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A NON-HAUSDORFF MULTIFUNCTION ASCOLI THEOREM FOR k_3-SPACES

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1. Introduction. A non-Hausdorff Ascoli theorem for continuous functions was established in [**6**]. The present purpose is to extend this result to point-compact continuous multifunctions, using Levine's generalization for closed subsets [**12**]. The paper is organized as follows: the object of section 2 is to establish the necessary multifunction lemmas and to introduce the notion of a Tychonoff set; section 3 generalizes to multifunction context the partial exponential law of R. H. Fox [**9**, p. 430], and establishes a special exponential law for multifunctions; section 4 concerns the crucial properties of even continuity for multifunctions, introduced in [**8**]; the main theorem of the paper is established in section 5. In this section we show that the main theorem of the paper contains all known Ascoli theorems for continuous functions or point-compact continuous mutifunctions.

Unexplained terminology is that of Kelley [**11**].

2. Multifunctions. We review the established definitions for multifunctions [**3**; **15**; **22**]. Let X, Y be non-empty sets. A *multifunction* is a point to set correspondence $f : X \rightarrow Y$ such that, for all $x \in X$, fx is a non-empty subset of Y. For $A \subseteq X$, $B \subseteq Y$ it is customary to write $f(A) = \bigcup_{x \in A} fx$, $f^-(B) = \{x : x \in X \text{ and } fx \cap B \neq \emptyset\}$ and $f^+(B) = \{x : x \in X \text{ and } fx \subseteq B\}$. If Y is a topological space, a multifunction $f : X \rightarrow Y$ is *point-compact* (*point-closed*) if fx is compact (closed) for all $x \in X$. If X, Y are topological spaces, a multifunction $f : X \rightarrow Y$ is *lower semi-continuous* (*upper semi-continuous*) if $f^-(U)$ ($f^+(U)$) is open in X whenever U is open in Y. A multifunction $f : X \rightarrow Y$ on a topological space X to a topological space Y is *continuous* if it is both lower semi-continuous and upper semi-continuous. Henceforth, the set of all point-compact continuous multifunctions, continuous functions, on a topological space X to a topological space Y will be denoted $\mathscr{C}(X, Y)$, $C(X, Y)$, respectively.

2.1. LEMMA. *Let f be a continuous multifunction on a topological X to a regular space Y and let U be an open subset of Y.*

(i) *If $x \in f^-(U)$ there exists a closed neighbourhood V of x such that $V \subseteq$ $^-(U)$.*

(ii) *If $x \in f^+(U)$ and f is point-compact, there exists a closed neighbourhood V of x such that $V \subseteq f^+(U)$.*

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Proof. (i) Let $y \in fx \cap U$. There exists an open neighbourhood W of y such that $\bar{W} \subseteq U$. Then $x \in f^-(W) \subseteq f^-(\bar{W}) \subseteq f^-(U)$, and since f is continuous, $V = f^-(\bar{W})$ is the required closed neighbourhood of x.

(ii) Since $fx \subseteq U$, fx is compact and Y is regular, there exists an open neighbourhood W of fx such that $fx \subseteq W \subseteq \bar{W} \subseteq U$. Then $x \in f^+(W) \subseteq f^+(\bar{W}) \subseteq f^+(U)$. Thus $V = f^+(\bar{W})$ is the required closed neighbourhood of x.

Let $\{Y_x\}_{x \in X}$ be a family of non-empty sets. The *m-product* $P\{Y_x : x \in X\}$ of the Y_x is the set of all multifunctions $f : X \to \bigcup_{x \in X} Y_x$ such that $fx \subseteq Y_x$ for all $x \in X$. In the case $Y_x = Y$ for all $x \in X$, the *m-product* of the Y_x, denoted Y^{mX}, is the set of all multifunctions on X to Y. For $x \in X$, the *x-projection* $pr_x : P\{Y_x : x \in X\} \to Y_x$ is the multifunction defined by $pr_x f = fx$. If the Y_x are topological spaces, the *pointwise topology* τ_p on $P\{Y_x : x \in X\}$ is defined to be the topology having as open subbase the sets of the forms $pr_x^-(U_x)$, $pr_x^+(U_x)$, where U_x is open in Y_x, $x \in X$ [**13**; **20**].

For $F \subseteq Y^{mX}$, $x \in X$, we write $F[x] = \bigcup_{f \in F} fx$. Let Y be a topological space. We say that a subset F of Y^{mX} is *pointwise bounded* if $F[x]$ has compact closure in Y for all $x \in X$. We say that a subset T of Y^{mX} is *Tychonoff* if, for every pointwise bounded subset F of T, $T \cap P\{\overline{F[x]} : x \in X\}$ is τ_p-compact. We cite four examples:

(1) Y^X is Tychonoff, by the classical Tychonoff theorem.

(2) Y^{mX} is Tychonoff, by the theorem of Lin [**13**, p. 400].

(3) The set of all point-closed members of Y^{mX} is Tychonoff, by [**7**, Corollary 2].

(4) $(Y^{mX})_0$, the set of all point-compact members of Y^{mX}, is Tychonoff, by [**7**, Corollary 3].

2.2 Lemma. *If F is a pointwise bounded subset of a Tychonoff set T, then the τ_p-closure of F in T is compact.*

Proof. Let $\bar{F}$ denote the τ_p-closure of F in T. Since $T \cap P\{\overline{F[x]} : x \in X\}$ is a τ_p-compact subset of T, it suffices to show that $\bar{F} \subseteq P\{\overline{F[x]} : x \in X\}$. Let $f \in \bar{F}$. We must show that, for $x \in X$, $y \in fx$ and an open neighbourhood V of y, $F[x] \cap V \neq \emptyset$. Since $M = \{h : h \in T \text{ and } hx \cap V \neq \emptyset\}$ is a τ_p-neighbourhood of f in T, there exists $h' \in M \cap F$. Then $h'x \cap V \neq \emptyset$ and $h'x \subseteq F[x]$, so $F[x] \cap V \neq \emptyset$.

Let X, Y be topological spaces. The multifunction $(f, x) \to fx$ on $Y^{mX} \times X$ to Y, or any restriction, will be denoted by the sumbol ω. Let $F \subseteq Y^{mX}$. A topology τ on F is said to be *jointly continuous* if $\omega : (F, \tau) \times X \to Y$ is continuous [**20**, p. 48]. The *compact open topology* τ_c on Y^{mX} is defined to be the topology having as open subbase the sets of the forms $\{f : f(K) \subseteq U\}$, $\{f : fx \cap U \neq \emptyset \text{ for all } x \in K\}$, where K is a compact subset of X and U is open in Y [**14**, p. 742; **20**, p. 47]. Obviously τ_c is larger than τ_p.

3. Exponential law. Let X, Y, Z be topological spaces. An element $f \in Z^{m(X\times Y)}$ determines the function $\tilde{f} : x \to f(x, \cdot)$ on X to Z^{mY}. The function $\mu : f \sim \tilde{f}$, called the *exponential map*, is a bijection of $Z^{m(X\times Y)}$ onto $(Z^{mY})^X$. When τ is a topology on $\mathscr{C}(X, Y)$, we say that (X, Y, Z, τ) satisfies the *partial exponential law* if

$$\mu(\mathscr{C}(X \times Y, Z)) \subseteq C(X, (\mathscr{C}(Y, Z), \tau)).$$

If, further, equality holds, (X, Y, Z, τ) is said to satisfy the *exponential law*.

We extend Lemma 1 of R. H. Fox [**9**, p. 430] to multifunctions:

3.1. LEMMA. *(X, Y, Z, τ_c) satisfies the partial exponential law.*

Proof. Let $f \in \mathscr{C}(X \times Y, Z)$. Let $x \in X$. Since $f(x, \cdot) = f \circ j$, where $j(y) = (x, y)$, $f(x, \cdot)$ is continuous [**22**, p. 35]. Thus $\tilde{f}$ maps X into $\mathscr{C}(Y, Z)$. It remains to show that $\tilde{f} : X \to (\mathscr{C}(Y, Z), \tau_c)$ is continuous.

Let $M = \{h : h \in \mathscr{C}(Y, Z) \text{ and } h(K) \subseteq U\}$, where K is a compact subset of Y and U is open in Z. Let $x_0 \in \tilde{f}^{-1}(M)$. Then $f(x_0, \cdot) \in M$, so $\{x_0\} \times K \subseteq f^+(U)$. By the theorem of Wallace, there is a neighbourhood V of x_0 such that $V \times K \subseteq f^+(U)$. Let $x \in V$. Then, for all $y \in K$, $\tilde{f}(x)y = f(x, y) \subseteq U$, so $\tilde{f}(x)(K) \subseteq U$. Thus $x \in \tilde{f}^{-1}(M)$, and we have shown that $\tilde{f}^{-1}(M)$ is open in X.

Let $M = \{h : h \in \mathscr{C}(Y, Z) \text{ and } hy \cap U \neq \emptyset \text{ for all } y \in K\}$, where K is a compact subset of Y and U is open in Z. Let $x_0 \in \tilde{f}^{-1}(M)$. Then $f(x_0, \cdot) \in M$, so $\{x_0\} \times K \subseteq f^-(U)$. There is a neighbourhood V of x_0 such that $V \times K \subseteq f^-(U)$. Let $x \in V$. Then, for all $y \in K$, $\tilde{f}(x)y \cap U \neq \emptyset$, so $\tilde{f}(x) \in M$. Thus $x \in \tilde{f}^{-1}(M)$, and we have shown that $\tilde{f}^{-1}(M)$ is open in X.

Let X, Y, Z be topological spaces. The symbol $S_k(X \times Y, Z)$ denotes the set of all multifunctions $f \in (Z^{m(X\times Y)})_0$ such that, for each $x \in X$ and each compact subset K of Y, the restrictions $f|\{x\} \times Y$, $f|X \times K$ are continuous.

3.2. LEMMA. *If Z is regular, then $\mu^{-1}(C(X, (\mathscr{C}(Y, Z), \tau_c)) \subseteq S_k(X \times Y, Z)$.*

Proof. Let $f \in \mu^{-1}(C(X, (\mathscr{C}(Y, Z), \tau_c))$. Since $f(x, \cdot)$ is point-compact for all $x \in X$, f is point-compact.

Let $x \in X$ be fixed. Consider $j(y) = (x, y)$ as the homeomorphism of Y onto $\{x\} \times Y$; then $f|\{x\} \times Y = \tilde{f}(x) \circ j^{-1}$ is continuous.

Let $(x_0, y_0) \in (f|X \times K)^-(U) = f^-(U) \cap (X \times K)$, where K is a compact subset of Y and U is open in Z. Then $y_0 \in (\tilde{f}(x_0))^-(U) \cap K$. By Lemma 2.1, there is a closed neighbourhood V of y_0 such that $V \subseteq (\tilde{f}(x_0))^-(U)$. Then $K' = V \cap K$ is a compact neighbourhood of y_0 in K, and

$$M = \{h : h \in \mathscr{C}(Y, Z) \text{ and } hy \cap U \neq \emptyset \text{ for all } y \in K'\}$$

is a neighbourhood of $\tilde{f}(x_0)$ in $(\mathscr{C}(Y, Z), \tau_c)$. Since $\tilde{f}$ is continuous, $\tilde{f}^{-1}(M) \times K'$ is a neighbourhood of (x_0, y_0) in $X \times K$, which is contained in $f^-(U) \cap (X \times K)$. Now let $(x_0, y_0) \in (f|X \times K)^+(U) = f^+(U) \cap (X \times K)$, where K is a compact subset of Y and U is open in Z. Then $y_0 \in (\tilde{f}(x_0))^+(U) \cap K$. There exists a closed neighbourhood V of y_0 contained in $(\tilde{f}(x_0))^+(U)$. Then $K' = V \cap K$ is a compact neighbourhood of y_0 in K, and

$$M = \{h : h \in \mathscr{C}(Y, Z) \text{ and } h(K') \subseteq U\}$$

is a neighbourhood of $\tilde{f}(x_0)$ in $(\mathscr{C}(Y, Z), \tau_c)$. Since $\tilde{f}$ is continuous, $\tilde{f}^{-1}(M) \times K'$ is a neighbourhood of (x_0, y_0) in $X \times K$, which is contained in $f^+(U) \cap (X \times K)$.

Let $X = (X, \tau)$ be a topological space. The *k-extension* of τ is the family $k(\tau)$ of all subsets U of X such that $U \cap K$ is open in K for every compact subset K of X. It is clear that $k(\tau)$ is a topology on X which is larger than τ. A topological space (X, τ) is called a *k-space* if $\tau = k(\tau)$ [**5**, p. 79]. Locally compact spaces and spaces satisfying the first countability axiom are familiar examples of k-spaces.

Let X, Y be topological spaces. A function $f : X \to Y$ is called *k-continuous* if its restriction to each compact subset of X is continuous [**4**, p. 275]. Henceforth, the set of all k-continuous functions on X to Y will be denoted $C_k(X, Y)$. It can be shown that a topological space X is a k-space if and only if $C_k(X, Y) = C(X, Y)$ for every topological space Y [**16**, Theorem 3.2].

A topological space X is a *k_3-space* if $C_k(X, Y) = C(X, Y)$ for every regular space Y [**19**, p. 195]. Thus a k-space is a k_3-space but not conversely. In fact, the product of uncountably many copies of the real line, which is not a k-space, is a k_3-space [**19**, Theorem 5.6 (i)].

3.3. LEMMA. *If X is compact, Y is a k_3-space and Z is regular, then* $S_k(X \times Y, Z) = \mathscr{C}(X \times Y, Z)$.

Proof. For topological spaces U, V, W let $\mathscr{C}_{k\times k}(U \times V, W)$ denote the set of all $f \in (W^{m(U\times V)})_0$ such that, whenever K, K' are compact subsets of U, V, respectively, $f|K \times K'$ is continuous. Then $\mathscr{C}(Y \times X, Z) \subseteq \mathscr{C}_{k\times k}(Y \times X, Z)$; we will prove the inverse inclusion.

Let $f \in \mathscr{C}_{k\times k}(Y \times X, Z)$. Let $y \in Y$. Let $j(x) = (y, x)$ be the homeomorphism of X onto $\{y\} \times X$. Since X is compact, $\tilde{f}(y) = (f|\{y\} \times X) \circ j$ is continuous, so $\tilde{f}$ maps Y into $\mathscr{C}(X, Z)$. Let K be a compact subset of Y, so that $g = f|K \times X$ is continuous. Then, by Lemma 3.1, $\tilde{g} : K \to (\mathscr{C}(X, Z), \tau_c)$ is continuous. But $\tilde{g} = \tilde{f}|K$, so $\tilde{f}$ is k-continuous, and we have shown that $\mathscr{C}_{k\times k}(Y \times X, Z) \subseteq \mu^{-1}(C_k(Y, (\mathscr{C}(X, Z), \tau_c)))$. Since Y is a k_3-space and $(\mathscr{C}(X, Z), \tau_c)$ is regular [**20**, p. 48], $C_k(Y, (\mathscr{C}(X, Z), \tau_c)) = C(Y, (\mathscr{C}(X, Z), \tau_c))$. By Lemma 3.2,

$$\mu^{-1}(C(Y, (\mathscr{C}(X, Z), \tau_c))) \subseteq S_k(Y \times X, Z).$$

Because X is compact, $S_k(Y \times X, Z) = \mathscr{C}(Y \times X, Z)$. We have shown that $\mathscr{C}_{k\times k}(Y \times X, Z) = \mathscr{C}(Y \times X, Z)$.

Because $(x, y) \to (y, x)$ is a homeomorphism of $X \times Y$ onto $Y \times X$, this last equation may be written $\mathscr{C}_{k\times k}(X \times Y, Z) = \mathscr{C}(X \times Y, Z)$. Then, by the definition of $S_k(X \times Y, Z)$, we have $\mathscr{C}(X \times Y, Z) \subseteq S_k(X \times Y, Z) \subseteq \mathscr{C}_{k\times k}(X \times Y, Z) = \mathscr{C}(X \times Y, Z)$.

The main result of this section now follows from Lemmas 3.1, 3.2 and 3.3:

3.4. THEOREM. *If X is compact, Y is a k_3-space and Z is regular, then (X, Y, Z, τ_c) satisfies the exponential law.*

4. Even continuity. Let X, Y be topological spaces and let $F \subseteq Y^{mX}$. We say that F is *evenly continuous* if, whenever $x \in X$, K is a compact subset of Y and V is a neighbourhood of K, there exist neighbourhoods U, W of x, K, respectively, such that

(a) $f \in F$ and $fx \cap W \neq \emptyset$ imply $U \subseteq f^-(V)$, and

(b) $f \in F$ and $fx \subseteq W$ imply $U \subseteq f^+(V)$.

This is an extension to multifunction context of the original Kelley-Morse definition [**11**, p. 235], where points of Y have been generalized to compact subsets of Y. It is easily verified that this definition extends also the multifunction extension of even continuity due to Y. F. Lin and D. A. Rose [**14**, p. 742]. If F is evenly continuous, then every member of F is lower semicontinuous; if, further, every member of F is point-compact, then every member of F is also upper semi-continuous, hence continuous.

4.1. LEMMA. *Let F be a set of point-compact multifunctions on a topological space X to a regular space Y, and let $\bar{F}$ denote the τ_p-closure of F in $(Y^{mX})_0$. If F is evenly continuous, then $\bar{F}$ is evenly continuous and τ_p on $\bar{F}$ is jointly continuous.*

Proof. Let $x \in X$, let K be a compact subset of Y, and let V be a closed neighbourhood of K. There exist open neighbourhoods U, W of x, K, respectively, such that, for all $f \in F$, $fx \cap W \neq \emptyset$ implies $U \subseteq f^-(V)$ and $fx \subseteq W$ implies $U \subseteq f^+(V)$. Let $g \in \bar{F}$ be such that $gx \cap W \neq \emptyset$. Let $\{g_\alpha\}$ be a net in F which is τ_p-convergent to g. Since $\{h : h \in (Y^{mX})_0$ and $hx \cap W \neq \emptyset\}$ is a τ_p-neighbourhood of g, $g_\alpha x \cap W \neq \emptyset$ eventually, so $U \subseteq g_\alpha^-(V)$ eventually. Suppose that $U \not\subseteq g^-(V)$. Then, for some $u \in U$, $gu \subseteq Y - V$, so $g_\alpha u \subseteq Y - V$ eventually, which is a contradiction. Now let $g \in \bar{F}$ be such that $gx \subseteq W$. Let $\{g_\alpha\}$ be a net in F which is τ_p-convergent to g. Since $\{h : h \in (Y^{mX})_0$ and $hx \subseteq W\}$ is a τ_p-neighbourhood of g, $g_\alpha x \subseteq W$ eventually, so $U \subseteq g_\alpha^+(V)$ eventually. Suppose that $U \not\subseteq g^+(V)$. Then, for some $u \in U$, $gu \cap (Y - V) \neq \emptyset$, so $g_\alpha u \cap (Y - V) \neq \emptyset$ eventually, which is a contradiction. This completes the proof of the even continuity of $\bar{F}$.

Let $\omega : (\bar{F}, \tau_p) \times X \to Y$. Suppose the $(f, x) \in \omega^-(G)$, where G is open in Y. Choose $y \in fx \cap G$. Since $\bar{F}$ is evenly continuous, there exist neighbourhoods U, W of x, y, respectively, such that $g \in \bar{F}$ and $gx \cap W \neq \emptyset$ imply $U \subseteq g^-(G)$. Then $\{h : h \in \bar{F}$ and $hx \cap W \neq \emptyset\} \times U$ is a neighbourhood of (f, x), which is contained in $\omega^-(G)$. Now suppose that $(f, x) \in \omega^+(G)$, where G is open in Y.

Since $\bar{F}$ is evenly continuous and fx is compact, there exist neighbourhoods U, W of x, fx, respectively, such that $g \in \bar{F}$ and $gx \subseteq W$ imply $U \subseteq g^+(G)$. Then $\{h : h \in \bar{F} \text{ and } hx \subseteq W\} \times U$ is a neighbourhood of (f, x), which is contained in $\omega^+(G)$.

4.2. LEMMA. *Let $f : X \times Y \rightarrow Z$ be a continuous multifunction. If X is compact and Z is regular, then the set $F = \{ f(x, \cdot) : x \in X\}$ is evenly continuous.*

Proof. Let $y \in Y$, let K be a compact subset of Z, and let V be an open neighbourhood of K. Let W be a closed neighbourhood of K which is contained in V. Since $f(\cdot, y)$ is continuous, $K_1 = f(\cdot, y)^-(W)$ and $K_2 = f(\cdot, y)^+(W)$ are closed in X, therefore compact. Thus the second projections $pr_2 : K_1 \times Y \rightarrow Y, pr_2 : K_2 \times Y \rightarrow Y$ are closed, so that $U_1 = Y - pr_2[(K_1 \times Y) - f^-(V)]$, $U_2 = Y - pr_2[(K_2 \times Y) - f^+(V)]$ are open in Y. Since $K_1 \subseteq f(\cdot, y)^-(V)$, $K_2 \subseteq f(\cdot, y)^+(V)$, we have $K_1 \times \{y\} \subseteq f^-(V)$, $K_2 \times \{y\} \subseteq f^+(V)$. Hence $y \notin pr_2[(K_1 \times Y) - f^-(V)], y \notin pr_2[(K_2 \times Y) - f^+(V)]$, that is, $y \in U_1 \cap U_2 = U$.

Let $g \in F$ be such that $gy \cap W \neq \emptyset$, so that $g = f(x, \cdot)$ for some $x \in K_1$. Let $u \in U$, so that $u \notin pr_2[(K_1 \times Y) - f^-(V)]$. Then $(x, u) \in f^-(V)$, that is, $gu \cap V \neq \emptyset$. Now let $g \in F$ be such that $gy \subseteq W$, so that $g = f(x, \cdot)$ for some $x \in K_2$. Let $u \in U$, so that $u \notin pr_2[(K_2 \times Y) - f^+(V)]$. Then $(x, u) \in f^+(V)$, that is, $gu \subseteq V$.

5. Ascoli theorem. Let X be a topological space and let $A \subseteq X$. Following [**12**], we say that A is *g-closed* if $\bar{A} \subseteq U$ whenever U is an open set containing A.

We are in position to establish the non-Hausdorff Ascoli theorem of the paper:

5.1. THEOREM. *Let X, Y be topological spaces. Let T be a Tychonoff subset of $(Y^{mX})_0$ and let $F \subseteq (T \cap \mathscr{C}(X, Y), \tau_c)$. If Y is regular, then the following conditions are sufficient for the compactness of F:*

(a) *F is g-closed in $T \cap \mathscr{C}(X, Y)$,*

(b) *F is pointwise bounded, and*

(c) *F is evenly continuous.*

If, further, X is a k_3-space, then the conditions (a), (b) *and* (c) *are necessary for the compactness of F.*

Proof. Sufficiency. Let $\bar{F}$ be the τ_p-closure of F in T. Since $T \subseteq (Y^{mX})_0$, (c) implies, by Lemma 4.1, that $\omega : (\bar{F}, \tau_p) \times X \rightarrow Y$ is continuous and $\bar{F}$ is evenly continuous; in particular $\bar{F} \subseteq \mathscr{C}(X, Y)$. Then, by Lemma 3.1, $\tilde{\omega} : \bar{F} \rightarrow (\mathscr{C}(X, Y), \tau_c)$ is continuous. Since T is a Tychonoff set, (b) implies, by Lemma 2.2, that $\bar{F}$ is τ_p-compact, so $\tilde{\omega}(\bar{F}) = \bar{F}$ is a τ_c-compact subset of $T \cap \mathscr{C}(X, Y)$. But (a) implies, by Theorem 2.9 of [**12**], that F is g-closed in $(\bar{F}, \tau_c)$. Since $(\bar{F}, \tau_c)$ is compact, Theorem 3.1 of [**12**] implies that F is τ_c-compact.

Necessity. Since $(T \cap \mathscr{C}(X, Y), \tau_c)$ is regular, Theorem 3.5 of [**12**] implies that F is g-closed in $T \cap \mathscr{C}(X, Y)$. Let $x \in X$. Since F is τ_p-compact, $F[x] = pr_x(F)$ is compact [**3**, p. 116], so $F[x]$ has compact closure in Y. Since the inclusion $i : F \to (\mathscr{C}(X, Y), \tau_c)$ is continuous, Theorem 3.4 implies that $\mu^{-1}(i) = \omega : F \times X \to Y$ is continuous. Then, by Lemma 4.2, $F = \{\omega(f, \cdot) : f \in F\}$ is evenly continuous.

5.2. COROLLARY. *Let $F \subseteq (\mathscr{C}(X, Y), \tau_c)$. If Y is regular, then the following conditions are sufficient for the compactness of F:*

(a) *F is g-closed in $\mathscr{C}(X, Y)$,*

(b) *F is pointwise bounded, and*

(c) *F is evenly continuous.*

If, further, X is a k_3-space, then the conditions (a), (b) *and* (c) *are necessary for the compactness of F.*

5.3. COROLLARY. *Let $F \subseteq (C(X, Y), \tau_c)$. If Y is regular, then the following conditions are sufficient for the compactness of F:*

(a) *F is g-closed in $C(X, Y)$,*

(b) *F is pointwise bounded, and*

(c) *F is evenly continuous.*

If, further, X ix a k_3-space, then the conditions (a), (b) *and* (c) *are necessary for the compactness of F.*

Remarks. (1) Let R be the equivalence relation on a regular space introduced in [**6**, p. 634]. If E^* is closed then $E \subseteq \bar{E} \subseteq E^*$, and therefore E is g-closed. If K is compact, then K^* is closed. Consequently, Corollary 5.3 contains the Theorem 4.1 of [**6**, p. 635], which in turn contains the well known Ascoli theorems of Arens [**1**, p. 491], Myers [**17**, pp. 497–498], Gale [**10**, p. 304], Kelley-Morse [**11**, p. 236], Bagley-Yang [**2**, pp. 704–705] and Corollary 4.4 of Noble [**18**, p. 403].

(2) A non-Hausdorff Ascoli theorem for point-compact continuous multifunctions on a k-space was established in [**8**]. It was shown that this theorem contains the multifunction Ascoli theorems of Mancuso [**15**, p. 470] and Smithson [**21**, p. 259]. Since a k-space is also a k_3-space, our Theorem 5.1 contains this result.

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