BULL. AUSTRAL. MATH. SOC. 35A35, 35J65
VOL. 50 (1994) [405–424]

APPROXIMATION OF A QUASILINEAR ELLIPTIC EQUATION WITH NONLINEAR BOUNDARY CONDITION

T.R. CRANNY

We consider a quasilinear elliptic partial differential equation with nonlinear boundary condition under assumptions which do not allow the application of standard degree theory results or techniques such as the method of continuity. An approximation using mollifiers is introduced, allowing the application of Leray-Schauder degree theory, and homotopy arguments are then used to prove the existence of solutions to the approximating problems. A subsequent paper will discuss the question of the convergence of these approximate solutions to a classical solution of the original problem.

1. INTRODUCTION

We consider the quasilinear partial differential equation with nonlinear boundary condition

$$(1.1)\qquad \begin{aligned} Qu = a^{ij}(x,u,Du)D_{ij}u + a(x,u,Du) &= 0 \qquad \text{in } \Omega \\ Gu = g(x,u,Du) &= 0 \qquad \text{on } \partial\Omega, \end{aligned}$$

for Ω a suitable domain in $\mathbb{R}^n$, and seek conditions which ensure the existence of a classical solution of (1.1).

Two of the main techniques used to prove the existence of such a classical solution are the Leray-Schauder degree theory and the nonlinear method of continuity. The degree theory approach is not directly applicable for problems were the boundary operator G is nonlinear in the gradient term, while the method of continuity relies heavily on invertibility conditions such as: $a^{ij}_z = 0$, $a_z \leqslant 0$, $G_z < 0$ (see [**4, 8**]).

We describe an approach which has the advantage of dealing with nonlinearities in the boundary condition while relaxing (in some senses) the smoothness and invertibility conditions needed for the method of continuity. The boundary $\partial\Omega$ is assumed to be $C^{1,\alpha}$, while the regularity conditions imposed upon the domain and differential operator are similar to those of the best existing results using the method of continuity (that

Received 19th January, 1994
The results described in this paper, and its sequel [**2**], are based upon the author's doctoral thesis [**1**]. I wish to thank Dr. H.B.Thompson for his supervision during the course of my doctoral studies.

is, where the intermediate Schauder estimates of [**3, 7**] are used). The results do not require Q to be monotonic in z, however the dependence of G upon z is no better than that needed for the method of continuity, while the continuity conditions imposed upon G are stronger than those needed for the method of continuity in the intermediate Schauder formulation. This is despite the extensive use of the intermediate Schauder estimates in this paper. The situation may be summarised by saying that in comparison with the method of continuity, the results presented here constitute a relaxing of the conditions imposed upon the differential operator and a tightening of those imposed upon the boundary operator. In [**8**], Lieberman and Trudinger describe an approach which makes use of a version of the method of continuity to apply degree theory to problems in which the boundary operator is nonlinear in the gradient. This technique removes the need for the strong monotonicity conditions used by both the method of continuity and the results presented here, but again the use of the method of continuity requires more smoothness of the boundary than is needed here. Most of the comments directed at the method of continuity also apply to the techniques used by Lieberman in [**6**] and associated papers.

The techniques described herein have as their motivation the approach developed by Thompson [**10, 11**], for two point boundary value problems for ODE's, where it was noted that the existence of a solution of (1.1) is equivalent to the existence of a pair of functions (u, ω) such that

$$Qu = 0 \qquad \text{in } \Omega \tag{1.2}$$

$$u = \omega \qquad \text{on } \partial\Omega$$

$$g(x, \omega, Du) = 0 \qquad \text{on } \partial\Omega. \tag{1.3}$$

In order to allow the application of degree theory under quite general circumstances, we consider an approximating problem, in which $g(x, \omega, Du) = 0$ is replaced with the equation $g\left(x, \omega, (Du)_\eta\right) = 0$, where $(\ )_\eta$ is a regularisation operator.

The introduction of the function ω above serves to weaken the connection between the differential equation and the boundary condition, since the boundary condition (1.3) no longer depends explicitly upon the values of u upon $\partial\Omega$, but now upon the function ω. (It will still depend upon u through the $(Du)_\eta$ term.) We continue the 'disconnection' of the two facets of the original problem, deriving a Dirichlet problem for u which makes no reference to ω or the boundary operator G, and a condition upon ω unrelated to u, Du, or the differential equation. This disconnection is attained through a series of homotopies for the Leray-Schauder degree, and produces an existence result for the approximating mapping under quite mild assumptions.

In a subsequent paper [**2**], we shall show that standard $C^{1,\sigma}(\overline{\Omega})$ *a priori* estimates ensure the convergence of these 'approximate' solutions to a classical solution of the original problem (1.1), and describe results which give such *a priori* estimates.

The format of this article is as follows: In Section 2 we establish notation and construct a number of necessary technical results. In Section 3 we give the statement of the main results, and combine them to prove the existence result Theorem 3.7. Some of the longer and more technical proofs have been collected in the Appendix.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^n$ such that $\partial\Omega \in C^{1,\alpha}$ where $0 < \alpha < 1$. We let $\vec{n}$ denote the unit inward normal. We consider the problem (1.1) under the assumptions that a^{ij}, $a \in C^{0,1}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$, and there exist positive constants λ, Λ, χ such that

$$\begin{aligned} 0 < \lambda\, |\xi|^2 \leqslant a^{ij}(x,z,p)\xi_i\xi_j \leqslant \Lambda\, |\xi|^2 < \infty \qquad &\text{for all } \xi \in \mathbb{R}^n \setminus \{0\} \\ -g_p(x,z,p) \cdot \vec{n} > \max\{\chi\ |g_p(x,z,p)|\,,\, \chi\}. & \end{aligned} \tag{2.1}$$

Remark. The above assumption on g_p is stronger than the standard obliqueness assumption, requiring also that the angle between $-g_p$ and the inward normal be less than and bounded away from $\pi/2$. Such an assumption follows directly from obliqueness for linear boundary conditions, (and for the purposes of this paper, for quasilinear boundary conditions) and for many nonlinear boundary conditions. It will be shown in [**2**] that in many cases one can proceed using only obliqueness.

We assume that the boundary condition can be written in the form

$$g(x,z,p) = z - q(x,p) = 0 \quad \text{on } \partial\Omega, \tag{2.2}$$

where q, $q_p \in C^{1,\alpha}(\partial\Omega \times \mathbb{R}^n)$. Such a form for the boundary condition is possible for example if $g(x,z,p)$ is sufficiently smooth and $g_z(x,z,p) \neq 0$, conditions which are roughly similar to those used in the method of continuity, but by utilising intermediate Schauder estimates the method of continuity can proceed under more general conditions. (See for example the final section of [**8**].) A possible improvement in our results is indicated in the concluding remarks.

We shall always assume that Q satisfies the natural structure conditions

$$\begin{aligned} \Lambda &\leqslant \lambda\mu(|z|) \\ |a| &\leqslant \lambda\mu_0(|z|)\Big(1 + |p|^2\Big), \\ |a_x|\,,\, |a_z|\,,\, |p|\ |a_p| &\leqslant \lambda\mu_1(|z|)\Big(1 + |p|^2\Big), \\ \left|a^{ij}_x\right|, \left|a^{ij}_z\right|, (1 + |p|)\ \left|a^{ij}_p\right| &\leqslant \lambda\mu_1(|z|), \end{aligned} \tag{2.3}$$

where μ, μ_0, μ_1 are positive non-decreasing functions.

On a number of occasions we shall need to approximate a given function with a smooth function. To do so, consider $\rho(\cdot) \in C^\infty(\mathbb{R}^n, \mathbb{R})$ where $\rho(\cdot) \geqslant 0$ and $\rho(x) = 0$ if $|x| \geqslant 1$, and $\int \rho(\cdot)\,dx = 1$. For $f \in C^0(\Omega)$, and $\eta > 0$, consider the operator $\varphi_\eta *$ defined by

$$(\varphi_\eta * f)(x) = \eta^{-n} \int_\Omega \rho\left(\frac{x-y}{\eta}\right) f(y)\,dy,$$

provided that $dist(x, \partial\Omega) > \eta$. We call $\varphi_\eta * f$ the *mollification* or *regularisation* of f, and η the *mollification parameter.* The properties of the mollification operator are described in [**4**, **9**].

The set $\{(x,z,p) \in \partial\Omega \times \mathbb{R} \times \mathbb{R}^n \;\big|\; |z| + |p| < A\}$ is denoted by $\partial\Omega(A)$.

We shall say a function $\alpha(\cdot)$ (respectively $\beta(\cdot)$) $\in C^1(\overline{\Omega}) \cap C^2(\Omega)$ is a *lower (upper) solution* of (1.1) if $Q\alpha \geqslant 0$ $(Q\beta \leqslant 0)$ in Ω. The results which follow make extensive use of such functions, and we shall require additional geometric properties such as the following.

DEFINITION 2.1: The lower solution $\alpha(\cdot)$ and the upper solution $\beta(\cdot)$ are a *boundary diverging pair*, or BD pair, if $\alpha(\cdot) = -\beta(\cdot) < 0$ on $\overline{\Omega}$, $\beta(\cdot) \in C^{1,\alpha}(\overline{\Omega}) \cap C^2(\Omega)$, $\|D\beta\|_{0;\overline{\Omega}} < 1$ and there exist constants $S_1, \tau > 0$ such that $(S_1)^{-1} \geqslant D\beta \cdot \vec{n} \geqslant S_1 > 0$ and $D\beta \cdot g_p \leqslant -\tau$.

These properties will allow us to adapt the boundary condition while keeping essential properties such as obliqueness.

LEMMA 2.2. *If Equation 2.1 holds and $a(x,z,p)$ satisfies*

$$(sgn\ z)\, a(x,z,0) < 0 \text{ in } \overline{\Omega}, \tag{2.4}$$

for $|z| \geqslant M_1$ some nonnegative constant, then *there exists a BD pair of upper and lower solutions $\beta(\cdot)$ and $\alpha(\cdot)$.*

PROOF: It is clear that there exists an $\varepsilon > 0$ such that

$$\begin{aligned} \inf_{\overline{\Omega}} a(x,-M,0) &> \quad \varepsilon \\ \sup_{\overline{\Omega}} a(x,M,0) &< \quad -\varepsilon, \end{aligned} \tag{2.5}$$

for $M > M_1$, so $-M$ is lower solution while M is an upper solution. We show that these constant functions can be perturbed to give a BD pair. Since $\partial\Omega \in C^{1,\alpha}$, as in [**3**] there exists a regularised distance function $\rho(\cdot) \in H_2^{-(1+\alpha)}$. One can in fact generalise that result so that for any $\delta > 0$ one may choose $\rho^\delta(\cdot)$ such that $D\rho^\delta(x) \cdot \vec{n}(x) > (1-\delta)\,|D\rho^\delta(x)|$ on $\partial\Omega$. There also exists a constant $\nu_0 > 0$ such that

$$\nu_0 \leqslant D\rho^\delta \cdot \vec{n} \leqslant (\nu_0)^{-1} \quad \text{on } \partial\Omega. \tag{2.6}$$

Using (2.1) and the fact that $\chi > 0$, we may choose $\delta > 0$ sufficiently small that $D\rho^\delta(x) \cdot g_p < -(\chi/2)\,|D\rho^\delta(x)|\;|g_p|$, and drop the superscript.

We extend the function $\rho(\cdot)$ to $\widetilde{\Omega} \supset\supset \Omega$, and consider the mollification of $\rho(\cdot)$. We choose $\eta > 0$ sufficiently small that

$$\nu_0/2 \leqslant (D\varphi_\eta * \rho) \cdot \vec{n} \leqslant 2(\nu_0)^{-1}$$
$$D\varphi_\eta * \rho \cdot g_p < -(\chi/4)\;|D\varphi_\eta * \rho|\;|g_p|$$

and denote the appropriate mollification (restricted to $\overline{\Omega}$) by $c(\cdot)$. Since the $a^{ij}(x,z,p)$ and $a(x,z,p)$ are continuous, we may choose $\sigma > 0$ sufficiently small that

$$\begin{aligned} &\left|\sigma\, a^{ij}(x,-M-\sigma c(x),-\sigma Dc(x))D_{ij}c(x)\right| < \varepsilon/2 && \text{in } \overline{\Omega} \\ &\left|a(x,-M,0) - a(x,-M-\sigma c(x),-\sigma Dc(x))\right| < \varepsilon/2 && \text{in } \overline{\Omega} \\ &\|\sigma\, Dc\|_{0;\overline{\Omega}} < 1. \end{aligned} \tag{2.7}$$

It then follows from (2.5) that $\beta(\cdot) = M + \sigma c(\cdot)$ is an upper solution, and by taking $\alpha(\cdot) = -\beta(\cdot)$, the desired result holds for $S_1 = \sigma\nu_0/2$ and $\tau = \sigma\chi^2\nu_0/8$. □

REMARKS. **1.** The condition (2.4) is significantly weaker than the corresponding invertibility condition used in the method of continuity. **2.** When discussing $\alpha(\cdot)$ and $\beta(\cdot)$ a BD pair, we shall assume that both functions are of the form constructed above.

We fix $0 < \varepsilon_0 << 1$, and define α_ε to be $\inf_\Omega \alpha(x) - \varepsilon_0$ and β_ε to be $\sup_\Omega \beta(x) + \varepsilon_0$.

In order to use degree theory, we first need to define the open sets from which the functions $u(\cdot)$ and $w(\cdot)$ are to be chosen. Since a 'Du' term appears in the boundary condition, we assume that u is chosen from an open subset of the Hölder space $C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0,\alpha)$. We therefore assume $0 < \gamma < \rho < \alpha < 1$ and consider the set Υ (from which u will be chosen), to be

$$\Upsilon \;=\; \Upsilon(L_\Upsilon) \stackrel{\text{def}}{=} \{u \in C^{1,\gamma}(\overline{\Omega}) \mid \alpha_\varepsilon < u(x) < \beta_\varepsilon,\; \|u\|_{1,\gamma;\overline{\Omega}} < L_\Upsilon\}, \tag{2.8}$$

where we shall always assume $L_\Upsilon < \infty$. Similarly, we consider the set Δ, from which ω is chosen, to be

$$\Delta \;=\; \Delta(L_\Delta) \stackrel{\text{def}}{=} \{\omega \in C^{1,\rho}(\partial\Omega) \mid \alpha(x) < \omega(x) < \beta(x),\; \|\omega\|_{1,\rho;\partial\Omega} < L_\Delta\}, \tag{2.9}$$

where $L_\Delta < \infty$. We shall also make use of the set

$$\Delta_\infty \stackrel{\text{def}}{=} \{\omega \in C^{1,\rho}(\partial\Omega) \mid \alpha(x) \leqslant \omega(x) \leqslant \beta(x)\}.$$

We begin the disconnection process by introducing an operator which will ultimately be used to replace the 'Du' term in (1.3). For every $\omega \in \Delta$ (in other words, for every candidate for the boundary values of a solution), we assign to each point $x \in \partial\Omega$ a vector $\widetilde{\Psi}\omega(x)$. The operator used is constructed so as to facilitate the impending homotopy arguments.

DEFINITION 2.3: Given the set Δ, a function $\widetilde{\Psi} : \Delta_\infty \to C^0(\partial\Omega, \mathbb{R}^n)$ will be called *strongly inwardly pointing* if it satisfies 1–3 below:

1. $\widetilde{\Psi} : \Delta_\infty \to C^{1,\alpha}(\partial\Omega, \mathbb{R}^n)$ is continuous, and $\widetilde{\Psi}(\Delta)$ is bounded in $C^{1,\alpha}(\partial\Omega)$.
2. Given an $\omega \in \overline{\Delta}$ and an $x_1 \in \partial\Omega$ such that $\omega(x_1) = \alpha(x_1)$, we have

$$\left(\widetilde{\Psi}\omega(x_1) - D\alpha(x_1)\right) \cdot \vec{n}(x_1) > 0, \qquad \left(\widetilde{\Psi}\omega(x_1) - D\alpha(x_1)\right) \cdot g_p(x_1, \alpha(x_1), p) \leqslant 0.$$

for all $p \in \mathbb{R}^n$ such that $|p| < \left|\widetilde{\Psi}\omega\right|_{0;\partial\Omega} + 1$.

3. Given an $\omega \in \overline{\Delta}$ and an $x_1 \in \partial\Omega$ such that $\omega(x_1) = \beta(x_1)$, we have

$$\left(\widetilde{\Psi}\omega(x_1) - D\beta(x_1)\right) \cdot \vec{n}(x_1) < 0, \qquad \left(\widetilde{\Psi}\omega(x_1) - D\beta(x_1)\right) \cdot g_p(x_1, \beta(x_1), p) \geqslant 0.$$

for all $p \in \mathbb{R}^n$ such that $|p| < \left|\widetilde{\Psi}\omega\right|_{0;\partial\Omega} + 1$.

We shall soon need to restrict our attention to those $\widetilde{\Psi}$ which give a bound on $\widetilde{\Psi}(\Delta)$ even when a bound on L_Δ is unavailable.

DEFINITION 2.4: For $R > 0$, let $SIP(R)$ be the set of all strongly inwardly pointing vector field operators $\widetilde{\Psi}$ such that $\left\|\widetilde{\Psi}(\Delta)\right\|_{1,\alpha;\partial\Omega} < R$ for any L_Δ.

Note that if $\alpha(\cdot)$ and $\beta(\cdot)$ are a BD pair, then $SIP(R)$ is non-empty since $\widetilde{\Psi} \equiv 0$ and perturbations thereof are always in $SIP(R)$.

DEFINITION 2.5: Given $\alpha(\cdot)$, $\beta(\cdot)$ a BD pair and $R \in (0, \infty)$, we say that the boundary operator $g(x, z, p)$ is *R-compatible* if for all $M_0 > 0$ there exists an $M > M_0$ such that for $\Delta = \Delta(M)$

1. $g\left(\cdot, w(\cdot), \widetilde{\Psi}\omega(\cdot)\right) \neq 0 \qquad$ on $\partial\Omega$ if $\omega \in \partial\Delta$
2. The Leray-Schauder degree $d\left(g\left(x, \omega, \widetilde{\Psi}\omega\right), \Delta, 0\right) \neq 0$

for all $\widetilde{\Psi} \in SIP(R)$.

Note that if g is R-compatible with α and β, then $d\left(g\left(x, \omega, \widetilde{\Psi}\omega\right), \Delta, 0\right)$ is independent of the $\widetilde{\Psi} \in SIP(R)$ chosen, by virtue of homotopy invariance.

DEFINITION 2.6: Given $\alpha(\cdot)$ and $\beta(\cdot)$ a BD pair, we say that the boundary operator $g(x,z,p)$ is *BD compatible* if for all $R_0 > 0$ there exists an $R > R_0$ such that $g(x,z,p)$ is R-compatible with α and β.

LEMMA 2.7. *If the assumptions of Lemma 2.2 apply, and $g(x,z,p)$ is of the form $z - q(x,p)$ where $q \in C^{1,\alpha}(\partial\Omega \times \mathbb{R}^n)$* then *there exists a BD pair $\alpha(\cdot)$, $\beta(\cdot)$ such that $g(x,z,p)$ is BD compatible with $\alpha(\cdot)$ and $\beta(\cdot)$.*

PROOF: See Appendix.

The boundary operator is now in a suitable form for the use of a homotopy argument for the degree, but we have yet to phrase (1.2) in an appropriate form. Before we establish the notation and results required for re-writing the differential operator, we adapt it in a way which will simplify comparison with upper or lower solutions, without disturbing the search for solutions. This is done by leaving the differential operator unchanged between $\alpha(\cdot)$ and $\beta(\cdot)$ (the region in which we shall guarantee the existence of solutions), while adapting the operator outside the 'envelope' of $\alpha(\cdot)$ and $\beta(\cdot)$. We establish here the desired notation and results.

DEFINITION 2.8: Let $\pi(z,c,d)$ denote the *middle operator* given by

$$\pi(z,c,d) = min\{d, max\{z,c\}\}.$$

Consider an function $\mathcal{K} \in C^1(\mathbb{R} \times (0,\infty); [-1,1])$ such that $\mathcal{K}(\cdot,\varepsilon)$ is odd, $\mathcal{K}(t,\varepsilon) = 0$ if and only if $t = 0$, and $\mathcal{K}(t,\varepsilon) = 1$ for all $t \geqslant \varepsilon$. We denote $\|\mathcal{K}(\cdot,\varepsilon)\|_1$ by $M_1(\varepsilon)$.

DEFINITION 2.9: Taking $0 < \varepsilon < \varepsilon_0$ and $\mathcal{K}$ the above function, we define the operator $\mathcal{T}(\cdot\,;\varepsilon)$ by:

$$\mathcal{T}(y;\varepsilon)(x) = \mathcal{K}(y(x) - \pi(y(x),\alpha(x),\beta(x)),\varepsilon).$$

Note that for any $\beta \in [0,1]$, $y \in C^\beta(\overline{\Omega})$ implies $\mathcal{T}(y;\varepsilon) \in C^\beta(\overline{\Omega})$ and $\|\mathcal{T}(y;\varepsilon)\|_{0,\beta;\overline{\Omega}} < 2M_1(\varepsilon)\ \|y\|_{0,\beta;\overline{\Omega}}$.

DEFINITION 2.10: Let K be the differential operator defined by

$$\begin{aligned} Ku(x) = {} & a^{ij}(x,\pi(u(x),\alpha(x),\beta(x)),Du(x))D_{ij}u(x) \\ & + (1 - |\mathcal{T}(u;\varepsilon)(x)|)\ a(x,\pi(u(x),\alpha(x),\beta(x)),Du(x)) \\ & - \mathcal{T}(u;\varepsilon)(x)(|a(x,\pi(u(x),\alpha(x),\beta(x)),Du(x))| + \varepsilon). \end{aligned} \tag{2.10}$$

LEMMA 2.11. *If $u_{|\partial\Omega} \in \Delta_\infty$, then $Ku = 0$ implies that $Qu = 0$.*

PROOF: Assume u is such that $u_{|\partial\Omega} \in \Delta_\infty$ and $Ku = 0$. To show that $Qu = 0$, it suffices to show that u lies between α and β. Assume there exists a point x_1 where $(u - \alpha)(x)$ has a negative minimum. Clearly $x_1 \in \Omega$, so

$$a^{ij}(x_1,\alpha,D\alpha)D_{ij}u(x_1) \geqslant a^{ij}(x_1,\alpha,D\alpha)D_{ij}\alpha(x_1). \tag{2.11}$$

Now $Ku = 0$, $Du(x_1) = D\alpha(x_1)$, $\pi(u(x_1), \alpha(x_1), \beta(x_1)) = \alpha(x_1)$ and $\mathcal{T}(u;\varepsilon)(x_1) < 0$. Therefore

$$\begin{aligned} a^{ij}(x_1, \alpha, D\alpha) D_{ij} u(x_1) &= -a(x_1, \alpha, D\alpha) + \varepsilon\, \mathcal{T}(u;\varepsilon)(x_1) \\ &\quad + \mathcal{T}(u;\varepsilon)(x_1)\big[|a(x_1,\alpha,D\alpha)| - a(x_1,\alpha,D\alpha)\big] \\ &< -a(x_1,\alpha,D\alpha) \leqslant a^{ij}(x_1,\alpha,D\alpha) D_{ij}\alpha, \end{aligned}$$

contradicting (2.11). Therefore $u(\cdot) \geqslant \alpha(\cdot)$ on $\overline{\Omega}$. That $u(\cdot) \leqslant \beta(\cdot)$ follows similarly. □

Note that α_ε is a lower solution of $Ku = 0$, and β_ε is an upper solution. The use of K rather than Q will lead to some simplification in later techniques. The above result shows that the change in differential operator will not introduce solutions other than those we seek. This is sufficient, since solutions to $Ku = 0$ on Ω will be shown to exist.

We now simplify the notation by writing $Ku = b^{ij}(x, u, Du) D_{ij} u + b(x, u, Du)$, and consider the adapted problem

$$\begin{aligned} Ku &= 0 \quad &&\text{in } \Omega \\ u(x) &= \omega(x) \quad &&\text{on } \partial\Omega, \end{aligned} \tag{2.12}$$

where

$$\omega(x) - q(x, Du(x)) = 0 \quad \text{on } \partial\Omega. \tag{2.13}$$

As promised, we return now to the task of re-writing (2.12) in a form suitable for the application of degree theory methods. To do so we apply the standard approach, in which for a given $(u, \omega) \in \Upsilon \times \Delta$, $T(u, \omega)$ is defined to be the function v such that

$$\begin{aligned} b^{ij}(x, u(x), Du(x)) D_{ij} v + b(x, u(x), Du(x)) &= 0 \qquad \text{in } \Omega \\ v(x) &= \omega(x) \quad \text{on } \partial\Omega. \end{aligned} \tag{2.14}$$

REMARKS. **1.** It is easy to check via Intermediate Schauder estimates that for any $(u, \omega) \in \Upsilon \times \Delta$, $T(u, \omega) \in C^{2,\gamma}(\Omega) \cap C^{1,\rho}(\overline{\Omega})$ (see [3]). **2.** While this approach of defining an operator by 'freezing' coefficients is standard (see for example [4, Chapter 11]), we require the explicit dependence of the operator upon the boundary data ω, since we cannot restrict our attention to a single known boundary function.

Equation 2.12 can thus be written in the form $u - T(u, \omega)$, allowing the full problem to be written as

$$(I - K)\begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} u - T(u, \omega) \\ \omega - q(x, Du) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.15}$$

where it is assumed that $(u,\omega) \in \Upsilon \times \Delta$. Unfortunately, it is not possible to choose γ, ρ such that the operator K is compact (see [**1**]).

We find ourselves in the well-known position of being unable to apply degree theory directly because the problem cannot be written in a suitable form. The conventional approach at this point is to impose conditions which allow the use of degree theory, or to abandon degree theory and derive existence results by alternative methods such as the method of continuity. We investigate an indirect route by which the Leray-Schauder degree can be applied to equations which in some sense approximate (2.15). This approximation is achieved by using mollifiers to replace the Du term in the boundary condition with a smoother approximation.

We therefore consider the case in which the Du term in the equation for q is replaced to give the approximating problem

$$\begin{pmatrix} u \\ \omega \end{pmatrix} \Longrightarrow \begin{pmatrix} u - T(u,\omega) \\ \omega - q\Big(x, ((Du))_\eta\Big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.16}$$

where the notation $((\ ))_\eta$ is used to indicate a mollifier of a yet-unspecified nature. It is assumed that the mollifier parameter $\eta > 0$, and $(u,\omega) \in \Upsilon \times \Delta$ where $\Upsilon \subset C^{1,\gamma}(\overline{\Omega})$, $\Delta \subset C^{1,\rho}(\partial\Omega)$, and $\rho > \gamma$. Υ and Δ are **not** assumed to be independent of η.

We now adapt the mollification operator in order to allow arguments which make use of the strongly inwardly pointing nature of the vector valued argument of q. The gradient function is extended outside $\overline{\Omega}$ before mollification to allow evaluation on $\partial\Omega$. The details for such a process can be found in [**9**].

Theorem 2.12. *Assume that $\alpha(\cdot)$, $\beta(\cdot)$ are a BD pair. Given Υ, there exists a mollification operator, denoted by $(\ )_\eta$, and parameterised by $\eta \in (0,\eta_0)$ (where η_0 is some constant independent of L_Υ), with the following properties:*

1. *For any $x \in \partial\Omega$ such that $u(x) = \alpha(x)$, if $Du(x) - D\alpha(x) = a\vec{n}(x)$ where $a \geqslant 0$, then*

$$\Big((Du)_\eta(x) - D\alpha(x)\Big) \cdot \vec{n} > 0 \ \text{ and } \ \Big((Du)_\eta(x) - D\alpha(x)\Big) \cdot g_p(x, \alpha(x), p) \leqslant 0,$$

for all $p \in \mathbb{R}^n$ such that $|p| < |Du|_{0;\partial\Omega} + 1$.

2. *For any $x \in \partial\Omega$ such that $u(x) = \beta(x)$, if $Du(x) - D\beta(x) = -a\vec{n}(x)$ where $a \geqslant 0$, then*

$$\Big((Du)_\eta(x) - D\beta(x)\Big) \cdot \vec{n} < 0 \ \text{ and } \ \Big((Du)_\eta(x) - D\beta(x)\Big) \cdot g_p(x, \beta(x), p) \geqslant 0,$$

for all $p \in \mathbb{R}^n$ such that $|p| < |Du|_{0;\partial\Omega} + 1$.

3. $\left\|(Du)_\eta\right\|_{1,\alpha;\overline{\Omega}}$ *is bounded by* C_η, *a constant determined by* η *alone, independent of* L_Υ.
4. *For any* $u \in C^b(\overline{\Omega})$ *(where* $1 \leqslant b \leqslant 1+\alpha$*),* $(Du)_\eta \to Du$ *in* $C^{b-1}(\overline{\Omega})$

PROOF: See Appendix.

LEMMA 2.13. *If* $\alpha(\cdot), \beta(\cdot)$ *are a BD pair, and* $\gamma < \rho$, $(u,\omega) \in \Upsilon \times \Delta$, **then** *the operator*

$$(I - K_\eta)\begin{pmatrix} u \\ \omega \end{pmatrix} = \begin{pmatrix} u - T(u,\omega) \\ \omega - q\Big(x,(Du)_\eta\Big) \end{pmatrix} \tag{2.17}$$

is of the form 'Identity - compact' on $\Upsilon \times \Delta$ *for all* $0 < \eta < \eta_0$.

PROOF: The compactness of the 'first component' of the equation, $u - T(u,\omega) = 0$, follows from standard theory since $\rho > \gamma$. To show that $w - q\Big(x,(Du)_\eta\Big)$ is of the correct form, note that by construction if $u \in \Upsilon$ is bounded, we have an a priori bound on $\left\|(Du)_\eta\right\|_{1,\alpha;\partial\Omega}$. We therefore have that for each nonzero $\eta < \eta_0$, $q\Big(x,(Du)_\eta\Big)$ is bounded in $C^{1,\alpha}(\partial\Omega)$ for any $u \in \Upsilon$. The precompactness then follows as above. The mappings can be shown to be continuous from standard linear theory and the construction of the mollifier. ∎

The above mollification operator has the highly desirable feature that for solutions of (2.17), the approximation to Du can be embedded in a strongly inwardly pointing operator $\widetilde{\Psi} \in SIP(R)$ for some $R > 0$.

LEMMA 2.14. *For any* $0 < \eta < \eta_0$ *and* $(u_0, \omega_0) \in \overline{\Upsilon} \times \Delta_\infty$ *a solution of (2.17), there exists a strongly inwardly pointing operator* $\widetilde{\Psi}_{0,\eta}$ *and a positive constant* R *depending on* η *alone such that*

1. $\widetilde{\Psi}_{0,\eta} \in SIP(R)$.
2. $\widetilde{\Psi}_{0,\eta}\,\omega_0 \equiv (Du_0)_\eta$ *on* $\partial\Omega$.

PROOF: For any $0 < \eta < \eta_0$ define the operator $\widetilde{\Psi}_{0,\eta} : \Delta_\infty \to C^{1,\alpha}(\partial\Omega, \mathbb{R}^n)$ by

$$\widetilde{\Psi}_{0,\eta}\,\omega(x) \stackrel{\text{def}}{=} (DT(u_0,\omega))_\eta(x) \quad \text{on } \partial\Omega,$$

noting that for $\omega = \omega_0$, (2.17) implies

$$\begin{aligned} \widetilde{\Psi}_{0,\eta}\,\omega_0 &\stackrel{\text{def}}{=} (DT(u_0,\omega_0))_\eta \\ &= (Du_0)_\eta. \end{aligned}$$

Let $v(\omega) = T(u_0,\omega)$. By applying Theorem 2.12 to Dv and comparing with the requirements of Definition 2.3, we see that the operator $\widetilde{\Psi}_{0,\eta}$ is strongly inwardly pointing. That $\widetilde{\Psi}_{0,\eta} \in SIP(R)$ with R determined by η also follows from the properties of the $(\ )_\eta$ operator, since for any $(u,\omega) \in \Upsilon \times \Delta$, $T(u,\omega)$ is bounded in $C^0(\overline{\Omega})$ by $2\beta_\varepsilon$. ∎

3. Existence of Approximate Solution

Having modified the problem to allow the application of degree theory, we prove the existence of solutions to the modified problem by continuing the disconnection of the two facets of the problem, ultimately making the 'Dirichlet half' independent of ω and the 'boundary condition half' independent of u and Du. For clarity of exposition, this disconnection is done in three sections, with each stage using a homotopy for the Leray-Schauder degree to simplify the problem. The existence of solutions to the approximating equation $(I - K_\eta)(u,\omega) = 0$ is then proven using degree theory and the properties of strongly inwardly pointing operators.

The first simplifying homotopy replaces the gradient term with a strongly inwardly pointing operator depending upon ω.

Theorem 3.1. *Let (2.1)-(2.4) hold. If $\alpha(\cdot)$ and $\beta(\cdot)$ are a BD pair, and $g(x,z,p)$ is BD compatible with $\alpha(\cdot)$ and $\beta(\cdot)$, then for each $\eta \in (0,\eta_0)$, we may choose the constants $\gamma, L_\Upsilon, L_\Delta$ and the strongly inwardly pointing operator $\widetilde{\Psi}$ such that there is a homotopy for the Leray-Schauder degree on $\Upsilon \times \Delta$ between the equations*

$$\begin{pmatrix} u - T(u,\omega) \\ \omega - q\Big(x, (Du)_\eta(x)\Big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} u - T(u,\omega) \\ \omega - q\Big(x, \widetilde{\Psi}\omega(x)\Big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.1}$$

Proof: See Appendix.

Definition 3.2: Given $\omega \in \Delta$, we denote by $W(\omega)$ the function $u \in C^{1,\rho}(\overline{\Omega}) \cap C^{2,\alpha}(\Omega)$ which satisfies

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega \\ u(\cdot) &= \omega(\cdot) && \text{on } \partial\Omega. \end{aligned}$$

Theorem 3.3. *Let (2.1)-(2.4) hold, and let $\alpha(\cdot)$ and $\beta(\cdot)$ be a BD pair, and $g(x,z,p)$ be BD compatible with $\alpha(\cdot)$ and $\beta(\cdot)$. Then for each $\eta \in (0,\eta_0)$, one may choose the constants γ, L_Υ, and L_Δ such that there is a homotopy for the Leray-Schauder degree on $\Upsilon \times \Delta$ between the equations*

$$\begin{pmatrix} u - T(u,\omega) \\ \omega - q\Big(x, \widetilde{\Psi}\omega(x)\Big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} u - W(\omega) \\ \omega - q\Big(x, \widetilde{\Psi}\omega(x)\Big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.2}$$

Proof: See Appendix.

We continue the simplification and disconnection process by simplifying the differential equation from Laplace's equation to an equation of the form 'Identity - W_0', where W_0 is a constant function which lies strictly between $\inf_\Omega \alpha(x)$ and $\sup_\Omega \beta(x)$. Note that $W_0 \in \Upsilon$ for all choices of $L_\Upsilon > 0$.

Theorem 3.4. *Let (2.1)-(2.4) hold, and let $\alpha(\cdot)$ and $\beta(\cdot)$ be a BD pair such that $g(x,z,p)$ is BD compatible with $\alpha(\cdot)$ and $\beta(\cdot)$. Then for each $\eta \in (0,\eta_0)$, we may choose γ, L_Υ and L_Δ such that there is a homotopy for the Leray-Schauder degree on $\Upsilon \times \Delta$ between the equations*

$$\begin{pmatrix} u - W(\omega) \\ \omega - q\Big(x, \widetilde{\Psi}\omega(x)\Big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} u - W_0 \\ \omega - q\Big(x, \widetilde{\Psi}\omega(x)\Big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.3}$$

Proof: See Appendix.

Remark. It should be noted that one can easily choose $\widetilde{\Psi}, \gamma, L_\Delta$, and L_Υ in such a manner that all of the above homotopy arguments hold for the same choice of the $\widetilde{\Psi}, \gamma, L_\Delta$, and L_Υ, giving the following result.

Theorem 3.5. *Let (2.1)-(2.4) hold. If $\alpha(\cdot)$ and $\beta(\cdot)$ are a BD pair, and $g(x,z,p)$ is BD compatible with $\alpha(\cdot)$ and $\beta(\cdot)$, then for any $\eta \in (0,\eta_0)$, we may choose the constants $\gamma, L_\Upsilon, L_\Delta$ and the strongly inwardly pointing operator $\widetilde{\Psi}$ such that there is a homotopy for the Leray-Schauder degree on $\Upsilon \times \Delta$ between the equations*

$$\begin{pmatrix} u - T(u,\omega) \\ \omega - q\Big(x, (Du)_\eta(\cdot)\Big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} u - W_0 \\ \omega - q\Big(x, \widetilde{\Psi}\omega(x)\Big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

To derive an existence result for solutions of (2.17), we have the following result:

Lemma 3.6. *Assume that α and β are a BD pair. Then, for any $\eta \in (0,\eta_0)$, the constants $\gamma, L_\Upsilon, L_\Delta$ may be chosen so as to ensure that*

$$d\left(\begin{pmatrix} u - T(u,\omega) \\ \omega - q\Big(\cdot, (Du)_\eta(\cdot)\Big) \end{pmatrix}, \Upsilon \times \Delta, 0\right) = d\Big(\omega - q\Big(\cdot, \widetilde{\Psi}\omega(\cdot)\Big), \Delta, 0\Big). \tag{3.5}$$

Proof: The result follows from the preceding theorem and a degree reduction argument. The details can be found in [1].

Theorem 3.7. *If Q, G satisfy (2.1)–(2.4), then for each $\eta \in (0,\eta_0)$, the constants $\gamma, L_\Upsilon, L_\Delta$ and the strongly inwardly pointing operator $\widetilde{\Psi}$ may be chosen so that there exists at least one $u \in \Upsilon$ such that*

$$\begin{aligned} Qu &= 0 \qquad \text{in } \Omega \\ g\Big(x, u, (Du)_\eta\Big) &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.6}$$

Proof: Using BD compatibility, one may choose suitably large L_Δ and L_Υ so that $d\Big(g\Big(x,\omega,\widetilde{\Psi}\omega\Big), \Delta, 0\Big) \neq 0$. The previous result then gives existence. ∎

This solution we denote by u_η, since there is an obvious dependence upon the mollification parameter. By the standard 'boot-strapping' arguments it can be shown that $u \in C^{2,\alpha}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$.

Our ultimate aim is naturally to obtain a classical solution of the original problem by means of a limiting process. The following result, taken from [**2**], describes one approach.

Theorem 3.8. *Assume for some sequence of $\eta_i \searrow 0$ there exist u_{η_i} solving (3.6). If there exists constants $0 < \sigma$, $C < \infty$ such that*

$$\left\|u_{\eta_i}\right\|_{1,\sigma;\overline{\Omega}} \leqslant C,$$

then *there exists a classical solution $\overline{u}$ of Equation (1.1).*

The proof of this result, and a discussion of how to achieve it, can be found in [**2**].

4. Appendix: Proof of Main Results

Proof of Lemma 2.7. The existence of a BD pair is guaranteed by Lemma 2.2. By Definition 2.6, we must show that for any $R_0 > 0$ there exists an $R > R_0$ such that $g(x, z, p)$ is R-compatible with $\alpha(\cdot)$ and $\beta(\cdot)$. We therefore consider fixed R_0 and take $R > R_0$. We must show that for all $L_0 > 0$ there exists an $L > L_0$ such that for $\Delta = \Delta(L)$ we have

1. $g\Big(\cdot, w(\cdot), \widetilde{\Psi}\omega(\cdot)\Big) \not\equiv 0$ on $\partial\Omega$ if $\omega \in \partial\Delta$
2. $d\Big(g\Big(x, \omega, \widetilde{\Psi}\omega\Big), \Delta, 0\Big) \neq 0$

for all $\widetilde{\Psi} \in SIP(R)$. We therefore consider a specific $\widetilde{\Psi} \in SIP(R)$.

Since $g(x, \pm M, 0) = \pm M - q(x, 0)$, we may take M from (2.5) sufficiently large that

$$\begin{aligned} g(x, M, 0) &> \varepsilon_1 \\ g(x, -M, 0) &< -\varepsilon_1, \end{aligned} \tag{4.1}$$

where ε_1 is some positive constant. We impose the additional requirement that in the construction of $\alpha(\cdot)$ and $\beta(\cdot)$ in Lemma 2.2, σ is sufficiently small that

$$\begin{aligned} g(x, -M - \sigma c(x), -\sigma Dc(x)) &< -\varepsilon_1/2 \\ g(x, M + \sigma c(x), \sigma Dc(x)) &> \varepsilon_1/2. \end{aligned} \tag{4.2}$$

We begin by showing that one may choose L_Δ sufficiently large that for any $\omega \in \partial\Delta$, (where $\Delta = \Delta(L_\Delta)$), one has $g\Big(\cdot, \omega(\cdot), \widetilde{\Psi}\omega(\cdot)\Big) \not\equiv 0$ on $\partial\Omega$. The function $\omega(\cdot)$

https://doi.org/10.1017/S0004972700013526 Published online by Cambridge University Press

may be in the boundary of Δ because there exists an $x_1 \in \partial\Omega$ such that $\omega(x_1) = \alpha(x_1)$ (or $\beta(x_1)$), and/or because $\|\omega\|_{1,\rho;\partial\Omega} = L_\Delta$.

To deal with the case where ω 'touches' the lower solution, note that this implies that $\left(\widetilde{\Psi}\omega - D\alpha\right)(x_1)\cdot\vec{n}(x_1) > 0$ and $\left(\widetilde{\Psi}\omega - D\alpha\right)(x_1)\cdot g_p \leqslant 0$, so

$$g\left(x_1,\omega(x_1),\widetilde{\Psi}\omega(x_1)\right) = g\left(x_1,\alpha(x_1),\widetilde{\Psi}\omega(x_1)\right) \leqslant g(x_1,\alpha(x_1),D\alpha(x_1)) < -\varepsilon_1/2.$$

so $g\left(\cdot,\omega(\cdot),\widetilde{\Psi}\omega(\cdot)\right) \not\equiv 0$ on $\partial\Omega$.

The possibility that $\|\omega\|_{1,\rho;\partial\Omega} = L_\Delta$ can be simply ruled out by choosing

$$L_\Delta > \left\|q\left(\cdot,\widetilde{\Psi}\omega(\cdot)\right)\right\|_{1,\rho;\partial\Omega}, \tag{4.3}$$

which is possible since $q \in C^{1,\alpha}(\partial\Omega\times\mathbb{R}^n)$ is given and $\left\|\widetilde{\Psi}\omega\right\|_{1,\alpha;\partial\Omega} < R$ independent of L_Δ.

To show that $d\left(g\left(x,\omega,\widetilde{\Psi}\omega\right),\Delta,0\right) \neq 0$, it suffices to evaluate the degree for $\widetilde{\Psi}\equiv 0$. From basic degree theory we have $d\left(g\left(x,\omega,\widetilde{\Psi}\omega\right),\Delta,0\right) = d(I - q(x,0),\Delta,0) = 1$ since $q(x,0)\in\Delta$ by (4.3) and (4.1). The boundary operator g is therefore BD compatible with α and β. ∎

PROOF OF THEOREM 2.12.: We construct such an approximator. We assume $\alpha(\cdot)$ and $\beta(\cdot)$ are a BD pair such that $D\beta\cdot\vec{n} > S_1$, $D\beta\cdot g_p < -\tau < 0$ on $\partial\Omega$, and $\alpha(\cdot) = -\beta(\cdot)$.

The distinguishing feature of this situation is the simplicity with which one can obtain the desired properties by use of a 'scaling down' of the standard mollification. We accordingly consider a mollifier of the form $(Du)_\eta = S\varphi_\eta * Du$, where S is a constant determined by η and Du. A suitable choice of such a constant is given here.

Let $s_1(\eta,Du)$, $s_2(\eta,Du)$ be defined for $\eta > 0$ by

$$\begin{aligned} s_1(\eta,Du) &\stackrel{\text{def}}{=} 1 - \frac{\|(\varphi_\eta * Du - Du)\|_{0;\overline{\Omega}}}{S_1/2} \\ s_2(\eta,Du) &\stackrel{\text{def}}{=} \sup\{t\in[0,1] \mid (1-t)\tau \geqslant t\,\|\varphi_\eta * Du - Du\|_{0;\overline{\Omega}}\,\|g_p\|_{0;\partial\Omega(A)}\}, \end{aligned} \tag{4.4}$$

where $A = \|Du\|_{0;\partial\Omega} + \beta_\varepsilon + 1$. Note that $s_2(\eta,Du)$ is well defined since $t = 0$ is in the set. We shall take as the scaling constant

$$S = S(\eta,Du) \stackrel{\text{def}}{=} (1-\eta)\max\{0,\min\{s_1(\eta,Du),\ s_2(\eta,Du)\}\}, \tag{4.5}$$

(so $S(\eta,Du)\in[0,1]$) and consider the operator $(\ )_\eta : C^\gamma(\partial\Omega)\to C^{1,\alpha}(\partial\Omega)$ defined by

$$(Du)_\eta \stackrel{\text{def}}{=} S(\eta,Du)\,\varphi_\eta * Du. \tag{4.6}$$

We show that the above operator satisfies the required conditions. We accordingly consider a point $x_1 \in \partial\Omega$ and function u such that $Du(x_1) - D\alpha(x_1) = a\vec{n}(x_1)$ where $a \geqslant 0$. The claim that $S(\eta, Du)\varphi_\eta * Du(x_1) \cdot \vec{n}(x_1) > D\alpha(x_1) \cdot \vec{n}(x_1)$ follows from routine calculations once one notes that we can assume without loss of generality that $\varphi_\eta * Du(x_1) \cdot \vec{n} \leqslant 0$, since otherwise the desired result holds with any nonnegative constant. We can similarly assume that $\|\varphi_\eta * Du - Du\|_{0;\overline{\Omega}} \leqslant S_1/2$, since $S(\eta, Du) = 0$ otherwise, leading trivially to the desired result.

To show that $(S(\eta, Du)\varphi_\eta * Du(x_1) - D\alpha(x_1)) \cdot g_p \leqslant 0$, note that $S \leqslant s_2(\eta, Du)$. Since $Du(x_1) = D\alpha(x_1) + a\vec{n}(x_1)$ where $a \geqslant 0$, it follows that $(S\varphi_\eta * Du - D\alpha)(x_1) = (S-1)(D\alpha + a\vec{n})(x_1) - S(Du - \varphi_\eta * Du)(x_1) + a\vec{n}(x_1)$, so

$$\begin{aligned}(S\varphi_\eta * Du - D\alpha)(x_1) \cdot g_p &= (S-1)D\alpha \cdot g_p + aS\vec{n}(x_1) \cdot g_p - S(Du - \varphi_\eta * Du)(x_1) \cdot g_p \\ &\leqslant (S-1)\tau + S\,\|Du - \varphi_\eta * Du\|_{0;\overline{\Omega}}\,\|g_p\|_{0;A} < 0.\end{aligned}$$

The constructed approximation therefore satisfies the first two conditions required. The third condition follows trivially from the corresponding result for the original mollifier, since $S(\eta, Du) \in [0,1]$ is constant. We now need only show that the last condition, the convergence result, holds. This follows since for any continuous Du, as η goes to zero, $\|\varphi_\eta * Du - Du\|_{0,\overline{\Omega}}$ also goes to zero, so from (4.4) it follows that $S(\eta, Du) \to 1$ as $\eta \to 0$ (since $\tau > 0$ is constant) giving the convergence result from the equivalent result for the standard mollifier. ∎

PROOF OF THEOREM 3.1. We consider the problem

$$\begin{pmatrix} u - T(u,\omega) \\ \omega - q\Big(x, (1-\theta)(Du)_\eta(x) + \theta\widetilde{\Psi}\omega(x)\Big)\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \tag{4.7}$$

and obtain the desired homotopy by showing first that for some choice of $\widetilde{\Psi}, \gamma, L_\Delta$ and L_Υ, there are no solutions of (4.7) on the boundary of $\Upsilon \times \Delta$ for any $\theta \in [0,1]$. We then show that the above transformation is sufficiently smooth in θ. This is sufficient since the above equation is clearly of the form 'Identity - compact' for all $\theta \in [0,1]$.

Clearly $(u,\omega) \in \partial(\Upsilon \times \Delta)$ implies $\omega \in \partial\Delta$ while $u \in \overline{\Upsilon}$, or $u \in \partial\Upsilon$ while $\omega \in \overline{\Delta}$. We begin by considering the former of the two possibilities.

We therefore wish to choose $\widetilde{\Psi}, \gamma, L_\Delta$ and L_Υ in such a way that there cannot exist a solution (u_0, ω_0) to (4.7) where $\omega_0 \in \partial\Delta$ and $u_0 \in \overline{\Upsilon}$. We can therefore assume that $\alpha_\varepsilon \leqslant u_0(x) \leqslant \beta_\varepsilon$ and $\|u_0\|_{1,\gamma;\overline{\Omega}} \leqslant L_\Upsilon$, and that either there exists an $x_1 \in \partial\Omega$ such that $\omega_0(x_1) = \alpha(x_1)$ or $\beta(x_1)$, while $\|\omega_0\|_{1,\rho;\partial\Omega} \leqslant L_\Delta$, or $\|\omega_0\|_{1,\rho;\partial\Omega} = L_\Delta$ while $\alpha(x) \leqslant \omega_0(x) \leqslant \beta(x)$ on $\partial\Omega$. To consider the first of these subcases, we therefore assume that there exists a point $x_1 \in \partial\Omega$ such that

$\omega_0(x_1) = \alpha(x_1)$ while $\|\omega_0\|_{1,\rho;\partial\Omega} \leqslant L_\Delta$. We recall that Lemma 2.14 allows us to embed $(Du_0)_\eta$ in the strongly inwardly pointing operator $\widetilde{\Psi}_{0,\eta}$, where $\widetilde{\Psi}_0 \in SIP(R)$ for all $R > R_1 > 0$. Let $\widetilde{\Psi} \in SIP(R_0)$. Therefore $(1-\theta)\widetilde{\Psi}_0 + \theta\widetilde{\Psi}\omega_0 \in SIP(R)$ for $R > max\{R_0, R_1\}$. Since the boundary operator $g(x,z,p)$ is BD compatible with α and β, we may choose R suitably large while preserving the property that $g(x,z,p)$ is R-compatible with α and β. By Definition 2.5, we may therefore choose an arbitrarily large L_Δ such that $\omega_0 - q\Big(x, (1-\theta)\widetilde{\Psi}_0\omega_0 + \theta\widetilde{\Psi}\omega_0\Big) \not\equiv 0$, since $\omega_0 \in \partial\Delta$. Therefore $\omega_0 - q\Big(x, (1-\theta)(Du_0)_\eta + \theta\widetilde{\Psi}\omega_0\Big) \not\equiv 0$.

We now consider the other subcase. As before, we choose $\widetilde{\Psi} \in SIP(R)$ for some suitably large R. For notational convenience, we denote such a suitable R by C_ψ, and recall that C_η is the constant specified in Theorem 2.12. We shall also use the notation

$$\widetilde{\Omega}(A) \stackrel{\text{def}}{=} \{(x,p) \in \partial\Omega \times \mathbb{R}^n \mid |p| \leqslant A\}.$$

From (4.7)

$$
\begin{aligned}
\|\omega_0\|_{1,\rho;\partial\Omega} &= \Big\|q\Big(x, (1-\theta)(Du_0)_\eta + \theta\widetilde{\Psi}\omega_0\Big)\Big\|_{1,\rho;\partial\Omega} \\
&\leqslant C\,\|q\|_{1,\rho;\widetilde{\Omega}(C_\eta + C_\psi)}\Big(1 + \Big\|(1-\theta)(Du_0)_\eta + \theta\widetilde{\Psi}\omega_0\Big\|_{1,\rho;\partial\Omega}\Big) \\
&< C_q(1 + C_\eta + C_\psi),
\end{aligned}
\tag{4.8}
$$

where C_q is a constant determined by $q(\cdot)$ and $C_\eta + C_\psi$. By the BD compatibility of the boundary operator with α and β, one may choose $L_\Delta \geqslant C_q(1 + C_\eta + C_\psi)$, while preserving the results of the preceding subcase.

We now turn to the task of preventing solutions on that portion of $\partial(\Upsilon \times \Delta)$ which corresponds to $u_0 \in \partial\Upsilon$, while $\omega_0 \in \Delta$.

As with $\partial\Delta$, $u_0 \in \partial\Upsilon$ implies either the existence of a point $x_0 \in \overline{\Omega}$ such that $u_0(x_0) = \alpha_\varepsilon$ or β_ε, while $\|u_0\|_{1,\gamma;\Omega} \leqslant L_\Upsilon$, or alternatively $\|u_0\|_{1,\gamma;\Omega} = L_\Upsilon$ while $\alpha_\varepsilon \leqslant u_0(x) \leqslant \beta_\varepsilon$ on $\overline{\Omega}$. The first option is ruled out by the proof of Lemma 4.7, so we consider the second subcase. The established bound on L_Δ gives a $C^{1,\rho}(\partial\Omega)$ bound on solutions, so a result for the Dirichlet problem by Krylov in [5] (see also [12]) gives (for some $0 < \beta < \rho$) a bound on $\|u_0\|_{1,\beta;\overline{\Omega}}$ in terms of L_Δ. L_Υ is chosen to be greater than this bound, and γ is chosen to be β.

It has now been shown that there are no solutions of (4.7) on the boundary of $\Upsilon \times \Delta$ for any $\theta \in [0,1]$. There is a homotopy for the Leray-Schauder degree since the transformation (4.7) can easily be shown to be sufficiently continuous as a function of θ. ∎

PROOF OF THEOREM 3.3. We recall the definition of $T(u,\omega)$ from (2.14), and define $T_\theta(\cdot,\cdot)$ on $\Upsilon\times\Delta$ by taking $T_\theta(u,\omega)$ to be the solution v of the linear partial differential equation

$$(4.9)\qquad \begin{aligned} \left[\theta b^{ij}(x,u,Du)+(1-\theta)\delta^{ij}\right]D_{ij}v+\theta\, b(x,u,Du)&=0 &&\text{in } \Omega\\ v(x)&=\omega(x) &&\text{on } \partial\Omega, \end{aligned}$$

and consider solutions to the problem

$$(4.10)\qquad \begin{pmatrix} u-T_\theta(u,\omega)\\ \omega-q\left(x,\widetilde{\Psi}\omega\right)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

It is clear that $T_1(u,\omega)\equiv T(u,\omega)$, and $T_0(u,\omega)\equiv W(\omega)$. We show the homotopy independence of $T_\theta(\cdot,\cdot)$ for a suitable choice of γ, L_Δ and L_Υ. The first step is again to show that for some choice of γ, L_Δ, and L_Υ, there are no solutions on the boundary of $\Upsilon\times\Delta$ for any $\theta\in[0,1]$. By choosing the strongly inwardly pointing operator $\widetilde{\Psi}$ as in the previous homotopy, we exclude the possibility of solutions where $\omega_0\in\partial\Delta$.

If $u_0\in\partial\Upsilon$ while $\omega_0\in\Delta$, either there exists a point $x_0\in\overline{\Omega}$ such that $u_0(x_0)=\alpha_\varepsilon$ or β_ε, while $\|u_0\|_{1,\gamma;\Omega}\leqslant L_\Upsilon$, or $\|u_0\|_{1,\gamma;\Omega}=L_\Upsilon$ while $\alpha_\varepsilon\leqslant u_0(x)\leqslant\beta_\varepsilon$ on $\overline{\Omega}$. This first case is ruled out by a result given separately after the current proof.

Using the same choice of L_Δ as used in Theorem 3.1, we now rule out the second possibility by choosing suitable values of L_Υ and γ. We again do this by an application of the Krylov result for the Dirichlet problem to obtain $\|u\|_{1,\beta;\overline{\Omega}}<C$, where the constants C,γ are independent of θ. We choose $L_\Upsilon\geqslant C$ and $\gamma\leqslant\beta$.

We have now shown that there are no solutions on the boundary of $\Upsilon\times\Delta$ for any $\theta\in[0,1]$, and it remains to show that (4.10) is sufficiently continuous in θ. We write the equation in the form $(I-K_\theta)(u,\omega)=(0,0)$ and show that given an $\varepsilon>0$ and $\mathcal{D}$ a bounded subset of $\Upsilon\times\Delta$, there exists a $\delta(\varepsilon,\mathcal{D})>0$ such that

$$\|K_t(u,\omega)-K_s(u,\omega)\|=\|T_t(u,\omega)-T_s(u,\omega)\|_{1,\gamma;\Omega}\leqslant\varepsilon,$$

for any $(u,\omega)\in\mathcal{D}$ if $|t-s|<\delta$. Let $v(\cdot)$ denote $T_t(u,\omega)$, and $w(\cdot)$ denote $T_s(u,\omega)$. We see that $q=v-w$ satisfies

$$(4.11)\qquad \begin{aligned} b_t^{ij}(x)D_{ij}q&=(s-t)c(x) &&\text{in } \Omega\\ q&=0 &&\text{on } \partial\Omega, \end{aligned}$$

where $c(x)\overset{\text{def}}{=}b(x,u,Du)+\left[b^{ij}(x,u,Du)-\delta^{ij}\right]D_{ij}w$, and $b_t^{ij}(x)\overset{\text{def}}{=}tb^{ij}(x,u,Du)+(1-t)\delta^{ij}$. We note that for $(u,\omega)\in\mathcal{D}$, $b_t^{ij}(\cdot)\in C^{\alpha\gamma}(\overline{\Omega})$.

https://doi.org/10.1017/S0004972700013526 Published online by Cambridge University Press

To derive the desired *a priori* bound one may apply the Intermediate Schauder estimates of [3, Theorem 6.2] to obtain

$$\|q\|_{1,\gamma;\Omega} = |q|^{-(1+\gamma)}_{1+\gamma;\Omega} \leqslant C\,|q|^{-(1+\gamma)}_{2+\alpha\gamma/2;\Omega} \leqslant C_9\,|t-s|,$$

as desired.

We therefore have the desired homotopy for the Leray-Schauder degree. ∎

LEMMA 4.1. *Let* (u,ω) *be such that* $u = T_\theta(u,\omega)$ *and* $\omega \in \overline{\Delta}$. *Then* $\alpha_\varepsilon < u(x) < \beta_\varepsilon(x)$ *for any* $\theta \in [0,1]$.

PROOF: The result holds for $\theta = 1$ by the proof of Lemma 4.7, and for $\theta = 0$ by the maximum principle. We therefore assume that $\theta \in (0,1)$, and consider $u = T_\theta(u,\omega)$ such that there exists a point $x_1 \in \overline{\Omega}$ where $(u-\alpha_\varepsilon)(x)$ has a non-positive minimum. Since $u(\cdot) \equiv \omega(\cdot)$ on $\partial\Omega$, $x_1 \in \Omega$. Therefore $Du(x_1) = 0$, and $\Delta u(x_1) \geqslant 0$. By the nature of the b^{ij} and b (recalling the construction of the differential operator K), we have

(4.12)
$$\begin{aligned} b^{ij}(x_1,u,Du)D_{ij}u + b(x_1,u,Du) &= a^{ij}(x_1,\alpha(x_1),0)D_{ij}u(x_1) + |a(x_1,\alpha(x_1),0)| + \varepsilon \\ &\geqslant |a(x_1,\alpha(x_1),0)| + \varepsilon \;> 0. \end{aligned}$$

But since $\left[\theta b^{ij}(x,u,Du) + (1-\theta)\delta^{ij}\right] D_{ij}u + \theta\, b(x,u,Du) = 0$ in Ω and $\Delta u(x_1) \geqslant 0$, it follows from $\theta \in (0,1)$ that $b^{ij}(x_1,u,Du)D_{ij}u + b(x_1,u,Du) \leqslant 0$, contradicting (4.12). The desired result then follows. ∎

PROOF OF THEOREM 3.4.

We consider the problem

$$\begin{pmatrix} u - (1-\theta)W(\omega) - \theta W_0 \\ \omega - q\Big(x, \widetilde{\Psi}\omega(x)\Big) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.13}$$

We again show that the Leray-Schauder degree is independent of θ by first showing that there are no solutions on the boundary of the set $\Upsilon \times \Delta$ for a suitable choice of γ, L_Δ and L_Υ, and by then showing that the above problem is sufficiently continuous in θ.

As before, we need only consider the case where $u_0 \in \partial\Upsilon$, while $\omega_0 \in \Delta$. If for some $\theta \in [0,1]$ there exists a solution u_0 of (4.13) such that $u_0(x_1) = \alpha_\varepsilon$, for some $x_1 \in \overline{\Omega}$, then

$$u_0(x_1) = \alpha_\varepsilon = (1-\theta)W(\omega_0)(x_1) + \theta W_0(x_1), \tag{4.14}$$

which is impossible since both W_0 and $W(\omega_0)$ lie strictly between $\inf_\Omega \alpha(x)$ and $\sup_\Omega \beta(x)$.

https://doi.org/10.1017/S0004972700013526 Published online by Cambridge University Press

Recalling the constant C used as L_Υ in Theorem 3.3, we redefine L_Υ to be $L_\Upsilon \stackrel{\text{def}}{=} C + |W_0|$. It is clear that with this choice of L_Υ, if $u_0 = (1-\theta)W(\omega_0) + \theta W_0$, then $\|u_0\|_{1,\gamma;\Omega} < L_\Upsilon$ for any $\theta \in [0,1]$.

We have now shown that there are no solutions on the boundary of $\Upsilon \times \Delta$ for any $\theta \in [0,1]$. The uniform continuity with respect to θ can be shown trivially, so the desired homotopy has been constructed. □

CONCLUDING REMARKS. It may be possible to relax the regularity conditions imposed upon G by further use of mollifiers, bringing this aspect of the results more in line with the results in [**8**]. This option has not been pursued here since many of the techniques used would be dramatically complicated.

REFERENCES

[1] T.R. Cranny, *Leray-Schauder degree theory and partial differential equations under nonlinear boundary conditions*, Doctoral Thesis (Department of Mathematics, University of Queensland, 1992).

[2] T.R. Cranny, 'Convergence of approximate solutions of a quasilinear partial differential equation', *Bull. Austral. Math. Soc.* **50** (1994), 425–433.

[3] D. Gilbarg and L. Hörmander, 'Intermediate Schauder estimates', *Arch. Rational Mech. Anal.* **74** (1980), 297–318.

[4] D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order* (Springer-Verlag, Berlin, Heidelberg, New York, 1983).

[5] N.V. Krylov, 'On estimates for the derivatives of solutions of nonlinear parabolic equations', *Soviet Math. Dokl.* **29** (1984), 14–17.

[6] G.M.Lieberman, 'The nonlinear oblique derivative problem for quasilinear elliptic equations', *Nonlinear Anal.* **8** (1984), 49–65.

[7] G.M. Lieberman, 'Intermediate Schauder estimates for oblique derivative problems', *Arch. Rational Mech. Anal.* **93** (1985), 129–134.

[8] G.M. Lieberman and N.S. Trudinger, 'Nonlinear oblique boundary value problems for nonlinear elliptic equations', *Trans. Amer. Math. Soc.* **295** (1986), 509–546.

[9] E.M. Stein, *Singular integrals and differentiability properties of functions* (Princeton University Press, Princeton, New Jersey, 1970).

[10] H.B. Thompson, 'Second order ordinary differential equations with fully nonlinear two point boundary conditions', *Pacific J. Math.* (to appear).

[11] H.B. Thompson, 'Second order ordinary differential equations with fully nonlinear two point boundary conditions II', *Pacific J. Math.* (to appear).

[12] N.S. Trudinger, 'Boundary value problems for fully nonlinear elliptic equations', *Proc. Centre Math. Anal. Austral. Nat. Univ.* (1984), 65–83.

https://doi.org/10.1017/S0004972700013526 Published online by Cambridge University Press

Department of Mathematics
The University of Queensland
Queensland 4072
Australia

Current address:
Centre for Mathematics and it Applications
Australian National University
Canberra ACT 0200
Australia