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An oscillation theorem for a superlinear functional differential equation with general deviating arguments

Yuichi Kitamura and Takaŝi Kusano

An oscillation criterion is established for a class of functional differential equations including the generalized Emden-Fowler equation

$$x^{(n)}(t) + p(t)|x(g(t))|^{\gamma} \operatorname{sgn} x(g(t)) = 0 , \quad \gamma > 1 ,$$

as a special case. The deviating arguments involved may be retarded or advanced or otherwise. The result extends and improves known fundamental oscillation criteria for superlinear differential equations with retarded arguments.

The oscillatory behaviour of functional differential equations with deviating arguments has been the object of intensive studies in the last ten years. Most of the literature, however, is concerned with equations involving retarded arguments. For typical oscillation results regarding such equations we refer to an excellent survey article of Mitropol'skiĭ, Ševelo [7].

A systematic study of differential equations with general (not necessarily retarded) deviating arguments was proposed by the present authors; see, for example, [2] and [6]. As an illustration we mention the following theorem proven in [2].

THEOREM A. *Consider the superlinear equation*

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$$\text{(A)} \qquad x^{(n)}(t) + p(t)|x(g(t))|^{\gamma} \operatorname{sgn} x(g(t)) = 0 , \quad \gamma > 1 ,$$

where $p, g \in C[R_+, R_+]$, $R_+ = [0, \infty)$, *and* $\lim_{t\to\infty} g(t) = \infty$. *Suppose that there is a function* $h \in C^1[R_+, R_+]$ *such that*

$$\text{(1)} \qquad h(t) \le \min\{g(t), t\} , \quad h'(t) \ge 0 , \quad \lim_{t\to\infty} h(t) = \infty ,$$

$$\text{(2)} \qquad \int^{\infty} [h(t)]^{n-1} p(t)\,dt = \infty .$$

Then, if n *is even, every solution* $x(t)$ *of* (A) *is oscillatory, while, if* n *is odd, every solution* $x(t)$ *is either oscillatory or such that* $|x^{(i)}(t)| \downarrow 0$ *as* $t \uparrow \infty$, $i = 0, 1, \ldots, n-1$.

In this note we consider the differential equation

$$\text{(B)} \qquad x^{(n)}(t) + f(t, x(g_1(t)), \ldots, x(g_m(t))) = 0$$

in generalization of (A) and wish to establish an oscillation criterion for (B) which generalizes earlier standard oscillation criteria for retarded differential equations ([*3*, *4*, *5*, *8*, *9*, *10*]). Our result, when specialized to (A), turns out to be a substantial improvement of Theorem A stated above.

The conditions we assume for (B) are as follows:

(a) $g_i \in C[R_+, R]$, $R = (-\infty, \infty)$, and $\lim_{t\to\infty} g_i(t) = \infty$, $i = 1, \ldots, m$;

(b) $f \in C[R_+ \times R^m, R]$, $f(t, y_1, \ldots, y_m)$ is nondecreasing in each y_i , and $y_1 f(t, y_1, \ldots, y_m) > 0$ for $y_1 y_i > 0$, $i = 1, \ldots, m$;

(c) there exist functions $p, \phi, \psi \in C[R_+, R_+]$ such that $\phi(r)$ and $\psi(r)$ are nondecreasing and positive for $r > 0$, $|f(t, y, \ldots, y)| \ge p(t)\phi(|y|)$ for $(t, y) \in R_+ \times R$, and

$$\text{(3)} \qquad \int_{\delta}^{\infty} \frac{dr}{\phi(r)\psi(r)} < \infty \quad \text{for any } \delta > 0 .$$

THEOREM B. *Let conditions* (a), (b), *and* (c) *be satisfied. Put* $g_*(t) = \min\{g_1(t), \ldots, g_m(t), t\}$ *and suppose that*

$$\int^{\infty} \frac{[g_*(t)]^{n-1}p(t)}{\psi([g_*(t)]^{n-1})}\,dt = \infty \,. \tag{4}$$

Then, if n *is even, every solution* $x(t)$ *of* (B) *is oscillatory, while, if* n *is odd, every solution* $x(t)$ *is either oscillatory or such that* $|x^{(i)}(t)| \downarrow 0$ *as* $t \uparrow \infty$, $i = 0, 1, \ldots, n-1$.

Proof. Let $x(t)$ be a non-oscillatory solution of (B) defined on $[T_x, \infty)$. Without loss of generality we may suppose that $x(t) > 0$ on $[T_x, \infty)$. Choose a $t_0 \geq T_x$ so large that $g_*(t) \geq T_x$ for $t \geq t_0$. From (B), $x^{(n)}(t) < 0$ for $t \geq t_0$, so that by a lemma of Kiguradze [*1*, Lemma 2] there exists a $t_1 > t_0$ and an integer $k \in \{0, 1, \ldots, n-1\}$ such that $k \not\equiv n \pmod 2$ and

$$x^{(i)}(t) > 0 \quad \text{for} \quad t \geq t_1\,, \quad i = 0, 1, \ldots, k\,, \tag{5}$$

$$(-1)^{i-k}x^{(i)}(t) > 0 \quad \text{for} \quad t \geq t_1\,, \quad i = k+1, \ldots, n\,. \tag{6}$$

Let n be even. Clearly $k \geq 1$ and so $x'(t) > 0$ for $t \geq t_1$. We claim that

$$x'(t) \geq \frac{(t-t_1)^{k-1}}{(k-1)!}\int_t^{\infty} \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, \tilde{x}(\tilde{g}(s)))\,ds\,, \quad t \geq t_1\,, \tag{7}$$

where $\tilde{x}(\tilde{g}(s))$ stands for $(x(g_1(s)), \ldots, x(g_m(s)))$. Observe that

$$x^{(k)}(t) = \sum_{i=k}^{n-1} \frac{(t-T)^{i-k}}{(i-k)!} x^{(i)}(T) + (-1)^{n-k-1}\int_T^t \frac{(s-t)^{n-k-1}}{(n-k-1)!} x^{(n)}(s)\,ds$$

for any t, $T \geq t_1$. Since (6) holds and $n - k - 1$ is even, it follows from the above that

$$x^{(k)}(t) \geq -\int_t^T \frac{(s-t)^{n-k-1}}{(n-k-1)!} x^{(n)}(s)\,ds$$

for $T \geq t \geq t_1$, which in the limit as $T \to \infty$ gives

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(8) $$x^{(k)}(t) \geq \int_t^\infty \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, \tilde{x}(\tilde{g}(s)))ds , \quad t \geq t_1 .$$

This coincides with (7) if $k = 1$. Suppose $k \geq 2$. Then we have

(9) $$x'(t) \geq \frac{(t-t_1)^{k-1}}{(k-1)!} x^{(k)}(t) , \quad t \geq t_1 .$$

This follows from the equation

$$x'(t) = \sum_{i=1}^{k-1} \frac{(t-t_1)^{i-1}}{(i-1)!} x^{(i)}(t_1) + \int_{t_1}^{t} \frac{(t-s)^{k-2}}{(k-2)!} x^{(k)}(s)ds$$

with the aid of (5) and the decreasing nature of $x^{(k)}(t)$. Combining (8) with (9) yields (7) as claimed.

We now take a $t_2 > t_1$ so that $g_*(t) \geq t_1$ for $t \geq t_2$. Since $x^{(n)}(t) < 0$ for $t \geq t_0$, there is a constant $a \geq 1$ such that $x(t) \leq at^{n-1}$ for $t \geq t_1$, and hence if $s \geq t_2$, then

(10) $$x(t)/a \leq [g_*(s)]^{n-1} \quad \text{for} \quad t_1 \leq t \leq g_*(s) .$$

We divide both sides of (7) by $\phi(x(t)/a)\psi(x(t)/a)$ and integrate it over $[t_1, t_3]$, $t_3 > t_2$, obtaining

$$\int_{t_1}^{t_3} \frac{x'(t)}{\phi(x(t)/a)\psi(x(t)/a)} dt$$

$$\geq \int_{t_1}^{t_3} \frac{(t-t_1)^{k-1}}{(k-1)!\phi(x(t)/a)\psi(x(t)/a)} \int_t^{t_3} \frac{(s-t)^{n-k-1}}{(n-k-1)!} f(s, \tilde{x}(\tilde{g}(s)))dsdt$$

$$= \int_{t_1}^{t_3} \int_{t_1}^{s} \frac{(s-t)^{n-k-1}(t-t_1)^{k-1}}{(n-k-1)!(k-1)!} \frac{f(s,\tilde{x}(\tilde{g}(s)))}{\phi(x(t)/a)\psi(x(t)/a)} dtds$$

$$\geq \int_{t_2}^{t_3} \int_{t_1}^{g_*(s)} \frac{(g_*(s)-t)^{n-k-1}(t-t_1)^{k-1}}{(n-k-1)!(k-1)!} \frac{f(s,\tilde{x}(\tilde{g}(s)))}{\phi(x(t)/a)\psi(x(t)/a)} dtds .$$

Noting that $x(t)$ is increasing and using (b), (c), and (10), we see that

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if $t_2 \le s \le t_3$, then

$$\frac{f(s,\tilde{x}(\tilde{g}(s)))}{\phi(x(t)/a)\psi(x(t)/a)} \ge \frac{\phi(x(g_*(s)))}{\phi(x(t))}\frac{p(s)}{\psi(x(t)/a)}$$

$$\ge \frac{p(s)}{\psi([g_*(s)]^{n-1})} \quad \text{for} \quad t_1 \le t \le g_*(s) \ .$$

From the above observations it follows that

$$\text{(11)} \qquad \int_{t_2}^{t_3} \frac{[g_*(s)-t_1]^{n-1}p(s)}{\psi([g_*(s)]^{n-1})}ds \le a(n-1)! \int_{x(t_1)/a}^{x(t_3)/a} \frac{dr}{\phi(r)\psi(r)} \ .$$

In view of (3), letting $t_3 \to \infty$ in (11), we conclude that

$$\int_{t_2}^{\infty} \frac{[g_*(s)-t_1]^{n-1}p(s)}{\psi([g_*(s)]^{n-1})}ds < \infty \ ,$$

which contradicts (4). Thus, if n is even, then all solutions of (B) oscillate.

Let n be odd. If the integer k in (5) and (6) is positive, then the same argument as above leads to a contradiction. Therefore, k must be zero and we have

$$x(t) \downarrow c \ge 0 \ , \quad |x^{(i)}(t)| \downarrow 0 \quad \text{as} \quad t \uparrow \infty \ , \quad i = 1, \ldots, n-1 \ .$$

Suppose $c > 0$. Then from (B) we get

$$\text{(12)} \qquad x^{(n)}(t) + \phi(c)p(t) \le 0 \quad \text{for} \quad t \ge t_2 \ .$$

An integration of (12) multiplied by $(t-t_2)^{n-1}/(n-1)!$ yields

$$\phi(c)\int_{t_2}^{t} \frac{(s-t_2)^{n-1}}{(n-1)!}p(s)ds + \sum_{i=0}^{n-1} \frac{(t-t_2)^{i}}{i!}(-1)^{i}x^{(i)}(t) - x(t_2) \le 0$$

for $t \ge t_2$, from which, taking (6) into account, we obtain

$$\int_{t_2}^{\infty} (s-t_2)^{n-1}p(s)ds < \infty \ .$$

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But this clearly contradicts (4), and so c must be zero. Thus the proof is complete.

Applying Theorem B to the particular case where

$$f(t, y_1, \ldots, y_m) = p(t) \prod_{i=1}^{m} |y_i|^{\gamma_i} \operatorname{sgn} y_1 ,$$

we have the following

COROLLARY C. *Consider the differential equation*

$$x^{(n)}(t) + p(t) \prod_{i=1}^{m} |x(g_i(t))|^{\gamma_i} \operatorname{sgn} x(g_1(t)) = 0 , \tag{C}$$

where γ_i , $i = 1, \ldots, m$, *are nonnegative constants. If* $\gamma_1 + \ldots + \gamma_m > 1$ *and*

$$\int^{\infty} [g_*(t)]^{n-1} p(t) dt = \infty ,$$

then, for n *even, every solution* $x(t)$ *of* (C) *is oscillatory, and, for* n *odd, every solution* $x(t)$ *is either oscillatory or such that* $|x^{(i)}(t)| \downarrow 0$ *as* $t \uparrow \infty$, $i = 0, 1, \ldots, n-1$.

We give an example which shows that Corollary C actually improves Theorem A.

EXAMPLE. Consider the equation

$$x''(t) + t^{-7/4}[x(g(t))]^3 = 0 , \quad g(t) = t + (t-t^{\frac{1}{2}}) \sin t . \tag{13}$$

Since $g_*(t) = t$ for $2k\pi \le t \le (2k+1)\pi$, $k = 1, 2, \ldots$, we have

$$\int_{2\pi}^{\infty} t^{-7/4} g_*(t) dt \ge \sum_{k=1}^{\infty} \int_{2k\pi}^{(2k+1)\pi} t^{-7/4} \cdot t dt$$
$$\ge \pi \sum_{k=1}^{\infty} [(2k+1)\pi]^{-3/4} = \infty .$$

Hence, by Corollary C, all solutions of (13) are oscillatory.

On the other hand, Theorem A cannot be applied to (13), for there does not exist a function $h(t)$ which satisfies both (1) and (2). In fact, if $h(t)$ is any function satisfying (1), then

$$h(t) \le h(\{2k+(3/2)\}\pi) \le g(\{2k+(3/2)\}\pi) = [\{2k+(3/2)\}\pi]^{1/2}$$

for $\{2k-(1/2)\}\pi \le t \le \{2k+(3/2)\}\pi$, $k = 1, 2, \ldots$, so that we find

$$\int_{3\pi/2}^{\infty} t^{-7/4} h(t)dt = \sum_{k=1}^{\infty} \int_{(2k-1/2)\pi}^{(2k+3/2)\pi} t^{-7/4} h(t)dt$$

$$\le 2\pi \sum_{k=1}^{\infty} [\{2k-(1/2)\}\pi]^{-7/4} [\{2k+(3/2)\}\pi]^{1/2} < \infty .$$

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Department of Mathematics,
Faculty of Education,
Nagasaki University,
Nagasaki,
Japan;

Department of Mathematics,
Faculty of Science,
Hiroshima University,
Hiroshima,
Japan.

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