BULL. AUSTRAL. MATH. SOC. 73D10, 73G05
VOL. 8 (1973), 61-73.

Some dynamic properties of a prestressed incompressible hyperelastic material

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The work continues some earlier investigations into dynamic properties of prestressed incompressible elastic materials. Whereas the material was previously assumed to be a Mooney material, it is here allowed to have any strain energy function. Plane wave propagation and the motions caused by an impulsive line of traction are examined. The results obtained are compared with the earlier work.

1. Introduction

This work is an extension of some earlier work by Belward [1] concerning the small motions of an elastic solid about a state of finite pure homogeneous biaxial deformation. In the previous work the material was assumed to be a Mooney material; here it is only assumed to be isotropic, incompressible and hyperelastic. The results of the analysis concern plane wave propagation and the response of the material to line impulses of traction. The generalisation to any strain energy function makes the analysis more detailed but not intractable. Many of the results obtained in [1] are preserved and the results for the two problems can easily be compared.

The secular equation for the speeds of propagation of plane waves is derived. It is factored into two simple terms from which the two permissible wave speeds are deduced and the corresponding amplitudes

Received 4 September 1972.

https://doi.org/10.1017/S0004972700045470 Published online by Cambridge University Press

calculated. The two waves can be characterised by their dependence on the strain energy function. One of these involves the first order derivatives of the strain energy function only, while the other depends on both first and second order derivatives. Thus only one wave distinguishes between a Mooney material and one with a more general strain energy function. By requiring that there exist two real wave speeds in every direction in the material, the well known Baker-Ericksen conditions on the strain energy function are reproduced. The same type of argument is also used to determine restrictions on the constants which appear in a strain energy function proposed by Signorini [3].

The response of the material to impulsive lines of traction applied along the principal directions of the underlying strain is also examined. Some favourable uncoupling of the equations of motion occurs (this can be anticipated from the secular equation) and this enables the displacement and pressure fields at small and large times to be investigated in some detail.

2. Basic equations

Frequent reference will be made to the earlier paper by Belward [1] and the analysis, wherever it follows that in [1], will be given in outline only.

Consider a body of homogeneous, isotropic, incompressible, hyperelastic material with strain energy function W. Let the body be given an initial finite static pure homogeneous deformation in which a particle at X_i (in a rectangular cartesian coordinate system) moves to y_i. The Cauchy stress is t_{ij} where

$$t_{ij} = -p\delta_{ij} + \left(2W_1+2I_1W_2\right)B_{ij} - 2W_2B_{is}B_{sj} , \tag{2.1}$$

(repeated suffixes imply summation over $s = 1, 2, 3$), B_{ij} is the left Cauchy-Green strain tensor,

$$B_{ij} = \frac{\partial y_i}{\partial X_s}\frac{\partial y_j}{\partial X_s} , \tag{2.2}$$

W_1 and W_2 are derivatives of the strain energy function with respect to

https://doi.org/10.1017/S0004972700045470 Published online by Cambridge University Press

two invariants of the strain,

$$(2.3)\quad W_1 = \frac{\partial W}{\partial I_1}\ ,\quad I_1 = B_{ss}\ ;\quad W_2 = \frac{\partial W}{\partial I_2}\ ,\quad I_2 = \tfrac{1}{2}\left(B_{ss}B_{qq} - B_{sq}B_{sq}\right)\ ,$$

and p is a hydrostatic pressure.

Suppose that a particle at y_i now moves to x_i , and that this displacement is small in the sense that

$$(2.4)\qquad x_i - y_i = \varepsilon u_i$$

and all terms involving powers of ε greater than unity can be ignored. Now if the stress, strain energy, pressure, etc are all expanded in powers of ε thus:

$$(2.5)\qquad t_{ij}(\varepsilon) = t_{ij} + \varepsilon t_{ij}^{(1)} + \dots\ ,$$

the perturbed stress can be shown to be given by

$$\begin{aligned}(2.6)\quad t_{ij}^{(1)} = -p^{(1)}\delta_{ij} &+ J_1 W_2 B_{ij} + 2\left(W_1 + I_1 W_2\right)\left(u_{i,s}B_{sj} + u_{j,s}B_{si}\right)\\ &- 2W_2\left(u_{i,q}B_{sq}B_{sj} + u_{j,q}B_{is}B_{sq} + u_{s,q}\left(B_{iq}B_{sj} + B_{is}B_{jq}\right)\right)\\ &+ \left\{J_1 W_{11} + \left(J_1 I_1 + J_2\right)W_{12} + I_1 J_2 W_{22}\right\}B_{ij} - \left\{J_1 W_{12} + J_2 W_{22}\right\}B_{is}B_{sj}\ ,\end{aligned}$$

where

$$(2.7)\quad u_{i,j} = \frac{\partial u_i}{\partial y_j}\ ,\quad J_1 = 4u_{s,q}B_{sq}\ ,\quad J_2 = 4u_{s,q}\left(B_{sq}B_{rr} - B_{sr}B_{qr}\right)\ ,$$

and the quantities I, J, W and B are evaluated in the state of homogeneous deformation. (Some small changes have been made to the notation used in [1] and some minor misprints corrected.) For a pure homogeneous biaxial deformation with principal axes parallel to the coordinate axes we have

$$(2.8)\qquad y_1 = \mu X_1\ ,\quad y_2 = \mu X_2\ ,\quad y_3 = \lambda X_3\ ,$$

with $\mu^2\lambda = 1$, the incompressibility condition.

When (2.8) is introduced into equations (2.6) and (2.7) and the latter equation entered into the equations of motion the following form for the equations of motion for the perturbed displacements u_i is obtained:

https://doi.org/10.1017/S0004972700045470 Published online by Cambridge University Press

$$(2.9)\quad -p_{,1} + 2au_{1,11} + (a-b)u_{1,22} + cu_{1,33} + (a+b)u_{2,21} + (a+h)u_{3,31} = \\ = \rho u_{1,tt} - \rho f_1 \; ,$$

$$(2.10)\quad -p_{,2} + (a+b)u_{1,12} + (a-b)u_{2,11} + 2au_{2,22} + cu_{2,33} + (a+h)u_{3,22} = \\ = \rho u_{2,tt} - \rho f_2 \; ,$$

$$(2.11)\quad -p_{,3} + cu_{1,31} + cu_{2,32} + au_{3,11} + au_{3,22} + (2c+k)u_{3,33} = \\ = \rho u_{3,tt} - \rho f_3 \; .$$

The displacements must also satisfy the incompressibility condition,

$$(2.12)\qquad u_{1,1} + u_{2,2} + u_{3,3} = 0 \; .$$

The constants appearing in equations (2.9, 2.10 and 2.11) are given by

$$(2.13)\quad a = 2\mu^2\left(W_1+\mu^2W_2\right) \; , \quad b = 2W_2\mu^2\left(\mu^2-\lambda^2\right) \; , \quad c = 2\lambda^2\left(W_1+\mu^2W_2\right) \; ,$$

$$(2.14)\quad 4\left(\lambda^2-\mu^2\right) = h\mu^{-2}\left(W_{11}+\left(2\mu^2+\lambda^2\right)W_{12}+\mu^2\left(\mu^2+\lambda^2\right)W_{22}\right)^{-1} \\ = k\lambda^{-2}\left(W_{11}+3\mu^2W_{12}+2\mu^4W_{22}\right)^{-1} \; .$$

(The superscript (1) has been suppressed in the perturbed pressure, and the body force f_k .)

The equations of motion used in [1] correspond to the above equations with $k = h = 0$. Because h and k only appear associated with $u_{3,33}$ the more general case does not increase the complexity of the problem as greatly as might be anticipated. In particular in any motions independent of y_3 the results of [1] apply directly.

3. Propagation of plane waves

Plane wave solutions of equations (2.9)-(2.12) are sought by introducing the functions

$$(3.1)\qquad u_k = A_k\left(y_j\right)\exp\left(i\omega\left(y_s l_s-vt\right)\right) \; , \quad p = P\exp\left(i\omega\left(y_s l_s-vt\right)\right)$$

into the equations and setting $f_k = 0$. If $l_s l_s = 1$ then v is the speed of propagation of the wave. Four linear equations in A_k and P result and the determinantal condition that these have non-trivial solutions reduces to

$$(3.2)\quad \left[(a-b)\left(l_1^2+l_2^2\right)+cl_3^2-\rho v^2\right]\left[a\left(l_1^2+l_2^2\right)+cl_3^2+(k-h)\left(l_1^2+l_2^2\right)l_3^2-\rho v^2\right] = 0 \ .$$

This is the secular equation for the squared speeds of propagation of the wave forms (3.1) in the direction l_k . We note that only one of the two wave speeds exhibits a dependence on the second order derivatives of W . This is also true of the pressure and amplitude. When $\rho v^2 = (a-b)\left(l_1^2+l_2^2\right) + cl_3^2$, the pressure and amplitudes are given by

$$(3.3)\qquad P : l_1A_1 + l_2A_2 : A_3 = 0 : 1 : 0 \ ;$$

the wave is a constant pressure wave polarised in the planes parallel to the y_1y_2 plane. For $\rho v^2 = a\left(l_1^2+l_2^2\right) + cl_3^2 + (k-h)l_3^2\left(l_1^2+l_2^2\right)$,

$$(3.4)\quad P : A_1 : A_2 : A_3 = -i\left\{h\left(l_1^2+l_2^2\right)+(k-h)\left(l_1^2+l_2^2\right)l_3^2\right\} : l_1 : l_2 : l_3-l_3^{-1} \ .$$

(In [1] the counterpart of (3.4) is in error; it should be replaced by (3.4) with $h = k = 0$.)

In order that there exist two real wave speeds in each direction we require

$$(3.5)\quad (a-b)\left(l_1^2+l_2^2\right) + cl_3^2 > 0 \quad\text{and}\quad a\left(l_1^2+l_2^2\right) + cl_3^2 + (k-h)l_3^2\left(l_1^2+l_2^2\right) > 0 \ .$$

From $(3.5)_1$, in view of (2.13), we deduce that

$$(3.6)\qquad W_1 + \lambda^2W_2 \geq 0 \quad\text{and}\quad W_1 + \mu^2W_2 \geq 0 \ ,$$

with equality permitted in only one of (3.6). These are the Baker-Ericksen conditions on the strain energy function obtained by many previous writers and first for incompressible materials by Truesdell [4]. The inequality $(3.5)_2$ does not give information of any value for a general strain energy function, however it can be applied to the strain energy function

$$(3.7)\qquad W = \alpha(I_1-3) + \beta(I_2-3) + \gamma(I_2-3)^2$$

proposed by Signorini [3] and studied lately by Manacorda [2]. It is easy to show that the condition $(3.5)_2$ implies that the inequalities

(3.8) $$\alpha + 4\gamma > 0 \;, \quad \gamma > 0 \;,$$

are necessary and sufficient conditions for the existence of two real wave speeds in any direction.

4. The response to impulsive line tractions: (i) General remarks

The remainder of the paper is concerned with various aspects of the solution of equations (2.9)-(2.12) when $\rho f_k = w_k\delta(y_2)\delta(y_3)\delta(t)$, (w_k is a constant vector). This amounts to finding the response of the material to a line impulse of traction acting along the y_1 axis. Because the initial homogeneous deformation has equal extensions in the y_1 and y_2 directions, and because the equations of motion are autonomous in y_k and t , this is equivalent to finding the response of the material to any line of traction in the y_1y_2 plane. (As mentioned previously, if the force acts along the y_3 axis the results are the same as for the corresponding problem in [1].)

Derivatives with respect to y_1 disappear from the equations of motion and Fourier transforms are taken in y_2, y_3 and t thus

(4.1) $$(\bar{u}_k,\, \bar{p}) = \int_\Omega \exp\big(i(\xi_s y_s + \omega t)\big)(u_k,\, p)\,d\xi_k dt \;.$$

The resulting equations are linear in $\bar{u}_k$ and $\bar{p}$ and their solutions are

(4.2) $$\bar{u}_1 = \frac{w_1}{(a-b)\xi_2^2 + c\xi_3^2 - \rho\omega^2} \;,$$

(4.3) $$\frac{\bar{u}_2}{w_2\xi_3^2 + w_3\xi_2\xi_3} = \frac{\bar{u}_3}{w_2\xi_2\xi_3 - w_3\xi_2^2} = \Big[\big(\xi_2^2+\xi_3^2\big)\big(\rho\omega^2 - a\xi_2^2 - c\xi_3^2\big) - (k-h)\xi_2^2\xi_3^2\Big]^{-1} \;,$$

(4.4) $$\bar{p} = -\frac{i\xi_2\big(a\xi_2^2 + (c+k)\xi_3^2 - \rho\omega^2\big)w_2 + i\xi_3\big((a-h)\xi_2^2 + c\xi_3^2 - \rho\omega^2\big)w_3}{\big(\xi_2^2+\xi_3^2\big)\big(a\xi_2^2 + c\xi_3^2 - \rho\omega^2\big) + (k-h)\xi_2^2\xi_3^2} \;.$$

It follows that the response of the material consists of two travelling waves, one carried by u_1 and the other carried by u_2, u_3 and p . The inverse of $\bar{u}_1$ is simply a fundamental solution of the wave equation in two dimensions (*cf.* equation (5.7) in [1])

$$(4.5)\qquad u_1 = \begin{cases} \dfrac{w_1}{2\pi(a-b)^{\frac{1}{2}}c^{\frac{1}{2}}}\left(t^2 - \dfrac{\rho y_2^2}{a-b} - \dfrac{\rho y_3^2}{a}\right)^{-\frac{1}{2}} , & t > \left(\dfrac{\rho y_2^2}{a-b} + \dfrac{\rho y_3^2}{c}\right)^{\frac{1}{2}} , \\ 0 , & t < \left(\dfrac{\rho y_2^2}{a-b} + \dfrac{\rho y_3^2}{c}\right)^{\frac{1}{2}} . \end{cases}$$

At this stage the complexity involved in dealing with any strain energy function begins to obstruct the analysis. In order to determine u_2 and u_3 it is necessary to invert

$$(4.6)\qquad \bar{\Psi} = \left[\left(\xi_2^2+\xi_3^2\right)\left(a\xi_2^2+c\xi_3^2-\rho\omega^2\right)+(k-h)\xi_2^2\xi_3^2\right]^{-1} .$$

In [1], k and h were both zero and Ψ could be represented as a double integral. A closed form formula for Ψ seems impossible to obtain in this case, beyond the triple integral obtained by applying the Fourier inversion formula to equation (4.6). The comparison between the results from [1] and the case of any strain energy function will therefore be made by perturbing from $k = h = 0$ and calculating the first perturbation. To the first power in $k - h$,

$$(4.7)\qquad \bar{\Psi} = \left(\xi_2^2+\xi_3^2\right)^{-1}\left(a\xi_2^2+c\xi_3^2-\rho\omega^2\right)^{-1} - (k-h)\xi_2^2\xi_3^2\left(\xi_2^2+\xi_3^2\right)^{-2}\left(a\xi_2^2+c\xi_3^2-\rho\omega^2\right)^{-2} .$$

The inverse of the first term of the right hand side of equation (4.7) was encountered in [1]. The second term will be dealt with in the next section.

The inversion of $\bar{p}$ follows a similar pattern to that of $\bar{u}_2$ and $\bar{u}_3$. Expansion of (4.4) to first powers in h and k gives

$$(4.8)\quad \bar{p} = \frac{\Phi_1}{\left(\xi_2^2+\xi_3^2\right)} + \frac{h\Phi_2+k\Phi_3}{\left(\xi_2^2+\xi_3^2\right)\left(a\xi_2^2+c\xi_3^2-\rho\omega^2\right)} + \frac{(k-h)\Phi_4}{\left(\xi_2^2+\xi_3^2\right)^2\left(a\xi_2^2+c\xi_3^2-\rho\omega^2\right)} ,$$

where the Φ_i are functions of ξ_2 and ξ_3 such that each term above is homogeneous of degree -1 in ξ_2, ξ_3 and ω. The problems of inverting $\bar{p}$ are similar to those for Φ.

5. The response to impulsive line tractions: (ii) Solutions

The notation $F^{-1}(f(\xi_k, \omega))$ is used for the inverse transform of f,

$$\frac{1}{(2\pi)^3}\int_\Omega e^{-i(\xi_p y_p+\omega t)} f(\xi_k, \omega)\, d\xi_k d\omega .$$

Now

$$F^{-1}\left[\xi_2^2\left(\xi_2^2+\xi_3^2\right)^{-2}\right] = -\frac{1}{8\pi}\left[\log\left(y_2^2+y_3^2\right)+2y_2^2\left(y_2^2+y_3^2\right)^{-1}+K\right],$$

where K is an arbitrary constant. Also

$$F^{-1}\left[\left(a\xi_2^2+c\xi_3^2-\rho\omega^2\right)^{-2}\right] = \begin{cases} \dfrac{1}{4\pi\rho}\left(t^2-\dfrac{\rho y_2^2}{a}-\dfrac{\rho y_3^2}{c}\right)^{\frac{1}{2}}, & t > \left(\dfrac{\rho y_2^2}{a}+\dfrac{\rho y_3^2}{c}\right)^{\frac{1}{2}}, \\ 0, & t < \left(\dfrac{\rho y_2^2}{a}+\dfrac{\rho y_3^2}{c}\right)^{\frac{1}{2}}. \end{cases}$$

Hence

$$(5.1) \qquad F^{-1}\frac{\xi_2^2\xi_3^2}{\left[\xi_2^2+\xi_3^2\right]^2\left[a\xi_2^2+c\xi_3^2-\rho\omega^2\right]^2} = -\frac{1}{32\pi^2(ac)^{\frac{1}{2}}\rho}\frac{\partial^2\chi}{\partial y_3^2},$$

where, by the convolution theorem,

$$\chi = \int_{D_1}\left(t^2-\frac{\rho z_2^2}{a}-\frac{\rho z_3^2}{c}\right)^{\frac{1}{2}}\left\{\log\left[\left(y_2-z_2\right)^2+\left(y_3-z_3\right)^2\right] + \frac{2\left(y_2-z_2\right)^2}{\left(y_2-z_2\right)^2+\left(y_3-z_3\right)^2}\right\}dz_2dz_3 .$$

The constant K is suppressed because it vanishes on differentiating χ. D_1 is the set of points inside the ellipse $t^2 = \rho a^{-1}z_2^2 + \rho c^{-1}z_3^2$.

This integral is similar to one which appeared in [1] and the techniques used there can be applied in the present case. Except when $a = c$, it does not appear possible to evaluate χ in terms of elementary functions, however the asymptotic forms of the derivatives of χ can be found for $y_2^2 + y_3^2 \gg t^2$ and $y_2^2 + y_3^2 \ll t^2$. When $y_2^2 + y_3^2 \gg t^2$ the substitutions $z_2 = t\zeta_2$, $z_3 = t\zeta_3$ can be made into the integral (5.2). The integrand may then be expanded in a power series in t and integrated term by term. It then follows that

$$(5.3) \qquad \chi = 2\pi a^{\frac{1}{2}} c^{\frac{1}{2}} \left[\log\left(y_2^2+y_3^2\right) + 2y_2^2\left(y_2^2+y_3^2\right)^{-1}\right]\frac{t^3}{3} + O\left(t^4\right) .$$

Exact error bounds can be given when χ is approximated by truncating the series. Combining this result with the corresponding ones from [1] the expansions of u_2 and u_3 take the form

$$(5.4) \qquad u_2,\, u_3 = \left\{t\phi_1\left(y_2 y_3\right) + O\left(t^2\right)\right\} + (k-h)\left\{t^3\phi_2\left(y_2,\, y_3\right) + O\left(t^4\right)\right\} .$$

A similar technique can be applied to the pressure to give

$$(5.5) \qquad p = \delta(t)\pi_1\left(y_2,\, y_3\right) + \left(h\pi_2\left(y_2,\, y_3\right) + k\pi_3\left(y_2,\, y_3\right)\right)\left(t + O\left(t^2\right)\right) .$$

The functions ϕ_i and π_i are linear in w_2 and w_3. Each can be calculated explicitly if required; π_1 and ϕ_1 are given in [1]. For t small the operations of differentiation and series expansion commute.

The important results implied by equations (5.4) and (5.5) are that:

(a) for small times, to first order the displacements do not differ between those for a Mooney material and one with a more general strain energy function;

(b) for small times, a term of order t, which is absent for a Mooney material, appears in the pressure when the generalisation to any strain energy function is made.

When $t \gg \left(y_2^2+y_3^2\right)^{\frac{1}{2}}$ the problem is considerably more difficult. The field point $\left(y_2,\, y_3\right)$ lies inside D_1, the domain of integration of equation (5.2). The differentiations needed to calculate u_2 and u_3 cannot be taken under the integral sign since divergent integrals result, nor is the scaling of the variables to remove the dependence on t from the domain D_1 of any assistance. The expansion of the derivatives of χ in inverse powers of t is accomplished by proving that

$$(5.6)\quad \frac{\partial\chi}{\partial y_3} = \int_{D_2}\left(t^2 - \frac{\rho z_2^2}{a} - \frac{\rho z_3^2}{a}\right)^{\frac{1}{2}} \frac{\partial}{\partial y_3}\left[\log\left((z_2-y_2)^2+(z_3-y_3)^2\right) + \frac{2(z_2-y_2)^2}{(z_2-y_2)^2+(z_3-y_3)^2}\right] dz_2 dz_3$$

where D_2 is the interior of the ellipse $\frac{\rho z_2^2}{a} + \frac{\rho z_3^2}{c} = \frac{\rho y_2^2}{a} + \frac{\rho y_3^2}{c}$. Note that the dependence on t is removed from the domain of integration so that the term $\left(t^2 - \frac{\rho z_2^2}{a} - \frac{\rho z_3^2}{c}\right)^{\frac{1}{2}}$ can be expanded immediately in powers of t^{-1}. The lemma implied by the statement of equation (5.6) corresponds to a similar result in [1]. The proofs of the two lemmas have identical structures. An outline of the proof of (5.6) is given in the last section; for fuller details the reader is referred to [1].

When the square root in equation (5.6) is expanded in inverse powers of t the integrals obtained can all be evaluated in terms of elementary functions. It is found that

$$(5.7)\qquad \frac{\partial\chi}{\partial y_3} = \eta_1(y_2, y_3)t + \eta_2(y_2, y_3)\frac{1}{t} + \frac{\eta_3(y_2,y_3)}{t^3} + \dots ,$$

η_1 is linear in y_2 and y_3, η_2 is homogeneous of degree 3 in y_2 and y_3, and so on. An identical statement applies to $\frac{\partial\chi}{\partial y_2}$. The displacements u_2 and u_3 are linear combinations of the fourth derivatives of χ, thus the leading term in equation (5.7) is eliminated When the results from [1] are combined with these results the asymptotic form of u_2 and u_3 is obtained thus:

(5.8) $$u_2 \sim \frac{w_2}{2\pi}\left(\frac{1}{a^{\frac{1}{2}}(a^{\frac{1}{2}}+c^{\frac{1}{2}})} - \frac{(k-h)}{c^{\frac{1}{2}}(a^{\frac{1}{2}}+c^{\frac{1}{2}})^2}\right)\frac{1}{t} ,$$

(5.9) $$u_3 \sim \frac{w_3}{2\pi}\left(\frac{1}{c^{\frac{1}{2}}(a^{\frac{1}{2}}+c^{\frac{1}{2}})} + \frac{(k-h)}{a^{\frac{1}{2}}(a^{\frac{1}{2}}+c^{\frac{1}{2}})^2}\right)\frac{1}{t} .$$

Similar techniques can be used to show that the pressure is given by

(5.10) $$p \sim \frac{1}{\pi t^3}(\theta_1(y_2,\, y_3)k+\theta_2(y_2,\, y_3)h) ,$$

where

$$\theta_1 = -\frac{(a^{\frac{1}{2}}+2c^{\frac{1}{2}})w_2y_2+a^{\frac{1}{2}}w_3y_3}{a^{\frac{1}{2}}c^{\frac{1}{2}}(a^{\frac{1}{2}}+c^{\frac{1}{2}})^3} , \text{ and } \theta_2 = \frac{a^{\frac{1}{2}}w_2y_2+(c^{\frac{1}{2}}+2a^{\frac{1}{2}})w_3y_3}{a^{\frac{1}{2}}c^{\frac{1}{2}}(a^{\frac{1}{2}}+c^{\frac{1}{2}})^3} .$$

The important features of these results are:

(a) that the displacements are independent of position to order t^{-1} ,

(b) that the responses to the forces are uncoupled to order t^{-1} ,

(c) that the distinction between a Mooney material and an arbitrary hyperelastic material is detectable in the coefficient of t^{-1} in the displacements,

(d) that the pressure behaves as t^{-3} so that the t^{-1} term remains absent in the pressure for an arbitrary hyperelastic material.

The properties (c) and (d) invert those noted in the case of t small.

6. Differentiation of the function χ of equation (5.2) under the integral sign

It is clear that (5.6) follows from (5.2) if it can be proved that the contribution to χ of the integral taken over the elliptic annulus $D_1 - D_2$ is constant. For convenience the substitutions $y_2 = a^{\frac{1}{2}}Y_2$,

$z_2 = a^{\frac{1}{2}}Z_2$, $y_3 = c^{\frac{1}{2}}Y_3$, $z_3 = c^{\frac{1}{2}}Z_3$ are made, then it suffices to prove that

$$(6.1)\quad \chi_A = \int_A \left(\rho t^2 - Z_2^2 - Z_3^2\right)^{\frac{1}{2}} \left\{ \log\left[a(Y_2-Z_2)^2 + c(Y_3-Z_3)^2\right] + \frac{2a(Y_2-Z_2)^2}{a(Y_2-Z_2)^2 + c(Y_3-Z_3)^2} \right\} dZ_2 dZ_3$$

(where A is the annulus $Y_2^2+Y_3^2 < Z_2^2+Z_3^2 < \rho t^2$) is constant.

Suppose $0 < c < a$ ($c, a \geq 0$ by assumption). Then $-1 < \frac{c-a}{a} < 0$ and

$$(6.2)\qquad aU_2^2 + cU_3^2 \equiv a\left(U_2^2+U_3^2\right)\left(1 + \frac{c-a}{a}\,\frac{U_3^2}{U_2^2+U_3^2}\right) .$$

Now

$$\left|\frac{c-a}{a}\,\frac{U_3^2}{U_2^2+U_3^2}\right| < 1 ,$$

so (6.2) may be used to expand the integrand of (6.1). The series may legitimately be integrated termwise and a series of terms of the form

$$(6.3)\quad F_n(Y_2, Y_3) = \left(\frac{c-a}{a}\right)^{n-1} \int_A \left(\rho t^2 - Z_2^2 - Z_3^2\right)^{\frac{1}{2}} \left(\frac{(Y_2-Z_2)^2 (Y_3-Z_3)^{2n-2}}{\left[(Y_2-Z_2)^2 + (Y_3-Z_3)^2\right]^n}\right) dZ_2 dZ_3$$

is obtained. Such terms arose in the corresponding proof in [1]. It was proved there that each F_n is independent of Y_2 and Y_3 . This proves the assertion about χ_A , since χ_A being the sum of a series of constants, is itself constant.

The property of the F_n stated above depends on the form of the difference function in the integrand on the right hand side of (6.3). An examination of the proof given in [1] reveals that if the difference function is homogeneous of degree zero in (Y_2-Z_2) and (Y_3-Z_3) then F_n is necessarily constant. Hence the property used in this paper and in [1] is true for any integral which can be expanded in a series of terms F_n provided only that the resulting series of constants is convergent.

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