyl's theorem $\Rightarrow$ $a$-Weyl's theorem.

We say that $T$ obeys *generalized a-Browder's theorem* if

$$\sigma_{SBF_+^-}(T) = \sigma_{ap}(T) \setminus \pi^a(T).$$

Generalized Weyl's theorem has been studied in [**4**]. In particular it is shown that generalized Weyl's theorem implies Weyl's theorem. It has been extended from normal operators to hyponormal operators [**3**], to $p$-hyponormal and $M$-hyponormal operators by Cao *et al* [**6**] and to a large class of operators satisfying the SVEP by [**1**] and [**23**]. In [**4**], it is shown that

generalized Browder's theorem $\Rightarrow$ Browder's theorem,

generalized $a$-Browder's theorem $\Rightarrow$ $a$-Browder's theorem.

In [**4**] the authors asked:

PROBLEM 1. Does there exist an operator satisfying Browder's theorem but not generalized Browder's theorem?

PROBLEM 2. Does there exist an operator satisfying $a$-Browder's theorem but not generalized $a$-Browder's theorem?

In this note we answer their two problems negatively by showing that generalized Browder's (resp. $a$-Browder's) theorem holds for a Banach space operator if and only if Browder's (resp. $a$-Browder's) theorem does. We also give condition under which generalized Weyl's (resp. $a$-Weyl's) theorem is equivalent to Weyl's (resp. $a$-Weyl's) theorem.

**2. Main results.** In [**4**, Theorem 3.15] it is show that generalized Browder's theorem implies Browder's theorem and the authors asked if there exists some operator which obeys Browder's theorem but not generalized Browder's theorem (Problem 1 of [**4**]). In the following we answer this negatively.

THEOREM 2.1. *Let $T \in \mathcal{B}(X)$. Then the following are equivalent:*

(i) *Browder's theorem holds for $T$;*
(ii) *generalized Browder's theorem holds for $T$.*

*Proof.* (i) $\Rightarrow$ (ii) : By [**4**, Theorem 3.15].
(ii) $\Rightarrow$ (i) : Assume that Browder's theorem holds for $T$. Then

$$\sigma(T) \setminus \sigma_w(T) = \pi_0(T). \tag{2.1}$$

Let $\lambda \in \sigma(T) \setminus \sigma_{BW}(T)$. Then $T - \lambda$ is a B-Fredholm operator of index zero. For some integer $n$ large enough, $T - (\lambda + \frac{1}{n})$ is a Fredholm of index zero (see [**5**], Corollary 3.2 ]). That is $\lambda + \frac{1}{n} \notin \sigma_w(T)$. Hence by (2.1), $\lambda + \frac{1}{n} \in \pi_0(T)$. Thus $T - (\lambda + \frac{1}{n})$ is a Fredholm operator of index zero with finite ascent and descent. Hence by [**13**, Theorem 4.7] we have $p(T - \lambda) = q(T - \lambda) < \infty$. Hence $\lambda \in \pi(T)$.

Conversely assume that $\lambda \in \pi(T)$. Then from [**2**, Theorem 2.3], $T - \lambda$ is a B-Fredholm operator of index zero. Thus $\pi(T) \subseteq \sigma(T) \setminus \sigma_{BW}(T)$ and so we have $\sigma(T) \setminus \sigma_{BW}(T) = \pi(T)$. Therefore generalized Browder's theorem holds for $T$. □

The following corollary gives a necessarily and sufficient condition that generalized Weyl's theorem and Weyl's theorem are equivalent.

COROLLARY 2.1. *Let $T \in \mathcal{B}(X)$. Then the following are equivalent:*

(i) *generalized Weyl's theorem holds for $T$;*
(ii) *generalized Browder's theorem holds for $T$ and $E(T) = \pi(T)$;*
(iii) *Weyl's theorem holds for $T$ and $E(T) = \pi(T)$;*
(iv) *Browder's theorem holds for $T$ and $E(T) = \pi(T)$.*

*Proof.* The equivalence between (i) and (ii) is given in [**2**, Corollary 2.6]. (ii) is equivalent to (iv) by Theorem 2.1. Since Weyl's theorem implies Browder's theorem then (iii) implies (iv). Now (i) implies (iii) by [**2**, Theorem 2.5] and [**4**, Theorem 3.9]. □

A bounded linear operator $T$ is called *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of $T$. Let $\mathcal{H}(\sigma(T))$ denote the space of all analytic functions in an open neighborhood of $\sigma(T)$. The first part of the following corollary was established in [**1**, Proposition 2.10]. However the arguments are different.

COROLLARY 2.2. *Let $T \in \mathcal{B}(X)$. If $T$ or $T^*$ has the SVEP and $E(T) = \pi(T)$ then $T$ satisfies Weyl's and generalized Weyl's theorem. If in addition, $T$ is isoloid then $f(T)$ satisfies Weyl's and generalized Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$.*

*Proof.* If $T$ or $T^*$ has the SVEP then it follows from [**18**, Theorem 2.9] that $T$ satifies Browder's theorem. Hence we deduce from Corollary 2.1 that $T$ satisfies Weyl's and generalized Weyl's theorem. Now if in addition $T$ is isoloid then it follows from [**23**, Theorem 2.2] that $f(T)$ satisfies Weyl's and generalized Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$. □

Theorem 3.13 of [**4**] shows that if $T$ satisfies generalized $a$-Browder's theorem then it satisfies $a$-Browder's theorem. Problem 2 in [**4**] asked if the converse is not true. In the following theorem we show that there is equivalence.

THEOREM 2.2. *Let $T \in \mathcal{B}(X)$. Then the following are equivalent:*
(i) *$a$-Browder's theorem holds for $T$.*
(ii) *generalized $a$-Browder's theorem holds for $T$.*

*Proof.* (i) ⇒ (ii) : Theorem 3.13 of [**4**].
(ii) ⇒ (i): Assume that $a$-Browder's theorem holds for $T$. Then

$$\sigma_{ap}(T) \setminus \sigma_{SF_+^-}(T) = \pi_0^a(T). \tag{2.2}$$

Let $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$; then $T - \lambda$ is an upper semi-B-Fredholm and $ind(T - \lambda) \leq 0$. For $n$ large enough, it follows from [**5**, Corollary 3.2] that $T - (\lambda + \frac{1}{n})$ is an upper semi-Fredholm operator and $ind(T - (\lambda + \frac{1}{n})) \leq 0$. Then from (2.2), $\lambda + \frac{1}{n}$ belongs to $\pi_0^a(T)$. In particular, $p(T - (\lambda + \frac{1}{n})) < \infty$. Hence $p(T - \lambda) < \infty$ by Theorem 4.7 of [**13**]. Now, since $T - \lambda$ is semi-B-Fredholm, then there exists an integer $m$ such that $R((T - \lambda)^m)$ is closed and $(T - \lambda) \mid_{R((T-\lambda)^m)}$ is Fredholm. From the proof of Proposition 2.1 of [**5**], we conclude that we can assume that $m \geq p(T - \lambda)$. Since we have $R(T - \lambda) + N((T - \lambda)^{i+1}) = R(T - \lambda) + N((T - \lambda)^i)$ for every $i \geq p(T - \lambda)$ and $R((T - \lambda)^m)$ is closed, then by [**17**, Lemma 17], we get that $R((T - \lambda)^{p(T-\lambda)+1})$ is closed. Finally, $\lambda \in \pi^a(T)$. Thus

$$\sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T) \subseteq \pi^a(T).$$

For the reverse inclusion. If $\lambda \in \pi^a(T)$, then by [**4**, Remark 2.7] $\lambda$ is isolated in $\sigma_{ap}(T)$ and from [**4**, Theorem 2.8] $T - \lambda \notin \sigma_{SBF_+^-}(T)$. Thus $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T)$. Finally, $\sigma_{ap}(T) \setminus \sigma_{SBF_+^-}(T) = \pi^a(T)$. Thus generalized $a$-Browder's theorem holds for $T$. □

The following corollary gives a necessary and sufficient condition that generalized $a$-Weyl's theorem and $a$-Weyl's theorem are equivalent.

COROLLARY 2.3. *Let $T \in \mathcal{B}(X)$. Then the following are equivalent:*
(i) *generalized $a$-Weyl's theorem holds for $T$.*
(ii) *generalized $a$-Browder's theorem holds for $T$ and $E^a(T) = \pi^a(T)$.*
(iii) *$a$-Weyl's theorem holds for $T$ and $E^a(T) = \pi^a(T)$.*
(iv) *$a$-Browder's theorem holds for $T$ and $E^a(T) = \pi^a(T)$.*

*Proof.* (ii) is equivalent to (iv) by Theorem 2.2. The equivalence between (i) and (ii) follow from [**4**, Corollary 3.2]. Since $a$-Weyl's theorem implies $a$-Browder's theorem (see [**4**, Corollary 3.5]), then (iii) implies (iv). By Theorem 3.11 and Corollary 3.2 of [**4**], we get (i) implies (iii). □

A bounded linear operator $T$ is called *a-isoloid* if every isolated point of $\sigma_{ap}(T)$ is an eigenvalue of $T$. Note that every $a$-isoloid operator is isoloid and the converse is not true in general.

COROLLARY 2.4. *Let $T \in \mathcal{B}(X)$. If $T$ or $T^*$ has the SVEP and $E^a(T) = \pi^a(T)$ then $T$ satisfies a-Weyl's and generalized a-Weyl's theorem. If in addition, $T$ is a-isoloid then $f(T)$ satisfies a-Weyl's and generalized a-Weyl's theorem for every $f \in \mathcal{H}(\sigma(T))$.*

*Proof.* If $T$ or $T^*$ has the SVEP then $a$-Browder's theorem holds for $T$ (see [**19**, Proposition 2.3]). Then the first part follows form Corollary 2.3. The second from [**23**, Theorem 2.4]. □

ADDED IN PROOF. M. Berkani has informed us that in his forthcoming paper. 'On the equivalence of Weyl's and generalized Weyl's theorem,' *Acta Math. Sinica*, to appear, he has proved the equivalence between (i) and (iii) in Corollaries 2.1 and 2.3.

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