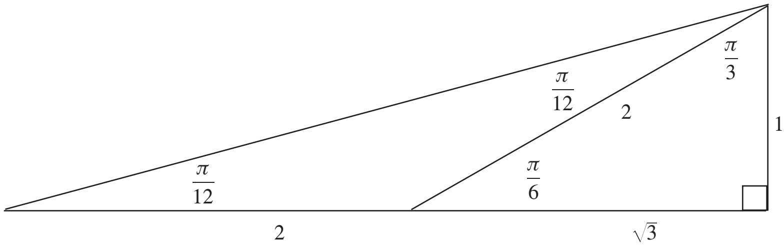

from the diagram the readers can readily see that

$$\tan\frac{5\pi}{12} = 2 + \sqrt{3} \qquad \text{and} \qquad \tan\frac{\pi}{12} = \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}.$$

There is another PWW from Garcia Capitan Francisco Javier [1]. Paul Stephenson [2] and Nick Lord [3] have offered other demonstrations of the identity of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$, for which Nick Lord gives four Proofs without words, with quite different ideas. For many useful principles and comments about Proofs without words, see [4].

References

1. F. J. Garcia Capitan, Proof without Words: tangents 15 and 17 degrees, *Coll. Math. J.* **48:1** (2017) p. 35.
2. P. Stephenson, Feedback: On what makes a good Proof without Words, *Math. Gaz.* **107** (March 2023) p. 165.
3. Nick Lord, Feedback, *Math. Gaz.* **107** (July 2023) p. 356.
4. G. Leversha, What makes a good Proof without Words, *Math. Gaz.* **105** (July 2021) pp. 271-281.

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108.11 Euler's limit—revisited

Let $e_n = \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N}$. It is well known that the sequence (e_n) is monotone increasing and bounded, hence it is convergent. The limit of this sequence is the famous Euler number e. Here we establish a generalisation of this limit.

Theorem: Let $\{a_n\}$ and $\{b_n\}$ be two sequences of positive real numbers such that $a_n \to +\infty$ and b_n satisfies the asymptotic formula $b_n \sim k \cdot a_n$, where $k > 0$. Then

$$\lim_{n \to \infty}\left(1 + \frac{1}{a_n}\right)^{b_n} = e^k.$$

https://doi.org/10.1017/mag.2024.28 Published online by Cambridge University Press

Proof: Let $f : (1, \infty) \to \mathbb{R}$ be defined by $f(x) = x - 1 - \ln x$, Since $f'(x) > 0$ for $x \in (1, \infty)$, thus f is increasing on $(1, \infty)$. Again, for the function $g : (1, \infty) \to \mathbb{R}$ which is defined by $g(x) = \ln x - 1 + \frac{1}{x}$, $g'(x) > 0$ for $x \in (1, \infty)$. Thus g is also increasing on $(1, \infty)$. Hence

$$1 - \frac{1}{x} < \ln x < x - 1 \text{ for } x > 1.$$

For a visual proof of the above inequality, see [2].

Since $a_n > 0$, thus $1 + \frac{1}{a_n} > 1$. Thus using the above inequality, we have

$$\frac{1}{1 + a_n} < \ln\left(1 + \frac{1}{a_n}\right) < \frac{1}{a_n}.$$

Since $b_n > 0$, we have

$$\frac{b_n}{1 + a_n} < b_n \cdot \ln\left(1 + \frac{1}{a_n}\right) < \frac{b_n}{a_n}.$$

Since $b_n \sim k \cdot a_n$, using the Sandwich Lemma ([1]), we have

$$\lim_{n \to \infty} \left(1 + \frac{1}{a_n}\right)^{b_n} = e^k.$$

It can be seen that by choosing $a_n = n$ and $b_n = n$, we get Euler's limit. Moreover, if $\frac{b_n}{a_n} \sim 0$, then $\lim_{n \to \infty} \left(1 + \frac{1}{a_n}\right)^{b_n} = 1$. Also, if $\frac{a_n}{b_n} \sim 0$, then $\lim_{n \to \infty} \left(1 + \frac{1}{a_n}\right)^{b_n} = \infty$.

Acknowledgements

We are thankful to the anonymous referee and the Editor for their careful reading of the manuscript and necessary suggestions.

References

1. R. G. Bartle and D. R. Sherbert, *Introduction to real analysis* (4th edn.), Wiley (2014).
2. Ananda Mukherjee, and Bikash Chakraborty, Yet Another Visual Proof that $\pi^e < e^\pi$, *Math Intelligencer* **41**(2), (2019) pp. 60.

10.1017/mag.2024.28
Published by Cambridge University Press
on behalf of The Mathematical Association

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