3

Families of Stable Varieties

We have defined stable and locally stable families over one-dimensional regular schemes in Sections 2.1 and 2.4. The first task in this chapter is to define these notions for families over more general base schemes. It turns out that this is much easier if there is no boundary divisor Δ. Since this case is of considerable interest, we treat it here before delving into the general setting in the next chapter. While restricting to the special case saves quite a lot of foundational work, the key parts of the proofs of the main theorems stay the same. To avoid repetition, we outline the proofs here, but leave detailed discussions to Chapter 4.

In Section 3.1 we review the theory of Chow varieties and Hilbert schemes. In general these suggest different answers to what a "family of varieties" or a "family of divisors" should be. The main conclusions, (3.11) and (3.13), can be summarized in the following principles:

- A *family of S_2 varieties* should be a flat morphism whose geometric fibers are reduced, connected, and satisfy Serre's condition S_2.
- Flatness is not the right condition for divisors on the fibers.

As in (2.46), a morphism $f\colon (X, \Delta) \to S$ is stable iff it is locally stable, proper and $K_{X/S} + \Delta$ is f-ample. Thus the key question is the right concept of local stability. There are many equivalent ways to define it when $\Delta = 0$.

Definition–Theorem 3.1 (Local stability over reduced schemes) Let S be a reduced scheme over a field of characteristic 0 and $f\colon X \to S$ a flat morphism of finite type whose fibers are semi-log-canonical (slc). Then f is *locally stable* iff the following equivalent conditions are satisfied:

(3.1.1) $K_{X/S}$ is $\mathbb{Q}$-Cartier.

(3.1.2) $\omega_{X/S}^{[m]}$ is an invertible sheaf for some $m > 0$.

(3.1.3) $\omega_{X/S}^{[m]}$ is flat with S_2 fibers for every $m \in \mathbb{Z}$.

(3.1.4) The restriction $\omega_{X/S}^{[m]} \to \omega_{X_s}^{[m]}$ is surjective for every $s \in S$ and $m \in \mathbb{Z}$.

(3.1.5) For every reduced W and morphism $q : W \to S$, the natural map

$$q_X^*(\omega_{X/S}^{[m]}) \to \omega_{X_W/W}^{[m]} \quad \text{is an isomorphism for every } m \in \mathbb{Z}.$$

(3.1.6) For every spectrum of a DVR T and morphism $q : T \to S$, the pullback $f_T : X_T \to T$ satisfies the above (1–5).

(3.1.7) There is a closed subset $Z \subset X$ such that $\operatorname{codim}(Z \cap X_s, X_s) \geq 3$ for every $s \in S$, and $f|_{X \setminus Z} : (X \setminus Z) \to S$ satisfies the above (1–6).

We prove the equivalence in (3.37). Over nonreduced bases, local stability is defined by (3.1.3); see (3.40). It implies all the other properties in (3.1), but is not equivalent to them; see Section 6.6 for such examples. The situation turns out to be much more complicated when $\Delta \neq 0$. Chapters 4 and 7 are entirely devoted to finding the right answers.

Let now $f : X \to S$ be a flat, projective family of S_2 varieties. It turns out that, starting in relative dimension 3, the set of points

$$\{s \in S : X_s \text{ is semi-log-canonical}\}$$

is not even locally closed; see (3.41) for an example. In order to describe the situation, in Section 3.2 we study functors that are representable by a locally closed decomposition (10.83).

We start the study of families of non-Cartier divisors in Section 3.3. As we have noted, this is one of the key new technical issues of the theory.

In Section 3.4 we use a representability theorem (3.36) to clarify the definition of stable and locally stable families, leading to the proof of (3.1). In Section 3.5 we bring these results together in (3.42) to prove the next main theorem of the chapter.

Theorem 3.2 (Local stability is representable) *Let S be a scheme over a field of characteristic 0 and $f : X \to S$ a projective morphism. Then there is a locally closed partial decomposition (10.83) $j : S^{ls} \to S$ such that the following holds.*

Let W be a scheme and $q : W \to S$ a morphism. Then the family obtained by base change $f_W : X_W \to W$ is locally stable iff q factors as $q : W \to S^{ls} \to S$.

Since ampleness is an open condition for a $\mathbb{Q}$-Cartier divisor, (3.2) implies the following.

Corollary 3.3 (Stability is representable) *Let S be a scheme over a field of characteristic 0 and $f : X \to S$ a projective morphism. Then there is a locally closed partial decomposition $j : S^{stab} \to S$ such that the following holds.*

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Let W be a reduced scheme and $q\colon W \to S$ a morphism. Then the family obtained by base change $f_W\colon X_W \to W$ is stable iff q factors as $q\colon W \to S^{stab} \to S$. □

Aside from some generalities, we have all the ingredients in place to construct the coarse moduli space of stable varieties. To formulate it, let $\mathcal{SV}$ (for stable varieties) denote the functor that associates to a scheme S the set of all stable families $f\colon X \to S$, up to isomorphism.

In order to get a moduli space of finite type, we fix the relative dimension n and the volume $v = \mathrm{vol}(K_{X_s}) := (K_{X_s}^n)$ of the fibers. This gives the subfunctor $\mathcal{SV}(n, v)$. The proof of the following is given in (6.18).

Theorem 3.4 (Moduli space of stable varieties) *Let S be a base scheme of characteristic 0 and fix n, v. Then the functor $\mathcal{SV}(n, v)$ has a coarse moduli space $\mathrm{SV}(n, v) \to S$, which is projective over S.*

Assumptions We work over arbitrary schemes in Sections 3.1–3.3, but over a field of characteristic 0 starting with Section 3.4.

3.1 Chow Varieties and Hilbert Schemes

What is a good family of algebraic varieties? Historically, two answers emerged to this question. The first one originates with Cayley (1860, 1862).[1] The corresponding moduli space is usually called the Chow variety. The second one is due to Grothendieck (1962); it is the theory of Hilbert schemes. For both of them, see Kollár (1996, chap.I), Sernesi (2006), or the original sources for details.

For the purposes of the following general discussion, a variety is a proper, geometrically reduced, and pure dimensional k-scheme.

The theory of Chow varieties suggests the following.

Definition 3.5 A *Cayley–Chow family* of varieties over a reduced base scheme S is a proper, pure dimensional (2.71) morphism $f : X \to S$, whose fibers X_s are generically geometrically reduced for every $s \in S$.

This is called an *algebraic family of varieties* in Hartshorne (1977, p.263). More general Cayley–Chow families are defined in Kollár (1996, sec.I.3).

It seems hard to make a precise statement, but one can think of Cayley–Chow families as being "topologically flat." That is, any topological

[1] The two papers have identical titles

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consequence of flatness also holds for Cayley–Chow families. This holds for the Zariski topology, but also for the Euclidean topology if we are over $\mathbb{C}$.

There are two disadvantages of Cayley–Chow families. First, basic numerical invariants, for example, the arithmetic genus of curves, can jump in a Cayley–Chow family. Second, the topological nature of the definition implies that we completely ignore the nilpotent structure of S. In fact, it really does not seem possible to define what a Cayley–Chow family should be over an Artinian base scheme S.

The theory of Hilbert schemes was introduced to solve these problems. It suggests the following definition.

Definition 3.6 A *Hilbert–Grothendieck family* of varieties is a proper, flat morphism $f : X \to S$ whose fibers X_s are geometrically reduced and pure dimensional. (Here S is allowed to be nonreduced.)

Every Hilbert–Grothendieck family is also a Cayley–Chow family, and technically it is much better to have a Hilbert–Grothendieck family than a Cayley–Chow family. However, there are many Cayley–Chow families that are not flat.

3.7 (Universal families) Both Cayley–Chow and Hilbert–Grothendieck families are preserved by pull-backs, thus they form a functor. In both cases, this functor has a fine moduli space if we work with families that are subvarieties of a given scheme Y/S.

Let us thus fix a scheme Y that is projective over a base scheme S. For general existence questions, the key case is $Y = \mathbb{P}^N_S$. For any closed subscheme $Y \subset \mathbb{P}^N_S$, the Chow variety (resp. the Hilbert scheme) of Y is naturally a subvariety (resp. subscheme) of the Chow variety (resp. the Hilbert scheme) of $\mathbb{P}^N_S$. The corresponding universal family is obtained by restriction. (See (3.15) or (Kollár, 1996, sec.I.5) for some cases when Y/S is not projective.)

3.7.1 (*Chow variety*) (See Section 4.8 or Kollár (1996, sec.I.3) for details, and (3.14) for comments on seminormality.) There is a seminormal S-scheme $\mathrm{Chow}^\circ(Y/S)$ and a universal family $\mathrm{Univ}^\circ(Y/S) \to \mathrm{Chow}^\circ(Y/S)$ that represents the functor

$$\mathcal{C}how^\circ(Y/S)(T) := \left\{ \begin{array}{c} \text{closed subsets } X \subset Y \times_S T \text{ such that} \\ X \to T \text{ is a Cayley–Chow family of varieties} \end{array} \right\}$$

on seminormal S-schemes $q : T \to S$. ($\mathrm{Chow}^\circ(Y/S)$ is the "open" part of the full $\mathrm{Chow}(Y/S)$, as defined in Kollár (1996, Sec.I.3).) If we also fix a relatively very ample line bundle $\mathcal{O}_Y(1)$, then we can write

$$\mathrm{Chow}^\circ(Y/S) = \amalg_n \mathrm{Chow}^\circ_n(Y/S) = \amalg_{n,d} \mathrm{Chow}^\circ_{n,d}(Y/S).$$

Here Chow°_n parametrizes varieties of dimension n and $\mathrm{Chow}^\circ_{n,d}$ varieties of dimension n and of degree d. Each $\mathrm{Chow}^\circ_{n,d}(Y/S)$ is of finite type, but usually still reducible.

3.7.2 (*Hilbert scheme*) (See Kollár (1996, Sec.I.1) or Sernesi (2006).) There is a universal family $\mathrm{Univ}^\circ(Y/S) \to \mathrm{Hilb}^\circ(Y/S)$ that represents the functor of Hilbert–Grothendieck families

$$\mathcal{H}ilb^\circ(Y/S)(T) := \left\{ \begin{array}{c} \text{closed subschemes } X \subset Y \times_S T \text{ such that} \\ X \to T \text{ is a flat family of varieties} \end{array} \right\}.$$

More generally, there is a universal family $\mathrm{Univ}(Y/S) \to \mathrm{Hilb}(Y/S)$ that represents the functor

$$\mathcal{H}ilb(Y/S)(T) := \left\{ \begin{array}{c} \text{closed subschemes } X \subset Y \times_S T \\ \text{such that } X \to T \text{ is flat} \end{array} \right\}.$$

We can write $\mathrm{Hilb}(Y/S) = \amalg_n \mathrm{Hilb}_n(Y/S) = \amalg_H \mathrm{Hilb}_H(Y/S)$. Here Hilb_n parametrizes subschemes of (not necessarily pure) dimension n, and Hilb_H subschemes with Hilbert polynomial $H(t)$. Each $\mathrm{Hilb}_H(Y/S)$ is projective, but usually still reducible.

3.8 (Comparing Chow and Hilb) Given a subscheme $X \subset Y$ of dimension $\le n$, we get an n dimensional cycle $[X] = \sum_i m_i[X_i]$, where X_i are the n-dimensional irreducible components and m_i is the length of $\mathcal{O}_X$ at the generic point of X_i. (Thus we completely ignore the lower dimensional irreducible components.)

If $m_i = 1$ for every i, then $[X] = \sum_i [X_i]$ can be identified with a point in $\mathrm{Chow}^\circ(Y/S)$. In order to make this map everywhere defined, we need to extend the notion of Cayley–Chow families to allow fibers that are formal linear combinations of varieties; see Kollár (1996, sec.I.3) for details. The end result is an everywhere defined, set-theoretic map $\mathrm{Hilb}_n(Y/S) \dashrightarrow \mathrm{Chow}_n(Y/S)$. Since $\mathrm{Hilb}_n(Y/S)$ is a scheme, but $\mathrm{Chow}_n(Y/S)$ is a seminormal variety, it is better to think of it as a morphism defined on the seminormalization

$$\mathcal{R}^H_C : \mathrm{Hilb}_n(Y/S)^{\mathrm{sn}} \to \mathrm{Chow}_n(Y/S). \qquad (3.8.1)$$

This is a very complicated morphism. As written, its fibers have infinitely many irreducible components for $n \ge 1$, since we can just add disjoint zero-dimensional subschemes to any variety $X \subset Y$ to get new subschemes with the same underlying variety. Even if we restrict to pure dimensional subschemes, we get fibers with infinitely many irreducible components. This happens, for instance, for the fiber over $m[L] \in \mathrm{Chow}_{1,m}(\mathbb{P}^3)$, where $L \subset \mathbb{P}^3$ is a line and $m \geq 2$.

It is much more interesting to understand what happens on

$$\overline{\mathrm{Hilb}^\circ_n}(Y/S) := \text{ closure of } \mathrm{Hilb}^\circ_n(Y/S) \text{ in } \mathrm{Hilb}_n(Y/S). \tag{3.8.2}$$

That is, $\overline{\mathrm{Hilb}^\circ_n}(Y/S)$ parametrizes n-dimensional subschemes that occur as limits of varieties. It turns out that the restriction of the Hilbert-to-Chow map

$$\mathcal{R}^H_C : \overline{\mathrm{Hilb}^\circ_n}(Y/S)^{\mathrm{sn}} \to \mathrm{Chow}_n(Y/S) \tag{3.8.3}$$

is a local isomorphism at many points. For smooth varieties this is quite clear from the definition of Chow-forms. Classical writers seem to have been fully aware of various equivalent versions, but I did not find an explicit formulation. The normal case, due to Hironaka (1958), is quite surprising; see Hartshorne (1977, III.9.11) for its usual form and (10.72) for a stronger version. These imply the following comparison of Hilbert schemes and Chow varieties.

Theorem 3.9 *Using the notation of (3.8), let $s \in S$ be a point and $X_s \subset Y_s$ a geometrically normal, projective subvariety of dimension n. Then the Hilbert-to-Chow morphism*

$$\mathcal{R}^H_C : \overline{\mathrm{Hilb}^\circ_n}(Y/S)^{sn} \to \mathrm{Chow}_n(Y/S)$$

is a local isomorphism over $[X_s] \in \mathrm{Chow}_n(Y/S)$. □

Informally speaking, for normal varieties, the Cayley–Chow theory is equivalent to the Hilbert–Grothendieck theory, at least over seminormal base schemes.

By contrast, $\mathrm{Hilb}(Y/S)$ and $\mathrm{Chow}(Y/S)$ are different near the class of a singular curve. For example, let $B \subset \mathbb{P}^3$ be a planar, nodal cubic. Then $[B]$ is contained in one irreducible component of $\mathrm{Hilb}_1(\mathbb{P}^3)$, but in two different irreducible components of $\mathrm{Chow}_1(\mathbb{P}^3)$. A general member of one component is a planar, smooth cubic. This component parametrizes flat deformations. A general member of the other component is a smooth, rational, nonplanar cubic. The arithmetic genus jumps, so these deformations are not flat. $\mathcal{R}^H_C$ is not a local isomorphism over $[B] \in \mathrm{Chow}_1(\mathbb{P}^3)$, but this is explained by the change of the genus. Once we correct for the genus change, (3.9) becomes stronger.

Definition 3.10 Let $X \subset \mathbb{P}^N$ be a closed subscheme of pure dimension n. The *sectional genus* of X is $1 - \chi(X \cap L, \mathscr{O}_{X\cap L})$, where $X \cap L$ is the intersection of X with $n-1$ general hyperplanes. Knowing the degree of X and its sectional genus is equivalent to knowing the two highest coefficients of its Hilbert polynomial.

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It is easy to see that the sectional genus is a constructible and upper semi-continuous function on $\mathrm{Chow}^{\circ}_{n}(Y/S)$; see (5.36). Thus there are locally closed subschemes $\mathrm{Chow}^{\circ}_{n,*,g}(Y/S) \subset \mathrm{Chow}^{\circ}_{n}(Y/S)$ that parametrize geometrically reduced cycles with sectional genus g; see (10.83). (The $*$ stands for the degree which we ignore in these formulas.) We can now define the Chow variety parametrizing families with locally constant sectional genus as

$$\mathrm{Chow}^{\mathrm{sg}}_{n}(Y/S) := \amalg_{n,g}\, \mathrm{Chow}^{\circ}_{n,*,g}(Y/S)^{\mathrm{sn}},$$

the disjoint union of the seminormalizations of the $\mathrm{Chow}^{\circ}_{n,*,g}(Y/S)$.

The sectional genus is constant in a flat family, thus we get the following strengthening of (3.9); see (5.36) and (10.71).

Theorem 3.11 *Using the notation of (3.8), let $s \in S$ be a point and $X_s \subset Y_s$ a geometrically reduced, projective, S_2 subvariety of pure dimension n. Then the Hilbert-to-Chow map*

$$\mathcal{R}^{H}_{C} : \overline{\mathrm{Hilb}^{\circ}_{n}}(Y/S)^{sn} \to \mathrm{Chow}^{sg}_{n}(Y/S)$$

is a local isomorphism over $[X_s] \in \mathrm{Chow}^{sg}_{n}(Y/S)$.

We can informally summarize these considerations as follows.

Principle 3.12 *For geometrically reduced, pure dimensional, projective, S_2 varieties, the Cayley–Chow theory is equivalent to the Hilbert–Grothendieck theory over seminormal base schemes, once we correct for the sectional genus.*

We are studying not just varieties, but slc pairs (X, Δ). The underlying variety is demi-normal, hence geometrically reduced and S_2. Thus (3.12) says that even if we start with the more general Cayley–Chow families, we end up with flat morphisms $f : X \to S$ with S_2 fibers. The latter is a class that is well-behaved over arbitrary base schemes.

However, the divisorial part is harder to understand. Although we have seen only a few examples supporting it, the following counterpart of (3.12) turns out to give the right picture.

Principle 3.13 *For stable families of slc pairs (X, Δ), the Hilbert–Grothendieck theory is optimal for the underlying variety X, but the Cayley–Chow theory is the "right" one for the divisorial part Δ.*

3.14 (Comment on seminormality) Hilbert schemes work well over any base scheme, but in Kollár (1996) the theory of Cayley–Chow families is developed only over seminormal bases. Following the methods of Section 4.8, it is possible to work out the Cayley–Chow theory of geometrically reduced cycles over reduced bases. In characteristic 0 this works for all cycles by Barlet (1975); Barlet and Magnússon (2020), but examples of Nagata (1955) suggest that, in positive characteristic, the restriction to seminormal bases may be necessary.

3.15 (Nonprojective cases) Let Y be an algebraic space over S. We define $\mathcal{H}ilb(Y/S)(T)$ as the set of all subspaces $X \subset Y \times_S T$ that are proper and flat over T. Artin (1969) proves that if $Y \to S$ is locally of finite presentation then the Hilbert functor is represented by an algebraic space $\operatorname{Hilb}(Y/S) \to S$ that is also locally of finite presentation.

Most likely similar results hold for $\operatorname{Chow}(Y/S)$; see Kollár (1996, sec.I.5). The complex analytic case is worked out in Barlet and Magnússon (2020).

3.2 Representable Properties

Let $\mathcal{P}$ be a property of schemes. For a morphism $f : X \to S$ consider the set $S(\mathcal{P}) := \{s \in S : X_s \text{ satisfies } \mathcal{P}\}$. Note that $S(\mathcal{P})$ depends on $f : X \to S$, so we use the notation $S(\mathcal{P}, X/S)$ if the choice of $f : X \to S$ is not clear.

In nice situations, $S(\mathcal{P})$ is an open or closed subset of S. For example satisfying Serre's condition S_m is an open condition for proper, flat morphisms by (10.12), and being singular is a closed condition.

Similarly, if $f : X \to S$ is a proper morphism of relative dimension 1, then $\{s \in S : X_s \text{ is a stable curve}\}$ is an open subset of S. However, we see in (3.41), that if $f : X \to S$ is a proper, flat morphism of relative dimension ≥ 3 then $\{s \in S : X_s \text{ is a stable variety}\}$ is not even a locally closed subset of S.

We already noted in Section 1.4 that flat morphisms with stable fibers do not give the right moduli problem in higher dimensions. One should look at stable families instead. Thus our main interest is in the class of morphisms $q : T \to S$ for which the pulled-back family $f_T : X_T \to T$ is stable. We then hope to prove that this happens in a predictable way. The following definition formalizes this.

Definition 3.16 Let $\mathcal{P}$ be a property of morphisms that is preserved by pull-back. That is, if $X \to S$ satisfies $\mathcal{P}$ and $q\colon T \to S$ is a morphism, then $f_T\colon X_T \to T$ also satisfies $\mathcal{P}$. Depending on the situation, pull–back can mean the usual fiber product $X_T := X \times_S T$, the hull pull-back to be defined in (3.27),

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the divisorial pull-back to be defined in (4.6), or the Cayley–Chow pull-back of Kollár (1996, I.3.18).

The functor of $\mathcal{P}$-pull-backs is defined for morphisms $T \to S$ by setting

$$Property(\mathcal{P})(T) := \begin{cases} \{\emptyset\} & \text{if } X_T \to T \text{ satisfies } \mathcal{P}\text{, and} \\ \emptyset & \text{otherwise.} \end{cases} \tag{3.16.1}$$

(That is, $Property(\mathcal{P})(T)$ is either empty or consists of a single element.) Thus a morphism $i_P : S^P \to S$ *represents* $\mathcal{P}$-pull-backs iff the following hold:

(3.16.2) $f^P : X^P := X_{S^P} \to S^P$ satisfies $\mathcal{P}$, and

(3.16.3) if $f_T : X_T \to T$ satisfies $\mathcal{P}$, then q factors as $q : T \to S^P \to S$, and the factorization is unique.

It is also of interest to understand what happens if we focus on special classes of bases. Let $\mathcal{R}$ be a property of schemes. We say that $i_P : S^P \to S$ *represents* $\mathcal{P}$-pull-backs for $\mathcal{R}$-schemes if S^P satisfies $\mathcal{R}$ and (3.16.3) holds whenever T satisfies $\mathcal{R}$. In this section we are mostly interested in the properties $\mathcal{R}$ = (reduced), $\mathcal{R}$ = (seminormal), and $\mathcal{R}$ = (normal).

If (3.16.3) holds for all T = (spectrum of a field), then $i_P : S^P \to S$ is geometrically injective (10.82). If (3.16.3) holds for all Artinian schemes, then i_P is a monomorphism (10.82).

In many cases of interest, $\mathcal{P}$ is invariant under base field extensions. Then $i_P : S^P \to S$ also preserves residue fields (10.82).

If $X \to S$ is projective, then we are frequently able to prove that $i_P : S^P \to S$ is a locally closed partial decomposition (10.83).

If $i_P : S^P \to S$ represents $\mathcal{P}$-pull-backs and i_P is of finite type (this will always be the case for us), then $S(\mathcal{P}) = \{s : X_s \text{ satisfies } \mathcal{P}\} = i_P(S^P)$ is a constructible subset of S. Constructibility is much weaker than representability, but we frequently need it in our proofs of representability.

Example 3.17 (Simultaneous normalization) Sometimes it is best to focus not on a property of a morphism, but on a property of its "improvement." We say that $f : X \to S$ has *simultaneous normalization* if there is a finite morphism $\pi : \bar{X} \to X$ such that $\pi_s : \bar{X}_s \to X_s$ is the normalization for every $s \in S$ and $f \circ \pi : \bar{X} \to S$ is flat. For example, consider the family of quadrics

$$X := (x_0^2 - x_1^2 + u_2 x_2^2 + u_3 x_3^2 = 0) \subset \mathbb{P}^3_{\mathbf{x}} \times \mathbb{A}^2_{\mathbf{u}}.$$

Then $\{(0,0)\} \amalg (\mathbb{A}^2_{\mathbf{u}} \setminus \{(0,0)\}) \to \mathbb{A}^2_{\mathbf{u}}$ represents the functor of simultaneous normalizations. In general, we have the following result, due to Chiang-Hsieh and Lipman (2006) and Kollár (2011b).

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Claim 3.17.1 Let $f : X \to S$ be a proper morphism whose fibers X_s are generically geometrically reduced. Then there is a morphism $\pi : S^{\mathrm{sn}} \to S$ such that, for any $g : T \to S$, the fiber product $X \times_S T \to T$ has a simultaneous normalization iff g factors through $\pi : S^{\mathrm{sn}} \to S$. □

Definition 3.18 Let $f : X \to S$ be a morphism and F a coherent sheaf on X. Given any $q : W \to S$, we get

$$\begin{array}{ccc} X \times_S W =: X_W & \xrightarrow{q_X} & X \\ \downarrow f_W & & \downarrow f \\ W & \xrightarrow{q} & S. \end{array} \tag{3.18.1}$$

As in (3.16.1), we have the *functor of flat pull-backs* $Flat(F)(*)$.

One of the most useful representation theorems is the following; see Mumford (1966, Lect.8) and Artin (1969).

Theorem 3.19 (Flattening decomposition) *Let $f : X \to S$ be a proper morphism and F a coherent sheaf on X. Then the functor of flat pull-backs $Flat(F)(*)$ is represented by a monomorphism $i^{flat} : S^{flat} \to S$ that is locally of finite type. If f is projective then i^{flat} is a locally closed decomposition.* □

Example 3.19.1 As a trivial special case, assume that $X = S$ is affine. Write F as the cokernel of a map of free sheaves $g : \mathscr{O}_S^n \to \mathscr{O}_S^m$. Then F is free of rank $m - r$ precisely on the subscheme $(\operatorname{rank} g \le r) \setminus (\operatorname{rank} g \le r - 1)$.

One can frequently check flatness using the following numerical criterion which is proved, but not fully stated, in Hartshorne (1977, III.9.9).

Theorem 3.20 *Let $f : X \to S$ be a projective morphism with relatively ample $\mathscr{O}_X(1)$ and F a coherent sheaf on X. The following are equivalent:*

(3.20.1) *F is flat over S.*

(3.20.2) *$f_*(F(m))$ is locally free for $m \gg 1$.*

If S is reduced then these are also equivalent to the following:

(3.20.3) *$s \mapsto \chi(X_s, F_s(m))$ is a locally constant function on S.* □

In Chapter 8 we need the following results.

Proposition 3.21 *Let $f : X \to S$ be a proper morphism and G a coherent sheaf on X, flat over S. The following properties of morphisms $q : T \to S$ are representable by locally closed subschemes:*

(3.21.1) *$(f_T)_* q_X^* G$ is locally free of rank r and commutes with base change.*

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(3.21.2) *$(f_T)_*q_X^*G$ is locally free of rank r, commutes with base change, and q_X^*G is relatively globally generated.*

Proof Using the notation of (3.24.1), locally we can write d_1 in as a matrix with entries in $\mathscr{O}_S$. Then $(\operatorname{rank} d_1 \le r) \subset S$ is the subscheme defined by the vanishing of the determinants of all $(r+1)\times(r+1)$-minors. With this definition we see that $(\operatorname{rank} d_1 \le \operatorname{rank} K^0 - r) \setminus (\operatorname{rank} d_1 \le \operatorname{rank} K^0 - r - 1)$ represents the functor (3.21.1).

For (3.21.2) we may assume that f_*G is locally free of rank r. Then (2) is represented by the open subscheme $S \setminus f(\operatorname{Supp}\operatorname{coker}(f^*f_*G \to G))$. □

Corollary 3.22 *Let $f : X \to S$ be a proper morphism and G a coherent sheaf on X, flat over S. Assume that $H^0(X_s, \mathscr{O}_{X_s}) \simeq k(s)$ for $s \in S$.*

Then there is a locally closed subscheme $S' \hookrightarrow S$ such that, a morphism $q : T \to S$ factors through S' iff q_X^*G is isomorphic to the pull-back of a line bundle from T.

Proof If q_X^*G is isomorphic to the the pull-back of a line bundle from T, then $\mathscr{O}_{X_T}$ is locally isomorphic to q_X^*G, hence X_T is flat over T. Thus S' factors through the flattening decomposition of f (3.19). We may thus assume that f is flat and S is affine. Since $H^0(S, \mathscr{O}_S) \to H^0(X_s, \mathscr{O}_{X_s})$ is surjectve, so is $H^0(X, \mathscr{O}_X) \to H^0(X_s, \mathscr{O}_{X_s})$, hence $f_*\mathscr{O}_X \simeq \mathscr{O}_S$ by cohomology and base change. So we are in the $r = 1$ case of (3.21.2). □

Remark 3.23 Being pure dimensional is an open property for flat, proper morphisms. Thus, using (3.19) we obtain that for any projective morphism $f : X \to S$ we have a locally closed partial decomposition $S^{\text{fp}} \to S$ that represents flat and pure dimensional pull-backs of f. Next let $\mathcal{P}$ be a property that implies flat and pure dimensional. Assume that $q : T \to S$ is a morphism such that $f_T : X_T \to T$ satisfies $\mathcal{P}$. Then $f_T : X_T \to T$ is also flat and pure dimensional, hence $q : T \to S$ factors through f^{fp}. Thus $S^{\text{P}} = (S^{\text{fp}})^{\text{P}}$.

In particular, if we want to prove that $S^{\text{P}} \to S$ exists for all projective morphisms, then it is enough to show that it exists for all flat, pure dimensional and projective morphisms. More generally, if $\mathcal{P}_1 \Rightarrow \mathcal{P}_2$ and S^{P_2} exists, then

$$S^{\text{P}_1} = (S^{\text{P}_2})^{\text{P}_1}. \tag{3.23.1}$$

3.24 (Semicontinuity) Let $f : X \to S$ be a proper morphism and G a coherent sheaf on X, flat over S. By a version of the semicontinuity theorem, there is a finite complex of locally free sheaves on S

$$K^\bullet := 0 \to K^0 \xrightarrow{d_1} K^1 \xrightarrow{d_2} \cdots \xrightarrow{d_{n-1}} K^n \to 0, \tag{3.24.1}$$

such that, for every morphism $h : T \to S$,

$$R^i(f_T)_* h_X^* G \simeq H^i(h^* K^\bullet). \tag{3.24.2}$$

(This form is stated and proved in Mumford (1970, §5); Hartshorne (1977, III.12.2) has a weaker statement, but the proof works to give this.)

This can be used to define

$$\det R^\bullet f_* G := (\textstyle\prod_{\text{even}} \det K^i) \otimes (\textstyle\prod_{\text{odd}} \det K^i)^{-1}. \tag{3.24.3}$$

This is independent of the choices made. If $R^i f_* G = 0$ for $i > 0$, then $\det R^\bullet f_* G = \det f_* G$. This is the main case that we use.

3.3 Divisorial Sheaves

We frequently have to deal with divisors $D \subset X$ that are not Cartier, hence the corresponding sheaves $\mathscr{O}_X(D)$ are not always locally free. Understanding families of such sheaves is a key aspect of the moduli problem. Many of the results proved here are developed for arbitrary coherent sheaves in Chapter 9.

Definition 3.25 (Divisorial sheaves) A coherent sheaf L on a scheme X is called a *divisorial sheaf* if L is S_2 and there is a closed subset $Z \subset X$ of codimension ≥ 2 such that $L|_{X \setminus Z}$ is locally free of rank 1.

We are mostly interested in the cases when X itself is demi-normal, but the definition makes sense in general, although with unexpected properties. For example, $\mathscr{O}_X$ is a divisorial sheaf iff X is S_2.

Set $U := X \setminus Z$ and let $j : U \hookrightarrow X$ denote the natural injection. Then $L = j_*(L|_U)$ by (10.6), thus L is uniquely determined by $L|_U$.

If $\dim X = 1$, then $Z = \emptyset$, so a divisorial sheaf is invertible. If D is a Mumford divisor, then $\mathscr{O}_X(D)$ is a divisorial sheaf. If X is demi-normal, then the $\omega_X^{[m]}$ are divisorial sheaves. Divisorial sheaves form a group, with

$$L [\otimes] M := j_*(L|_U \otimes M|_U). \tag{3.25.1}$$

For powers, we use the notation $L^{[m]} := (L^{\otimes m})^{[**]}$.

Let $H \subset X$ be a general member (depending on L, M) of a base point free linear system. Then $L|_H, M|_H$ are divisorial sheaves and $(L [\otimes] M)|_H = L|_H [\otimes] M|_H$; see (10.18).

Let $f : X \to S$ be a morphism. A coherent sheaf L on X is a *flat family of divisorial sheaves,* if L is flat over S and its fibers are divisorial sheaves. (L need not be a divisorial sheaf on X.)

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Given any $q : T \to S$ with induced $q_X : X \times_S T \to X$, the pull-back $q_X^* L$ is again a flat family of divisorial sheaves.

Let $f : X \to S$ be a morphism. We frequently need to deal with properties that hold not everywhere, but only on open subsets of each fiber.

Definition 3.26 Let $f : X \to S$ be a morphism and F a coherent sheaf on X. We say that F is *generically flat* (resp. *mostly flat*) over S, if there is a dense, open subset $j : U \hookrightarrow X$ such that

(3.26.1) $F|_U$ is flat over S, and

(3.26.2) $\operatorname{Supp} F_s \setminus U$ has codimension ≥ 1 (resp. ≥ 2) in $\operatorname{Supp} F_s$ for $s \in S$.

We usually set $Z := X \setminus U$.

A subscheme $Y \subset X$ is generically (resp. mostly) flat iff $\mathscr{O}_Y$ is.

Definition 3.27 (Hull and hull pull-back) With $j : U \hookrightarrow X$ as in (3.26), let F be a mostly flat family of coherent sheaves. Assume that $F|_U$ has S_2 fibers. We imagine that F is the “correct” object over U, but a mistake may have been made over $Z = X \setminus U$. We correct F by replacing it with its *hull*

$$F^H := j_*(F|_U). \tag{3.27.1}$$

Under mild conditions (for example, when X is excellent), F^H is a coherent sheaf on X; see Chapter 9 for a detailed treatment of hulls.

Let $q : W \to S$ be a morphism. We get a fiber product diagram as in (3.18.1). Then $F_W := q_X^* F$ has S_2 fibers over $q_X^{-1}(U)$. Its hull F_W^H is called the *hull pull-back* of F. If confusion is likely, we use $(F_W)^H$ to denote the hull of the pull-back and $(F^H)_W$ to denote pull-back of the hull F^H.

We are especially interested in the maps

$$r_W^S : (F^H)_W \to (F_W)^H. \tag{3.27.2}$$

We have already encountered these in (2.75) when $W = \{s\}$ is a point. For applications the key is to understand when F^H is flat. The following basic observations guide us:

(3.27.3) F^H is flat with S_2 fibers over a dense, open $S^\circ \subset S$ by (10.11).

(3.27.4) We see in (9.36) that F^H is flat with S_2 fibers $\Leftrightarrow$ r_W^S is an isomorphisms for every $q : W \to S$ $\Leftrightarrow$ r_s^S is surjective for every $s \in S$.

Definition 3.28 Using the notation of (3.26), F is a *mostly flat family of S_2 sheaves* if $F|_U$ is flat with S_2 fibers and $F = F^H$.

L is a *mostly flat family of divisorial sheaves* if L is invertible on U.

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For now, we study these problems for divisorial sheaves. The first main result is the following special case of (9.40), the second is (4.32).

Theorem 3.29 *Let $f : X \to S$ be a projective morphism and L a mostly flat family of divisorial sheaves on X (3.28). Then there is a locally closed decomposition $j : S^{H\text{-}flat} \to S$ such that, for every morphism $q : W \to S$, the hull pull-back L_W^H is a flat family of divisorial sheaves (3.25) on X_W, iff q factors as $q : W \to S^{H\text{-}flat} \to S$.* □

Corollary 3.30 *Let $f : X \to S$ be a flat, projective morphism with S_2 fibers and L a mostly flat family of divisorial sheaves on X. Then there is a locally closed partial decomposition $j : S^{inv} \to S$ such that, for every morphism $q : W \to S$, the hull pull-back L_W^H is invertible, iff q factors as $q : W \to S^{inv} \to S$.*

Proof For flat morphisms with S_2 fibers, an invertible sheaf is also a flat family of divisorial sheaves. Thus if L_W^H is invertible, then q factors through $S^{\text{H-flat}} \to S$. So, by (3.23.1), $S^{\text{inv}} = (S^{\text{H-flat}})^{\text{inv}}$. For a flat family of sheaves, being invertible is an open condition, thus S^{inv} is open in $S^{\text{H-flat}}$. □

The following variant turns out to be very useful in (3.42) and (6.24).

Proposition 3.31 *Let $f : X \to S$ be a flat, projective morphism with S_2 fibers and $N_1, \dots, N_s, L_1, \dots, L_r$ mostly flat families of divisorial sheaves. Then there is a locally closed partial decomposition $S^{NL} \to S$ such that, a morphism $q : T \to S$ factors through S^{NL} iff the following hold:*

(3.31.1) *The hull pull-backs $(L_j)_T^H$ are invertible, and*

(3.31.2) *the $(N_i [\otimes] L_1^{[m_1]} [\otimes] \cdots [\otimes] L_r^{[m_r]})_T^H$ are flat families of divisorial sheaves for every $m_i \in \mathbb{Z}$.*

Proof We apply (3.29) to each N_i and (3.30) to each L_j to get locally closed partial decompositions $S^{N_i} \to S$ and $S^{L_j} \to S$ that represent the functors of flat hull pull-backs with S_2 fibers for N_i and L_j, plus invertibility for the L_j. Let $S^* \to S$ denote the fiber product of all of them.

It is clear that S^{NL} factors through S^*. Tensoring with an invertible sheaf preserves flat families of divisorial sheaves, thus $S^{NL} = S^*$. □

The following analog of (3.20) is a special case of (9.36), where for polynomials we use the ordering $f(*) \le g(*) \iff f(t) \le g(t)\ \forall t \gg 1$ as in (5.14).

Theorem 3.32 *Let S be a reduced scheme, $f : X \to S$ a projective morphism with ample $\mathcal{O}_X(1)$ and L a mostly flat family of divisorial sheaves on X. Then*

(3.32.1) *$s \mapsto h^0(X_s, L_s^H)$ is constructible and upper semi-continuous,*

(3.32.2) *$s \mapsto \chi(X_s, L_s^H(*))$ is constructible, upper semi-continuous, and*

(3.32.3) *L is a flat family of divisorial sheaves (3.25) iff $s \mapsto \chi(X_s, L_s^H(*))$ is locally constant on S.* □

Remark 3.32.4. Recall that by (3.20) a coherent sheaf G is flat over S iff $s \mapsto \chi(X_s, G_s(*))$ is locally constant on S. However, the assumptions of (3.32) are quite different since L_s^H is not assumed to be the fiber of L over s. In fact, usually there is no coherent sheaf on X whose fiber over s is isomorphic to L_s^H for every $s \in S$. The map $r_s^S : L_s \to L_s^H$ is an isomorphism over U_s, but both its kernel and the cokernel can be nontrivial. They have opposite contributions to the Euler characteristic.

3.33 (Hilbert function of divisorial sheaves) Let X be a proper scheme of dimension n and L, M line bundles on X. The Hirzebruch–Riemann–Roch theorem computes $\chi(X, L \otimes M^r)$ as a polynomial of r. Its leading terms are

$$\chi(X, L \otimes M^r) = \frac{r^n}{n!}(M^n) + \frac{r^{n-1}}{2(n-1)!}\Big((\tau_1(X) + 2L) \cdot M^{n-1}\Big) + O(r^{n-2}), \quad (3.33.1)$$

where τ_1 is the first Todd class.

Assume next that L is invertible only outside a subset $Z \subset X$ of codimension ≥ 2. By blowing up L, we get a proper birational morphism $\pi : X' \to X$ and a line bundle L' such that $\pi_* L' = L$. Thus we can compute $\chi(X, L \otimes M^r)$ as $\chi(X', L' \otimes \pi^* M^r)$, modulo an error term which involves the sheaves $R^i\pi_* L'$. These are supported on Z, hence the $\chi(X, R^i\pi_* L' \otimes M^r)$ all have degree $\leq n-2$. Thus we again obtain (3.33.1), and, if X is demi-normal, then $\tau_1(X) = -K_X$.

If, in addition, $L^{[m]}$ is locally free for some $m > 0$, then applying (3.33.1) to $L \mapsto L^{[a]}$ for all $0 \leq a < m$ and $M = L^{[m]}$ we end up with the expected formula

$$\chi(X, L^{[r]}) = \frac{r^n}{n!}(L^n) - \frac{r^{n-1}}{2(n-1)!}(K_X \cdot L^{n-1}) + O(r^{n-2}). \quad (3.33.2)$$

Further note that $\chi(X, L^{[r]})$ is a polynomial on any translate of $m\mathbb{Z}$, so one can write the $O(r^{n-2})$ summand as $\sum_{i=0}^{n-2} a_i(r) r^i$, where the $a_i(r)$ are periodic functions that depend on X and L.

3.34 (Hilbert function of slc varieties) Let X be a proper, slc variety of dimension n. We are especially interested in $r \mapsto \chi(X, \omega_X^{[r]})$, which we call the *Hilbert function* of ω_X. By (3.33), we can write it as

$$\chi(X, \omega_X^{[r]}) = \frac{r^n}{n!}(K_X^n) - \frac{r^{n-1}}{2(n-1)!}(K_X^n) + \sum_{i=0}^{n-2} a_i(r) r^i, \quad (3.34.1)$$

where the $a_i(r)$ are periodic functions with period = index(X).

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By (11.34), if ω_X is ample and the characteristic is 0, then, for $i, r \geq 2$,

$$h^0(X, \omega_X^{[r]}) = \chi(X, \omega_X^{[r]}), \quad \text{and} \quad h^i(X, \omega_X^{[r]}) = 0. \tag{3.34.2}$$

Comment on the terminology It might seem natural to call $r \mapsto h^0(X, \omega_X^{[r]})$ the Hilbert function. However, (3.34.1) is not a polynomial in general. For stable varieties the two variants differ only for $r = 1$ by (3.34.2).

3.4 Local Stability

Definition 3.35 (Relative canonical class) Let $f : X \to S$ be a flat, projective morphism with demi-normal fibers. The relative canonical sheaf $\omega_{X/S}$ was constructed in (2.68).

Let $Z \subset X$ be the subset where the fibers are neither smooth nor nodal. Set $j : U := X \setminus Z \hookrightarrow X$. Then $X_s \cap Z$ has codimension ≥ 2 for every fiber X_s and $\omega_{U/S}$ is locally free. Thus $\omega_{X/S}$ is a mostly flat family of divisorial sheaves. The corresponding divisor class is denoted by $K_{X/S}$. As in (3.25), we define its reflexive powers by the formula

$$\omega_{X/S}^{[m]} := j_*(\omega_{U/S}^m) \simeq \mathscr{O}_X(mK_{X/S}). \tag{3.35.1}$$

All these also hold for flat, finite type morphisms (that are not necessarily projective) by (2.68.7).

If the fibers of $f : X \to S$ are slc, then $\omega_{X/S}$ is a flat family of divisorial sheaves by (2.67). However, its reflexive powers are usually only mostly flat over S. Applying (3.30) to $\omega_{X/S}^{[m]}$ gives the following, which turns out to be the key to our treatment of local stability over reduced schemes.

Corollary 3.36 *Let $f : X \to S$ be a flat, projective family of demi-normal varieties and fix $m \in \mathbb{Z}$. Then there is a locally closed decomposition $j : S^{[m]} \to S$ such that the following holds.*

Let $q : W \to S$ be a morphism. Then $\omega_{X_W/W}^{[m]}$ is a flat family of divisorial sheaves iff q factors as $q : W \to S^{[m]} \to S$. □

In applications of (3.36), a frequent problem is that $S^{[m]}$ depends on m, even if we choose m to be large and divisible; see (2.45) for such an example.

3.37 (Proof of 3.1) Assertions (3.1.1) and (3.1.2) say the same using different terminology. The equivalences of (3.1.3–5) follow from (9.17).

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Assume (3.1.3) and pick $s \in S$. Since X_s is slc, $\omega_{X_s}^{[m_s]}$ is locally free for some $m_s > 0$. In a flat family of sheaves being invertible is an open condition, thus $\omega_{X/S}^{[m_s]}$ is invertible in an open neighborhood $X_s \subset U_s \subset X$. Finitely many of these U_{s_i} cover X. Then $m = \operatorname{lcm}\{m_{s_i}\}$ works for (3.1.2).

It is clear that (3.1.1) implies (3.1.6) and for (3.1.6) $\Rightarrow$ (3.1.3) we argue as follows. We need to prove that $\omega_{X/S}^{[m]}$ is a flat family of divisorial sheaves. This is a local question on S, hence we may assume that $(0 \in S)$ is local.

Let us discuss first the case when f is projective. By (3.36), the property

$$\mathcal{P}^{[\mathrm{m}]}(W) := (\omega_{X_W/W}^{[m]} \text{ is a flat family of divisorial sheaves})$$

is representable by a locally closed decomposition $i_m : S^{[\mathrm{m}]} \to S$. We aim to prove that i_m is an isomorphism.

For each generic point $g_i \in S$, choose a local morphism $q_i : (0_i \in T_i) \to (0 \in S)$ that maps the generic point $t_i \in T_i$ to g_i. By assumption, $X_{T_i} \to T_i$ is locally stable, hence $\omega_{X_{T_i}/T_i}^{[m]}$ is a flat family of divisorial sheaves by (2.79.2). Thus q_i factors through $i_m : S^{[\mathrm{m}]} \to S$. Therefore, $i_m : S^{[\mathrm{m}]} \to S$ is an isomorphism by (10.83.2), completing the proof for projective morphisms.

This argument also works in the nonprojective case, provided $i_m : S^{[\mathrm{m}]} \to S$ exists. As we discuss in Section 9.8, the latter is unlikely. However, if S is local, complete, and we aim to represent flat hull pull-backs for local morphisms, then $i_m : S^{[\mathrm{m}]} \to S$ exists; see (9.44) for details. The rest of the argument now works as before; see also (3.38).

Finally, if any of the properties (3.1.1–6) holds for X, then it also holds for $X \setminus Z$. The surprising part is the converse. By using (3.1.6) both for X and for $X \setminus Z$, it is enough to see that (3.1.7) $\Rightarrow$ (3.1.1) holds when S is the spectrum of a DVR. The latter is proved in (2.7). □

Corollary 3.38 *Let S be a reduced scheme over a field of characteristic 0 and $f : X \to S$ a flat family of demi-normal varieties. Let $T \to S$ be faithfully flat. Then $X \to S$ is locally stable iff $X_T \to T$ is.* □

Corollary 3.39 *Let $f : X \to S$ be a flat, proper morphism of finite type with demi-normal fibers such that $K_{X/S}$ is $\mathbb{Q}$-Cartier. Then*

$$S^{slc} := \{s : X_s \textit{ is slc}\} \subset S \quad \textit{is open.} \tag{3.39.1}$$

Proof By (10.14), a set $U \subset S$ is open iff it is closed under generalization and U contains a dense open subset of $\bar{s}$ for every $s \in U$.

For S^{slc}, the first of these follows from (2.3). In order to see the second, assume first that X_s is lc. Then mK_{X_s} is Cartier for some $m > 0$, hence $mK_{X/S}$ is Cartier over an open neighborhood of $s \in U_s \subset \bar{s}$. Next consider a log

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resolution $p_s : Y_s \to X_s$. It extends to a simultaneous log resolution $p^\circ : Y^\circ \to X^\circ$ over a suitable $U_s^\circ \subset \bar{s}$. Thus, if $E^\circ \subset Y^\circ$ is any exceptional divisor, then $a(E_t, X_t) = a(E^\circ, X^\circ) = a(E_s, X_s)$ for every $t \in U_s^\circ$. This shows that all fibers over U_s° are lc.

If X_s is not normal, one can use either a simultaneous semi-log-resolution (Kollár, 2013b, sec.10.4) or normalize first, apply the above argument, and descend to X, essentially by definition (11.37). □

3.5 Stability Is Representable I

Focusing on the property (3.1.3), over nonreduced bases we get the definition of local stability, due to Kollár and Shepherd-Barron (1988).

Definition 3.40 (Local stability and stability II) Let S be a scheme over a field of characteristic 0 and $f : X \to S$ a flat morphism of finite type with demi-normal fibers. Then f is *locally stable* iff the fibers X_s are slc and $\omega_{X/S}^{[m]}$ is a flat family of divisorial sheaves (3.25) for every $m \in \mathbb{Z}$.

Furthermore, f is *stable* iff, in addition, f is proper and $\omega_{X/S}$ is f-ample.

The next example shows that being stable is not a locally closed condition.

Example 3.41 In $\mathbb{P}^5_{\mathbf{x}} \times \mathbb{A}^2_{st}$, consider the family of varieties

$$X := \left(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \leq 1\right).$$

We claim that the fibers X_{st} are normal, projective with rational singularities and for every s, t the following equivalences hold:

(3.41.1) X_{st} is lc $\Leftrightarrow$ X_{st} is klt $\Leftrightarrow$ $K_{X_{st}}$ is $\mathbb{Q}$-Cartier $\Leftrightarrow$ $3K_{X_{st}}$ is Cartier $\Leftrightarrow$ either $(s, t) = (0, 0)$ or $st \neq 0$.

All these become clear once we show that there are three types of fibers.

(3.41.2) If $st \neq 0$ then, after a linear coordinate change, we get that

$$X_{st} \simeq X_{11} \simeq \left(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_4 & x_5 & x_3 \end{pmatrix} \leq 1\right).$$

This is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2$, hence smooth. The self-intersection of its canonical class is -54.

(3.41.3) If $s = t = 0$ then we get the fiber

$$X_{00} := \left(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix} \leq 1\right).$$

This is the cone (with $\mathbb{P}^1$ as vertex-line) over the rational normal curve $C_3 \subset \mathbb{P}^3$. The singularity along the vertex-line is isomorphic to $\mathbb{A}^2/\frac{1}{3}(1,1) \times \mathbb{A}^1$, hence log terminal. The canonical class of X_{00} is $-\frac{8}{3}H$, where H is the hyperplane class and its self-intersection is $-512/9 < -54$.
(3.41.4) Otherwise either s or t (but not both) are zero. After possibly permuting s, t, and a linear coordinate change, we get the fiber

$$X_{0t} \simeq X_{01} \simeq \left(\operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_4 & x_3 \end{pmatrix} \leq 1\right).$$

This is the cone over the degree 3 surface $S_3 \simeq \mathbb{F}_1 \hookrightarrow \mathbb{P}^4$. Its canonical class is not $\mathbb{Q}$-Cartier at the vertex, so this is not lc.

This is a locally stable example. Taking a general double cover ramified along a general, sufficiently ample hypersurface gives a stable example.

Thus the best one can hope for is that local stability is representable. From now on the base scheme is assumed to be over a field of characteristic 0.

3.42 (Proof of 3.2) Being flat is representable by (3.19) and being demi-normal is an open condition for flat morphisms by (10.42). So, using (3.23.1), we may assume that $f : X \to S$ is flat, of pure relative dimension n and its fibers are demi-normal.

Now we come to a surprisingly subtle part of the argument. If X_s is slc then K_{X_s} is $\mathbb{Q}$-Cartier, thus the next natural step would be the following.

Question 3.42.1 Is $\{s \in S : K_{X_s}$ is $\mathbb{Q}$-Cartier$\}$ a constructible subset of S?

We saw in (2.45) that this is not the case, not even for families of normal varieties. The key turns out to be the following immediate consequence of (4.44); the latter is the hardest part of the proof.

Claim 3.42.2 Let $f : X \to S$ be a flat, proper family of demi-normal varieties. Then $\{\operatorname{index}(X_s) : X_s$ is slc$\}$ is a finite set. □

We can now complete (3.2). Let M be a common multiple of the indices of the slc fibers. We apply (3.31) with $N_i := \omega_{X/S}^{[i]}$ for $1 \leq i < M$ and $L_1 := \omega_{X/S}^{[M]}$. We get $S^{NL} \to S$ such that the $\omega_{X^{NL}/S^{NL}}^{[m]}$ are flat families of divisorial sheaves for every m, and $\omega_{X^{NL}/S^{NL}}^{[M]}$ is invertible. Finally (3.39) gives that S^{ls} is an open subscheme of S^{NL}. □

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