ON THE C-PROJECTIVITY OF IDEALS IN BANACH ALGEBRAS

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The notion of projective Banach module was defined by Helemskii in [1]—the paper which properly founded the homological theory of Banach algebras. The same author introduced the definition of the (relatively) flat Banach module in [2]. Recently M. C. White [3] modified both of those definitions, introducing so called C-projective and C-flat Banach modules.

For a given constant \( C > 0 \) the Banach module \( X \) over a Banach algebra \( A \), (abbreviated below as “module”), is called \( C \)-projective [3] if for arbitrary modules \( Y, Z \) and morphism \( \phi : X \to Z \), epimorphism \( \sigma : Y \to Z \), and bounded linear operator \( j : Z \to Y \) such that \( \sigma j = 1 \), there exists a morphism \( \iota : Z \to Y \) such that \( \iota \sigma = \phi \) and \( \|\iota\| < C \|\sigma\| \|\phi\| \|j\| \). As well as [1], the paper [3] gives us the more useful equivalent definition of \( C \)-projectivity. Namely, a module \( X \) is \( C \)-projective if and only if the morphism of external multiplication

\[
\pi : A \otimes X \to X, \text{ defined by the formula } \pi(a \otimes x) = ax,
\]

has a right inverse morphism \( \rho \) such that \( \|\rho\| \leq C \). Here the symbol \( \otimes \) denotes the projective tensor product of Banach spaces [4]. If \( \|\rho\| = C \) and there is no right inverse with a norm smaller than \( C \), then it is natural to say that \( X \) is exactly \( C \)-projective. In this paper we give answers to two questions that (directly or not) were put in [3]. First, for arbitrary \( C > 1 \), we give an example of an exactly \( C \)-projective Banach \( A \)-module. (Moreover, it is a maximal ideal in a uniform algebra \( A \).) Note that \( C \)-projectivity is impossible for \( C < 1 \) and for \( C = 1 \) there exist trivial examples: consider for example any maximal ideal in the disc-algebra, corresponding to an inner point of the disc. Second, we shall show that \( C \)-projectivity does not possess the same “continuity property” as \( C \)-flatness [3]: that is, there exists a module (again a maximal ideal in the uniform algebra) that is \((C + \epsilon)\)-projective for all \( \epsilon > 0 \) but not \( C \)-projective.

As usual, we denote by \( A(E) \) the uniform algebra of functions that are continuous on the given compact subset \( E \subset \mathbb{C} \) and analytic in its interior.

EXAMPLE 1. Consider the compact subset \( K = D \cup E \) of \( \mathbb{C} \times \mathbb{R} \), where \( D = \{(z, 0) : |z| \leq 1\} \) is the closed disc and \( E = \{(z, t) : |z|^{-1} \leq |z| \leq 1, 0 < t \leq 1\} \) is the cylindric annulus. (We denote by \( E_t \) its section, where \( t \) is constant.) Consider the uniform algebra

\[
A = \{f \in C(K), f(z, 0) \in A(D), f(z, t) \in A(E_t) \text{ for } t \in (0, 1]\}
\]

PROPOSITION 1. For the maximal ideal \( M \subset A \), corresponding to the point \( O = (0, 0) \),

(1) \( M \) is a \( C \)-projective Banach \( A \)-module,

(2) \( M \) is not \( k \)-projective Banach \( A \)-module for any \( k < C \).

Proof. (1) Consider two functions: \( h \in M \) such that \( h(z, t) = z \) for \( (z, t) \in K \) and \( f(z, t) = 1/z \) for \( (z, t) \in K \setminus O \). Note that \( \|h\| = 1 \) and, for each \( m \in M \), \( fm \) is defined on \( K \setminus O \) and we extend the definition to \( K \) by continuity. We have \( \|fm\| \leq C\|m\| \), because \( |f| \leq C \) on

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E\text{\text{,}} and multiplication by z preserves the uniform norm in A(D). Now define the morphism \( \rho : M \to A \otimes M \) by the formula \( \rho(m) = mf \otimes h \). Obviously \( \pi(\rho(m)) = mfh = m \) and \( \|\rho(m)\| = \|mf\|\|h\| \leq C\|m\| \). Hence part (1) is proved.

(2) Assume the contrary; then for the epimorphism \( \omega : A \to C \) of evaluation at the point \( O \) (which has a right inverse operator \( j \) given by the natural injection) and morphism \( \phi : M \to C \) given by the formula \( \phi(f) = \frac{f}{m}(O) \) there exists a morphism \( \psi : M \to A \) such that \( \omega \psi = \phi \) and \( \|\psi\| \leq k\|\phi\| \|f\| = k \). But each morphism \( \psi : M \to A \) is a multiplication by some function \( g \in C(K \setminus O) \). See, for example, the standard argument in [5]. Since \( \|gm\| \leq k\|m\| \) we can conclude that \( |g| \leq k \) on each annulus \( E_{i} \). Since \( \omega \psi = \phi \) it is evident that \( g = \frac{1}{z} + a \) on \( D \setminus O \), where \( a \in A(D) \). By continuity \( \frac{1}{z} + a \leq k \) on \( E_{0} \) and hence on the circle \( T = \{ (z, 0) : |z| = 1/C \} \). Therefore \( |1 + az| = |gz| \leq k|z| = k/C < 1 \) on \( T \). This contradicts the maximum modulus principle, and so \( M \) is an exactly \( C \)-projective \( A \)-module.

**Example 2.** Consider the compact subset \( K = D \cup (\cup_{n \geq C + 1} E_{n}) \) of \( C \times R \), where \( D \) is the same disc and \( E_{n} = \{(z, 1/n) : 1/C - 1/n \leq |z| \leq 1 + 1/n \} \) is the closed annulus. Then

\[
A = \left\{ f \in C(K) : f(z, 0) \in A(D), f(z, 1/n) \in A(E_{n}), f\left(\frac{1}{z}, 0\right) = f\left(\frac{1}{C}, 0\right), \forall n > C + 1 \right\}
\]

is a uniform algebra. Let \( M \) be the maximal ideal, corresponding to the point \( O = (0, 0) \).

**Proposition 2.** For the maximal ideal \( M \subset A \), corresponding to the point \( O = (0, 0) \),

(1) \( M \) is a \( (C + \varepsilon) \)-projective Banach \( A \)-module for all \( \varepsilon > 0 \).

(2) \( M \) is not a \( C \)-projective Banach \( A \)-module.

**Proof.** (1) Fix \( n > C + 1 \) and let two functions \( h \in M \) and \( f \in C(K \setminus O) \) be defined by \( h(z, t) = 1 \) on \( E_{k} \) \((C < k \leq n - 1)\), but \( h(z, t) = z \) on \( E_{k} \) \((k \geq n)\), and \( f(z, t) = 1 \) on \( E_{k} \) \((C < k \leq n - 1)\), but \( f(z, t) = t \) on \( E_{k} \) \((k \geq n)\) and on \( D \setminus O \). Note that \( \|h\| = 1 + \frac{1}{n} \) and, for each \( m \in M \), we have \( \|fm\| \leq nC/(n - C)\|m\| \), because \( |f| \leq nC/(n - C) \) on \( E_{n} \) and the multiplication by \( z \) preserves the uniform norm in \( A(D) \). Now define the morphism \( \rho : M \to A \otimes M \) by the formula \( \rho(m) = mf \otimes h \). Obviously \( \pi(\rho(m)) = mfh = m \) and \( \|\rho(m)\| = \|fm\|\|h\| \leq nC/(n - C)\|m\| (1 + \frac{1}{n}) = C[1 + (C + 1)/(n - C)]\|m\| \). Since \( n \) is arbitrarily large part (1) is proved.

(2) Repeating the argument from Proposition 1 we obtain a function \( g \in C(K \setminus O) \) such that \( |g| \leq C \) on each annulus \( E_{n} \) and \( g = \frac{1}{z} + a \) on \( D \setminus O \), where \( a \in A(D) \). As the inner circles \( T_{n} \) of \( E_{n} \) tend to the circle \( T \) of radius \( 1/C \) from \( D \), by continuity we obtain \( |g| \leq C \) on \( T \). Hence \( |1 + az| = |gz| \leq C \). \( \frac{1}{C} = 1 \) on \( T \). Using the maximum modulus principle we conclude that \( a \equiv 0 \). Thus \( g \equiv \frac{1}{z} \) on \( D \) and so \( g\left(\frac{1}{C}, 0\right) \equiv C \); also by definition of the algebra \( A \), \( g\left(\frac{1}{C}, 1/n\right) \equiv C \), for all \( n \). Applying the maximum modulus principle to each annulus \( E_{n} \), we conclude that \( g \equiv C \) on \( E_{n} \). By continuity \( g \equiv C \) on \( T \) giving a contradiction.

Note that both examples represent so-called non-idempotent maximal ideals; (that is \( M \not\equiv \overline{M}^{2} \)). We know almost nothing about the exact estimates of \( C \)-projectivity in the idempotent case. If we analyse Helemskii’s original proof of the projectivity of the algebra of convergent sequences \( c_{0} \) (and the algebra \( l_{1} \) of summable sequences) one can see that both these algebras are \( 1 \)-projective [1]. The author can generalize this result to the algebra \( C(K) \), where \( X \) is a semi-discrete compact set. Let \( X \) be a compact set; denote by \( X' \) the set of its
accumulation points and $X^{(n+1)} = X^{(n)}$ ($n \in \mathbb{N}$). If $X^{(n)}$ is empty for some $n \in \mathbb{N}$, we say $X$ is a semidiscrete compact set. As for the algebra $C[0; 1]$ we can only see from [1] that the maximal ideals in it are 2-projective, but the constant 2 seems not to be the best possible.

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REFERENCES


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