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ASYMPTOTIC EXISTENCE OF TIGHT ORTHOGONAL MAIN EFFECT PLANS

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ABSTRACT. Our main result is showing the asymptotic existence of tight OMEPs. More precisely, for each fixed number k of rows, and with the exception of OMEPs of the form $2 \times 2 \times \cdots 2 \times 2s \,/\!/\, 4s$ with s odd and with more than three rows, there are only a finite number of tight OMEP parameters for which the tight OMEP does not exist.

1**. Introduction.** An *Orthogonal Main Effect Plan*, or OMEP, is a matrix having k rows (or factors), n columns (or runs), s_i symbols in row i, for $1 \le i \le k$, and which satisfies the property: If $1 \le i < j \le k$, and if x is any symbol in row i, and y is any symbol in row j, then the number of columns with an x in row i and a y in row j equals the number of times x appears in row i, multiplied by the number of times y appears in row j, divided by n. We call the matrix an $s_1 \times s_2 \times \cdots \times s_k \,/\!/\, n$ OMEP. The number of times symbol x occurs in row i is often denoted by r_{ix}. These numbers are called the *replication numbers* of the OMEP. OMEPs with $s_1 = s_2 = \cdots = s_k = s$ and $n = \lambda s^2$, having all replication numbers equal to λs, are orthogonal arrays of strength two and index λ.

OMEPs have been considered by many authors, in part because they are useful in constructing statistical designs. For a recent survey on OMEPs and related structures, see [6]. For an application of tight OMEPs, see [4].

Suppose D is an $s_1 \times s_2 \times \cdots \times s_k \,/\!/\, n$ OMEP, and that $n = p_1^{m_1} p_2^{m_2} \cdots p_d^{m_d}$ is the prime power factorization of n. Let

$$g_i = \gcd\{r_{ix} \mid x \text{ a symbol in row } i\}.$$

Since for an OMEP we have

$$n | r_{ix} r_{jy} \quad \text{for } i \neq j,\ x \text{ in row } i,\ y \text{ in row } j,$$

it follows

$$n | g_i g_j \quad \text{for all } 1 \le i < j \le k\,. \tag{1}$$

For each prime p_t dividing n, let l_t be the greatest integer such that $p_t^{l_t} | g_j$ for each j, and choose c_t so that $p_t^{l_t}$ exactly divides g_{c_t}. (Note that c_t is not necessarily uniquely determined.) Then, by (1), we see $p_t^{m_t - l_t}$ divides g_j for $j \neq c_t$. If $p_t^{m_t - l_t}$ exactly divides g_j

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for $j \neq c_t$, and furthermore if $s_j = n/g_j$ for each $j \in \{1, 2, \ldots k\}$, then we call the OMEP *tight*. In this case we have $l_t \leq m_t/2$.

If D itself is not tight, then the l_t's and the c_t's still exist, and these determine the parameter set of a tight OMEP, say $s'_1 \times s'_2 \times \cdots \times s'_k \,//\, n$. In this case, we have $g_i \geq g'_i$, and so $s_i \leq s'_i$ for each i. (In fact it also follows that if D is not tight then some $s'_i > s_i$.) Hence, if this tight OMEP exists, then an OMEP with the same parameters as D can be obtained by collapsing levels in the tight OMEP. Therefore, it is useful to know when tight OMEPs exist. Tight OMEPs were first discussed in [4].

From the preceding discussion, tight OMEPs have parameters of the form $\lambda_1 g \times \lambda_2 g \times \cdots \times \lambda_k g \,//\, \lambda_1\lambda_2 \cdots \lambda_k g^2$, with the λ_i's pairwise relatively prime. Not all OMEPs with these parameters are tight; it may be that $r_i \neq n/s_i = \lambda_1\lambda_2 \cdots \lambda_k g/\lambda_i$. However, whenever $n = \lambda_1\lambda_2 \cdots \lambda_k g^2$ and $s_i = \lambda_i g$, it is at least possible in principle that there is a tight OMEP with these parameters, that is, with $r_i = n/s_i$. When we say that $s_1 \times s_2 \times \cdots \times s_k \,//\, n$ is a tight parameter set, we mean that by taking $r_{ix} = n/s_i$, the replication numbers satisfy the necessary arithmetic conditions for the existence of a tight OMEP, so it is at least conceivable that there exists a tight $s_1 \times s_2 \times \cdots \times s_k \,//\, n$ OMEP. It may be that the OMEP still does not exist, for example $6 \times 6 \times 6 \times 6 \,//\, 36$ is a tight parameter set, yet the OMEP does not exist since it would correspond to two MOLS of order 6.

We refer to some common design theory structures in this paper. For example a transversal design $\mathrm{TD}_\lambda(k, g)$ is equivalent a tight $g \times g \times \cdots \times g \,//\, \lambda g^2$ OMEP with k rows. An *RBIBD*(v, k, λ) is a resolvable balanced incomplete block design on v points with blocks of size k. For further information on these and similar structures, see any good book on design theory, for example [2].

2. **Asymptotic existence of tight OMEPs.** Asymptotic existence of tight OMEPs is established in this section. As an intermediate step, asymptotic existence of resolvable transversal designs is also established. In [4], it is shown that every tight OMEP parameter set on 3 or fewer rows has a corresponding tight OMEP, so we make the implicit assumption $k \geq 4$.

We first outline some common constructions for OMEPs.

THEOREM 2.1 (PRODUCT CONSTRUCTION). *If an $s_1 \times s_2 \times \cdots \times s_k \,//\, n$ OMEP exists, and an $s'_1 \times s'_2 \times \cdots \times s'_k \,//\, n'$ OMEP exists, then an $s_1s'_1 \times s_2s'_2 \times \cdots \times s_ks'_k \,//\, nn'$ OMEP exists.*

Also, we need a concatenation construction. See [4] for details.

THEOREM 2.2 (CONCATENATION CONSTRUCTION). *Suppose D is an $s_1 \times s_2 \times \cdots \times s_k \,//\, n$ OMEP, and D' is an $s_1 \times s_2 \times \cdots s_{k-1} \times s'_k \,//\, n'$ OMEP, with replication numbers r_{ix} and r'_{jy} respectively. Further suppose that these OMEPs have the same symbol sets in the first $k-1$ rows, $r_{ix}/n = r'_{ix}/n'$ when $1 \leq i \leq k-1$, and for the remaining row, the symbols in the first OMEP are all different from the symbols of the second OMEP. Then the concatenation of these matrices is an $s_1 \times s_2 \times \cdots \times s_{k-1} \times (s_k + s'_k) \,//\, (n + n')$ OMEP.*

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The following incidence structure is useful.

DEFINITION 2.3. Let $S = \{v_{ij} \mid 1 \leq i \leq k, 1 \leq j \leq g\}$. Let B be a set of subsets (called blocks) of S. The pair (S, B) is called an $R(g, k, \mu, \lambda)$-design if the block set can be partitioned into parallel classes and if pairs of points v_{ix}, v_{jy} are in no blocks if $i = j$, in λ blocks if $i \neq j$ and $x \neq y$, and in μ blocks if $i \neq j$ and $x = y$.

LEMMA 2.4. *Let g and k be fixed. Then an* $R(g, k, g^{k-2} - 1, g^{k-2})$*-design exists.*

PROOF. Let the point set be $\{(i, j) \mid 1 \leq i \leq g, 1 \leq j \leq k\}$. The set of blocks

$$\big\{\{(p_1, 1), (p_2, 2), \ldots, (p_k, k)\} \mid p_i\text{'s not all equal}\big\}$$

is a $R(g, k, g^{k-2} - 1, g^{k-2})$-design. ∎

If we do not exclude blocks with all p_i equal, we get a $\mathrm{RTD}_{g^{k-2}}(k, g)$. We state this well known result formally here.

REMARK 2.5. For any fixed g and k, a $\mathrm{RTD}_{g^{k-2}}(k, g)$ exists.

LEMMA 2.6. *If a* $\mathrm{RTD}_{\lambda_1}(k, g)$ *and a* $\mathrm{RTD}_{\lambda_2}(k, g)$ *exists, with* $\gcd(\lambda_1, \lambda_2) = 1$, *then a* $\mathrm{RTD}_\lambda(k, g)$ *exists for all* $\lambda \geq \lambda_1\lambda_2$. *Hence if a* $\mathrm{RTD}_\mu(k, g)$ *with* $\gcd(\mu, g) = 1$ *exists, then a* $\mathrm{RTD}_\lambda(k, g)$ *exists for all* λ *sufficiently large.*

PROOF. The first statement holds since $\lambda = s\lambda_1 + t\lambda_2$ has a nonnegative integral solution in s, t for all $\lambda \geq \lambda_1\lambda_2$. The second follows by using Lemma 2.4. ∎

THEOREM 2.7. *If an* $\mathrm{RBIBD}(v, k, \lambda)$ *and a* $\mathrm{RTD}_\mu(v/k, g)$ *exist, then a* $R\big(g, v, \lambda\mu(g + (v - k)/(k - 1)\big), \lambda\mu\big(v - k)/(k - 1)\big)$*-design exists.*

PROOF. We construct blocks on the point set

$$S = \{(i, j) \mid 1 \leq i \leq g, 1 \leq j \leq v\}.$$

Assume that the $\mathrm{RTD}_\mu(v/k, g)$ is on the points $\{(i, j) \mid 1 \leq i \leq g, 1 \leq j \leq v/k\}$, and the groups are $G_j = \{(i, j) \mid 1 \leq i \leq g\}$. Assume the RBIBD is on the point set $\{1, 2, \ldots, v\}$. For each parallel class of the RTD, say $\{B_1, B_2, \ldots, B_g\}$, and each parallel class of the RBIBD, say $\{B'_1, B'_2, \ldots, B'_{\frac{v}{k}}\}$, we construct a parallel class on S as follows. If

$$B_i = \left\{(\delta_{i,1}, 1), (\delta_{i,2}, 2), \ldots, \left(\delta_{i,\frac{v}{k}}, \frac{v}{k}\right)\right\},$$

then our parallel class on S has blocks $\{\beta_j \mid 1 \leq j \leq g\}$ defined by

$$\beta_j = (\{\delta_{j,1}\} \times B'_1) \cup (\{\delta_{j,2}\} \times B'_2) \cup \cdots \cup (\{\delta_{j,\frac{v}{k}}\} \times B'_{\frac{v}{k}}).$$

It is easy to check that these blocks give the desired design. ∎

COROLLARY 2.8. *Let* $g \geq 4$ *be a fixed number not divisible by* 3, *and let k be fixed. Then for all* λ *large enough, a* $\mathrm{RTD}_\lambda(k, g)$ *exists.*

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PROOF. Choose i such that $3^{i+1} \geq k$. Apply Theorem 2.7 using an $\mathrm{RBIBD}\big(3^{i+1}, 3^i, (3^i-1)/2\big)$ and a $\mathrm{RTD}(3, g)$ to obtain an $R(g, 3^{i+1}, g(3^i-1)/2 + 3^i, 3^i)$-design which we truncate to a $R\big(g, k, g(3^i-1)/2 + 3^i, 3^i\big)$-design. Now take $g(3^i-1)/2$ copies of the blocks of a $R(g, k, g^{k-2}-1, g^{k-2})$ and one copy of the blocks of our $R\big(g, k, g(3^i-1)/2 + 3^i, 3^i\big)$-design, to give a $\mathrm{RTD}_\mu(k, g)$, where $\mu = g^{k-1}(3^i-1)/2 + 3^i$. Since μ is relatively prime to g, Lemma 2.6 gives the result. ■

LEMMA 2.9. *For any k, any m, and any λ sufficiently large, a* $\mathrm{RTD}_\lambda(k, 3^m)$ *exists.*

PROOF. The proof is as above, but for $m > 1$, we use a $\mathrm{RTD}(4, 3^m)$ and a $\mathrm{RBIBD}\big(4^{i+1}, 4^i, (4^i-1)/3\big)$ as our "ingredient" designs, and for $m = 1$, we use a $\mathrm{RTD}_2(4, 3)$ and a $\mathrm{RBIBD}\big(4^{i+1}, 4^i, (4^i-1)/3\big)$ as our "ingredient" designs. ■

COROLLARY 2.10. *For any k and any g with $3|g$, and all λ sufficiently large, a* $\mathrm{RTD}_\lambda(k, g)$ *exists.*

PROOF. We first consider the case $g = 6$. In this case, choose i such that $5^{i+1} \geq k$. Applying Theorem 2.7 using an $\mathrm{RTD}_5(5, 6)$ and an $\mathrm{RBIBD}(5^{i+1}, 5^i, (5^i-1)/4)$ gives an $R(6, M, 6\cdot 5(5^i-1)/4 + M, M)$-design, where $M = 5^{i+1}$. Adding $6 \cdot 5(5^i-1)/4$ copies of the blocks of an $R(6, M, 6^{M-2}-1, 6^{M-2})$-design gives an $\mathrm{RTD}_\lambda(M, 6)$, where $\lambda = 6 \cdot 5 \cdot \frac{5^i-1}{4} \cdot 6^{M-2} + M$ is relatively prime to 6. Thus Lemma 2.6 now gives the result.

For $g \neq 6$, write $g = 3^m g'$, with $3 \not| g'$. Since $g \neq 6$, $g' \neq 2$. From Lemma 2.9, there exists a $\mathrm{RTD}_{\lambda_1}(k, 3^m)$ with $\gcd(\lambda_1, g) = 1$, and by Corollary 2.8 there is a $\mathrm{RTD}_{\lambda_2}(k, g')$ with $\gcd(\lambda_2, g) = 1$. The direct product of these is a $\mathrm{RTD}_{\lambda_1\lambda_2}(k, g)$. Since $\gcd(\lambda_1\lambda_2, g) = 1$, Lemma 2.6 now gives the result. ■

These last few observations show that for fixed k and $g \geq 3$, a $\mathrm{RTD}_\lambda(k-1, g)$ exists for all λ large enough, say all $\lambda \geq M(g, k)$. Hence a tight $\lambda g \times g \times \cdots \times g \,//\, \lambda g^2$ (having k rows) exists for all $\lambda \geq M(g, k)$. Therefore, (using the product theorem) for any set of λ_i's pairwise relatively prime with at least one $\lambda_i \geq M(g, k)$ a tight

$$\lambda_1 g \times \lambda_2 g \times \cdots \times \lambda_k g \,//\, \lambda_1\lambda_2\cdots\lambda_k g^2 \tag{2}$$

OMEP exists. Since (for fixed k, g) there are only a finite number of parameters of the form in (2) with the λ_i's all less than λ, we see that there are at most a finite number of such tight OMEP parameters for which the tight OMEP does not exist. Furthermore, since for all sufficiently large g a $\mathrm{TD}(k, g)$ exists, and for such g and for any choice of the λ_i's a $\lambda_1 g \times \lambda_2 g \times \cdots \times \lambda_k g \,//\, \lambda_1\lambda_2\cdots\lambda_k g^2$ OMEP exists. Thus, there are only a finite number of parameter sets of the form in (2) with $g \neq 2$ for which the OMEP does not exist. It remains to show that there are only a finite number which do not exist when $g = 2$. Since a $\mathrm{TD}(k, 2\alpha)$ exists for some α odd (depending on k), by collapsing levels in it we obtain a $2\alpha \times 2\alpha \times 2 \times \cdots \times 2 \,//\, 4\alpha^2$ OMEP. Also, there is a tight $2\alpha' \times 2 \times \cdots \times 2 \,//\, 4\alpha'$ OMEP for α' (depending on k) a sufficiently large power of 2, and hence a tight $2\alpha' \times 2\alpha \times \cdots \times 2 \,//\, 4\alpha'\alpha$ OMEP. By using the concatenation construction (Theorem 2.2) we obtain a tight $2\mu \times 2\alpha \times 2 \cdots \times 2 \,//\, 4\mu\alpha$ OMEP for all

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$\mu \geq \alpha\alpha'$. Further, for such μ, there is a tight $2\mu \times 2\alpha' \times 2 \times \cdots \times 2 \,//\, 4\mu\alpha'$ OMEP. Again using concatenation we obtain a tight $2\mu \times 2\mu' \times 2 \cdots \times 2 \,//\, 4\mu\mu'$ OMEP for all $\mu' \geq \alpha\alpha$. Thus, for any choice of μ_i's pairwise relatively prime with at least two of the μ_i's at least $\alpha\alpha'$, there is a tight

$$2\mu_1 \times 2\mu_2 \times \cdots \times 2\mu_k \,//\, 4\mu_1\mu_2 \cdots \mu_k \tag{3}$$

OMEP. We must now consider OMEPs of the form in (3) but where all but one of the μ_i's are less than $\alpha\alpha'$. We need some lemmas first.

LEMMA 2.11. *For any k, there is an odd λ such that a* $2\lambda \times 4 \times 2 \times 2 \cdots \times 2 \,//\, 8\lambda$ OMEP *exists (having k rows).*

PROOF. Choose i such that $3^{i+1} \geq k-2$ and i is even. For neatness define $M = 3^{i+1}$. Let D_1 be a $2 \times 4 \times 2 \times 2 \times 2 \,//\, 8$ OMEP, and let D_2 be an RBIBD$(3^{i+1}, 3^i, (3^i-1)/2)$. Let

$$\mathcal{T} = \{\text{all } M\text{-tuples using 0,1 except } (0,0,\ldots,0) \text{ and } (1,1\ldots,1)\}\,.$$

Let the j-th parallel class of D_2 be $\{B_{j1}, B_{j2}, B_{j3}\}$. We construct an OMEP on $M+2$ rows, with rows labeled $\infty, 0, 1, 2, \ldots, M$. We construct the OMEP so that the symbols in row ∞ are $\mathcal{T} \times \{1, 2, \ldots, (3^i-1)/2\} \cup \{1, 2, \ldots, M-1\}$, the symbols row 0 are $\{0, 1, 2, 3\}$, and the symbols in each other row are $\{0, 1\}$. Assume the symbols in the rows of D_1 are $\{0,1\}$, $\{0,1,2,3\}$, $\{0,1\}$, $\{0,1\}$, and $\{0,1\}$, respectively. Assume the point set of D_2 is $\{1, 2, \ldots, M\}$. For each column $(p_\infty, p_0, p_1, p_2, p_3)^T$ of D_1 and each parallel class $\{B_{j1}, B_{j2}, B_{j3}\}$ of D_2 we construct a column with $2j - p_\infty$ in the row ∞, p_0 in row 0, p_1 in each row indexed in B_{j1}, p_2 in each row indexed in B_{j2}, and p_3 in each row indexed in B_{j3}. (Since $\{B_{j1}, B_{j2}, B_{j3}\}$ is a parallel class this defines the entire column.) Further, for each $\alpha \in \{1, 2, \ldots, (3^i-1)/2\}$, each M-tuple $T = (t_1, t_2, \ldots, t_M)$ in $\mathcal{T}$, and each $s \in \{0, 1, 2, 3\}$ we construct a column with (T, α) in row ∞, s in row 0, and $t_l + s$ in row l for each row l, $1 \leq l \leq M$ (where addition is done modulo 2). These columns together form an OMEP where symbols from row ∞ and row 0 occur together once, symbols from row ∞ and row l ($1 \leq l \leq M$) occur together twice, symbols from row 0 and row l ($1 \leq l \leq M$) occur together λ times, where $\lambda = ((3^{i+1}-1)/2 + (2^{M-1}-1)(3^i-1)/2$, and symbols from any pair of distinct rows with labels between 1 and M occur together 2λ times. (Since i is even, $(3^i-1)/2$ is even, and $(3^{i+1}-1)/2$ is odd, so λ is odd.) Thus this is a $2\lambda \times 4 \times 2 \times 2 \times \cdots \times 2 \,//\, 8\lambda$ OMEP on $M+2$ rows, which gives the desired OMEP, possibly after removing some rows. ∎

LEMMA 2.12. *For any k, there is a λ which is a power of* 2 *such that a tight* $2\lambda \times 4 \times 2 \times 2 \cdots \times 2 \,//\, 8\lambda$ OMEP *exists (having k rows).*

PROOF. Choose i so that $4^i \geq k-1$. A $4^i \times 4 \times 4 \times \cdots \times 4 \,//\, 4^{i+1}$ OMEP exists, having $4^i + 1$ rows. By collapsing levels we obtain a $4^i \times 4 \times 2 \times \cdots \times 2 \,//\, 4^{i+1}$ OMEP. Taking $\lambda = 2^{2i-1}$, we see this is a tight $2\lambda \times 4 \times 2 \times \cdots \times 2 \,//\, 8\lambda$ OMEP, having $4^i + 1 \geq k$ rows. ∎

COROLLARY 2.13. *For any k, and for all sufficiently large λ, a tight* $2\lambda \times 4 \times 2 \times 2 \cdots \times 2 \,//\, 8\lambda$ OMEP *on k rows exists.*

PROOF. This follows from Lemma 2.11, Lemma 2.12, and Lemma 2.6. ∎

We now show asymptotic existence of OMEPs with parameters as in (3). Again recall that k is some fixed number of rows.

In the first case at least one μ_i is even, say $\mu_2 = 2\mu_2'$. Then by (2.12), a tight $2\lambda \times 4 \times 2 \times 2 \cdots \times 2 \,//\, 8\lambda$ OMEP on k rows exists for all λ large, say $\lambda \geq M'(k)$. Using the product construction we see a tight $2\lambda \times 2\mu_2 \times 2 \times 2 \cdots \times 2 \,//\, 4\lambda\mu_2$ OMEP exists for $\lambda \geq M'(k)$, and so again using the product construction we see a tight $2\lambda \times 2\mu_2 \times 2\mu_3 \times \cdots \times 2\mu_k \,//\, 4\lambda\mu_2\mu_3 \cdots \mu_k$ OMEP exists for such λ. Thus if some $\mu_i \geq M'(k)$, and some μ_j is even, then a tight $2\mu_1 \times 2\mu_2 \times \cdots \times 2\mu_k \,//\, 4\mu_1\mu_2 \cdots \mu_k$ OMEP exists. Hence there are at most a finite number of OMEP parameters in the first case for which the OMEP does not exist.

In the second case, no μ_i is even. If $k \leq 3$ then all possible tight parameter sets have corresponding tight OMEPs (See [4]). We know if $k \geq 4$ and at most one μ_i is greater than one then the OMEP cannot exist, and in this case the parameters have the form $2 \times 2 \times \cdots 2 \times 2s \,//\, 4s$ for s odd. (See [4], for example.) Otherwise at least two μ_i's are greater than one. Suppose $\mu_1 \geq \mu_2 > 1$. By the earlier results a $2\lambda'\mu_2 \times 2\mu_2 \times 2\mu_2 \times \cdots \times 2\mu_2 \,//\, 4\lambda'\mu_2^2$ OMEP exists for a (large) odd λ', and so by collapsing levels a $2\lambda'\mu_2 \times 2\mu_2 \times 2 \times 2 \times \cdots \times 2 \,//\, 4(\lambda'\mu_2)\mu_2$ OMEP exists. Also a $2^i \times 2\mu_2 \times 2 \times \cdots \times 2 \,//\, 2^{i+1}\mu_2$ OMEP exists for large enough i, since for large i a $2^i \times 2 \times 2 \times \cdots \times 2 \,//\, 2^{i+1}$ OMEP on k rows exists. Thus again by an argument similar to the proof of Lemma 2.6 we see a $2\lambda \times 2\mu_2 \times 2 \times 2 \times \cdots \times 2 \,//\, 4\lambda\mu_2$ OMEP exists for all large λ, and so a $2\lambda \times 2\mu_2 \times 2\mu_3 \times 2\mu_4 \times \cdots \times 2\mu_k \,//\, 4\lambda\mu_2\mu_3 \cdots \mu_k$ OMEP exists for all large λ. Thus if μ_1 is sufficiently large the OMEP exists, and hence at most a finite number of OMEP parameters arise in the second case for which the OMEP does not exist.

These are the only possible cases and so there are at most a finite number of OMEP parameters with the form in (3) for which the OMEP does not exist, with the one exception of parameters of the type $2s \times 2 \times 2 \cdots \times 2 \,//\, 4s$ with s odd and with four or more rows.

Combining all these results we see that for any fixed k, and with the exception of parameters of the form $2 \times 2 \times 2 \times \cdots \times 2 \times 2s \,//\, 4s$ with s odd and having 4 or more rows, there are a finite number of tight OMEP parameters on k rows for which the OMEP does not exist.

3. **Application.** With these results we can show that the Jacroux's lower bound on the number of runs n needed to construct an $s_1 \times s_2 \times \cdots \times s_k \,//\, n$ OMEP is "almost asymptotically tight". To explain what we mean here we need to make some observations.

Jacroux's [3] lower bound on the number of columns in a $s_1 \times s_2 \times \cdots \times s_k \,//\, n$ OMEP is as follows.

THEOREM 3.1. *Suppose that an* OMEP D *has* $k \geq 3$ *factors in which factor* i *has* s_i *levels,* $i = 1 \cdots k$, *with* $s_i \geq s_{i+1}$, *and* n *experimental runs. If* $n = s_1's_2'$ *for* s_1', s_2' *satisfying*

$$s_1's_2' = \min_{x \geq s_1, y \geq s_2} xy, \; xy < 2s_1s_2, \; s_3 \leq \gcd(x, y),$$

then D *is a minimal* OMEP.

Essentially we are bounding the number of runs required by bounding the number of runs required for the truncated $s_1 \times s_2 \times s_3 \,//\, n$ OMEP.

Street [5] has extended Jacroux's result when $k = 3$:

THEOREM 3.2. *If $l \le m \le p$ and z is the least number of times a pair of symbols from rows two and three occur together in a column, then an $l \times m \times p \,//\, \big((m+x)(p+y)/z\big)$* OMEP *cannot exist unless $l \le g_1 g_2$, where $g_1 = \gcd(z, m+x, p+y)$ and $g = \gcd\big((m+x)/g_1, (p+y)/g_2\big)$.*

The concept of a tight OMEP quickly leads to the above results, as follows. In [4], it is shown that the minimal n for which an $s_1 \times s_2 \times s_3 \,//\, n$ OMEP exists is the minimal n for which a tight $s_1' \times s_2' \times s_3' \,//\, n$ OMEP exists with $s_i' \ge s_i$ for $i = 1, 2, 3$. Let $d = \gcd(s_1', s_2', s_3')$, and let $u_i = n/s_i'$ for $i = 1, 2, 3$. Now since we are dealing with three row OMEPs, $s_1' \times s_2' \times s_3' \,//\, n$ is the parameter set of a tight OMEP if and only if u_1, u_2, u_3 are pairwise relatively prime, and $n = d^2 u_1 u_2 u_3$. All tight three-factor OMEPs exist, so the minimal n for which a tight $s_1' \times s_2' \times s_3' \,//\, n$ OMEP exists is given by

$$
\begin{aligned}
&\min d^2 u_1 u_2 u_3\\
&\text{subject to}\\
&u_i d \ge s_i,\\
&\gcd(u_i, u_j) = 1 \quad \text{for } i \ne j.\\
&u_i, \quad d \text{ positive integers.}
\end{aligned}
$$

Assuming $s_1 \ge s_2 \ge s_3$, elementary methods show this integral problem has an optimal solution with $u_3 = 1$. Taking $s_i' = u_i d$, we see that there is an optimal solution with $n = s_1' s_2'$, and $s_3' = \gcd(s_1', s_2')$.

Thus Jacroux's lower bound is actually telling us the smallest n for which there is a tight OMEP parameter set $s_1' \times s_2' \times s_3' \,//\, n$ with $s_i' \ge s_i$. Furthermore, since the above integral system has an optimal solution with $u_3 = 1$, we see that the smallest n for which there is a tight OMEP parameter set $s_1' \times s_2' \times \cdots \times s_k' \,//\, n$ with $s_i' \ge s_i$ can be assumed to have the form $n = \mu_1 \mu_2 g^2$, and the tight parameter set can be assumed to have the form

$$
\mu_1 g \times \mu_2 g \times g \times \cdots \times g \,//\, \mu_1 \mu_2 g^2 \tag{4}
$$

Now if $g \ge 3$ then there are at most a finite number of parameters with the form (4) for which the tight OMEP does not exist. Thus if $s_1 \ge s_2 \cdots \ge s_k$, and $s_3 \ge 3$, then there are at most a finite number of choices for the other s_i for which Jacroux's bound is not tight. Even if $s_3 = 2$ and both s_1, s_2 are greater than 2 then there are still at most a finite number of cases where Jacroux's bound is not tight. This is what we mean by the "almost asymptotically tight" phrase in the beginning of this section.

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