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DENSITY OF RESONANCES FOR STRICTLY CONVEX ANALYTIC OBSTACLES

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with an appendix by
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ABSTRACT. We estimate the density of resonances close to a critical curve, for strictly convex obstacles with analytic boundary. Contrary to the C^∞-case, already treated with Zworski, the estimates are in terms of dynamical quantities. A new feature in the proof is a certain averaging procedure.

RÉSUMÉ. Nous estimons la densité des résonances près d'une courbe critique, pour des obstacles strictement convexes à bord analytique. Contrairement au cas C^∞, déjà traité avec Zworski, les estimations font appel aux quantités dynamiques. Une procédure de moyennisation est un aspect nouveau dans la démonstration.

0. **Introduction.** Let $O \subset\subset \mathbf{R}^n$, $n \geq 2$, be an open convex set with smooth (C^∞) boundary. (In the main theorem below and its proof we will always assume that the boundary is analytic.) Assume that O is strictly convex in the sense that the second fundamental form, $Q(\nu)$ on $T\partial O$ is positive definite at every point of ∂O. We consider the Dirichlet Laplacian $-\Delta = -\sum \frac{\partial^2}{\partial x_j^2}$ on $\mathbf{R}^n \setminus O$ and recall that the scattering poles (also called resonances) in a sector $\Re k > 0, 0 \leq -\arg k \leq \theta, \theta < \frac{\pi}{2}$, can be defined as the poles of the meromorphic extension of $k \longmapsto (-\Delta - k^2)^{-1} : L^2_{\mathrm{comp}}(\mathbf{R}^n \setminus O) \longrightarrow (H^1_0 \cap H^2)_{\mathrm{loc}}(\mathbf{R}^n \setminus O)$ from the upper half-plane across the positve real axis.

In [SZ1] Zworski and the author showed by a method of complex scaling up to the boundary, that the number of scattering poles (counted with their natural multiplicity) in an angle $1 \leq |k| \leq r, 0 \leq -\arg k \leq \theta$ is $O(1)\theta^{\frac{3}{2}} r^n$, $r \geq r(\theta)$. A little later, Hargé-Lebeau [HaL], used this type of complex scaling up to the boundary for evolution equations and showed that there is a constant $C_0 > 0$, such that there are only finitely many resonances in a domain of the form: $1 \leq \Re k \leq r$, $\Im k \geq -C_0(\Re k)^{\frac{1}{3}}$. This result was previously known in the cases $n = 2, 3$ (Filippov-Zayev [FZ], Babich-Grigoreva [BG]) and in the analytic case, where one can relax the strict convexity of the obstacle to non-trapping (Lebeau [L], Popov [P], Bardos-Lebeau-Rauch [BaLR]). In [SZ2] we returned to the question of estimating the density of resonances near the real axis for strictly convex obstacles and we also established an explicit result about the pole free region: If

$$C_{0,\infty} = 2^{-\frac{1}{3}}\left(\cos\frac{\pi}{6}\right)\zeta_1 \inf_{\nu \in S\partial O} Q(\nu)^{\frac{2}{3}}, \tag{0.1}$$

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where $-\zeta_1 > -\zeta_2 > \cdots$ are the zeros of the Airy function and $S\partial O$ is the tangent sphere bundle, then there is a constant $C > 0$, such that there are no resonances in

$$\{k \in \mathbf{C}\,;\, \Re k \geq 1, \Im k \geq -C_{0,\infty}(\Re k)^{\frac{1}{3}} + C\}. \tag{0.2}$$

We also gave upper bounds on the number of resonances in regions of the form

$$\{k \in \mathbf{C}\,;\, 1 \leq \Re k \leq r, \Im k \geq -C_{0,\infty}(\Re k)^{\frac{1}{3}} - C(\Re k)^{\beta}\}$$

with $0 \leq \beta \leq 1$, which permitted us to recover the result of [SZ1] as the special case when $\beta = 1$ and C is small. When $\beta = \frac{1}{3}$ this number of resonances is $\mathcal{O}(r^{n-1})$, and when in addition C becomes very small, it is possible to estimate the constant in the "$\mathcal{O}$" by means of the volume of the set of ν for which $2^{-\frac{1}{3}}(\cos\frac{\pi}{6})\zeta_1 Q(\nu)^{\frac{2}{3}}$ is close to $C_{0,\infty}$. An explicit result of this type with $\beta \leq \frac{1}{3}$ is Theorem 2 in [SZ2].

The significance of these estimates with $\beta = \frac{1}{3}$, $C \ll 1$ or with $\beta < \frac{1}{3}$ is not clear in general, since the optimality of the pole free region result, is not known for many obstacles and for genuinely non-analytic obstacles, it may be expected that the resonances closest to the real axis are generated by a mechanism which uses the non-analyticity in an essential way and hence is out of reach of "classical" WKB-methods. Before seriously attacking the questions of optimality it therefore seems of interest to have a closer look at the upper bounds on the density of resonances in the strictly convex analytic case, because we then have a pole free region result with a constant which in general is better than in the C^∞ case: Put

$$C_{0,\mathrm{a}} = 2^{-\frac{1}{3}}\Big(\cos\frac{\pi}{6}\Big)\zeta_1 \lim_{T\to\infty}\inf_{\gamma}\frac{1}{T}\int_0^T Q\big(\dot\gamma(t)\big)^{\frac{2}{3}}\,dt, \tag{0.3}$$

where γ varies in the set of boundary geodesics with speed 1. Notice that $C_{0,\mathrm{a}} \geq C_{0,\infty}$. Then for every $\epsilon > 0$ there are at most finitely many resonances in a region of the form,

$$\{k \in \mathbf{C}\,;\, \Re k \geq 1, \Im k \geq -(C_{0,\mathrm{a}} - \epsilon)(\Re k)^{\frac{1}{3}}\}. \tag{0.4}$$

This result appears to be implicit in [BaLR] and will in any case be proved below.

Put

$$\zeta_j(\nu) = \big(2Q(\nu)\big)^{\frac{2}{3}}\zeta_j. \tag{0.5}$$

Let $\Phi_t\colon S\partial O \longrightarrow S\partial O$ be the geodesic flow, and put

(0.6)

$$\zeta_1^T(\nu) = \frac{1}{T}\int_0^T \zeta_1\big(\Phi_t(\nu)\big)\,dt, \quad \zeta_2^T(\nu) = \zeta_1^T(\nu) + \big(2Q(\nu)\big)^{\frac{2}{3}}(\zeta_2 - \zeta_1), \quad \zeta_{j,\min}^T = \inf_{S\partial O}\zeta_j^T.$$

By Birkhoff's ergodic theorem, $\zeta_1^\infty(\nu) = \lim_{T\to\infty}\zeta_1^T(\nu)$ exists almost everywhere on $S\partial O$. It is easy to see (Appendix A) that

$$\sup_{T>0}\zeta_{1,\min}^T = \lim_{T\to\infty}\zeta_{1,\min}^T \leq \operatorname{ess\,inf}\zeta_1^\infty. \tag{0.7}$$

Put

$$(0.8)\qquad W_\infty(\mu) = \int_{S\partial O} \Big(\mu - \Big(\cos\frac{\pi}{6}\Big)\zeta_1^\infty(\nu)\Big)_+^{\frac{1}{2}} S(d\nu),$$

where $S(d\nu)$ is the natural measure on $S\partial O$ (defined in Section 5). (0.8) is a disguised phase space volume, as will be clear from the discussion in Section 4. We have

$$\liminf_{T\to\infty} \zeta_{2,\min}^T - \lim_{T\to\infty} \zeta_{1,\min}^T \geq \inf_{\nu\in S\partial O} \big(2Q(\nu)\big)^{\frac{2}{3}} (\zeta_2 - \zeta_1) > 0.$$

The main result of this work is:

THEOREM. *Let $C_0 > 0$, $r_0 > 0$. For $0 \leq \mu$ with*

$$\cos\frac{\pi}{6} \lim_{T\to\infty} \zeta_{1,\min}^T + 3\Big(\mu - \cos\frac{\pi}{6} \lim_{T\to\infty} \zeta_{1,\min}^T\Big) \leq \cos\frac{\pi}{6} \liminf_{T\to\infty} \zeta_{2,\min} - 1/C_0,$$

and $k_0 \geq k_0(C, r_0) > 0$, the number of resonances k with $\Im k \geq f_{k_0,r_0,\mu}(\Re k)$, is $\leq$

$$(0.9)\qquad \frac{\sqrt{2r_0}}{(2\pi)^{n-1}} k_0^{n-1-\frac{1}{3}} \Bigg(W_\infty\Big(\cos\frac{\pi}{6} \lim_{T\to\infty} \zeta_{1,\min}^T + 3\Big(\mu - \cos\frac{\pi}{6} \lim_{T\to\infty} \zeta_{1,\min}^T\Big)\Big) + o(1)\Bigg),$$

as $k_0 \longrightarrow \infty$, uniformly with respect to μ. Here $f_{k_0,r_0,\mu}$ is the unique quadratic polynomial, such that the parabola $\Im k = f_{k_0,r_0,\mu}(\Re k)$ passes through the three points $k_0 \pm \sqrt{\frac{\mu r_0}{2}} k_0^{\frac{2}{3}}$, $k_0 - i\frac{\mu}{2} k_0^{\frac{1}{3}}$.

It would be nice to be able to eliminate the factor 3 in (0.9), which would give the simpler and more natural term $W_\infty(\mu)$. That might however be as difficult as to prove that we also have a lower bound of the same (simplified) form.

Using some other estimates from Section 4, we get a slightly different variant of the theorem: For a fixed $C > 1$ and for $\mu \geq 0$ with,

$$\lim_{T\to\infty} (\zeta_{1,\min}^T) \cos\frac{\pi}{6} + \frac{C}{C-1}\Big(\mu - \lim_{T\to\infty} (\zeta_{1,\min}^T) \cos\frac{\pi}{6}\Big) \leq \liminf_{T\to\infty} (\zeta_{2,\min}^T) \cos\frac{\pi}{6} - \frac{1}{C_0},$$

the number of resonances in $\Im k \geq f_{k_0,r_0,\mu}(\Re k)$ is $\leq$

$$\frac{\sqrt{2r_0}}{(2\pi)^{n-1}} k_0^{n-1-\frac{1}{3}} \Bigg(CW_\infty\Big(\lim_{T\to\infty} (\zeta_{1,\min}^T) \cos\frac{\pi}{6} + \frac{C}{C-1}\Big(\mu - \lim_{T\to\infty} (\zeta_{1,\min}^T) \cos\frac{\pi}{6}\Big)\Big) + o(1)\Bigg),$$

when $k_0 \longrightarrow \infty$.

Large parts of the proof are as in [SZ2]; as there we make exterior complex scaling up to the boundary and we also employ many estimates of that paper. A new feature here is that we also use microlocal weights over the boundary of the form $e^{h^{1/3} G(x',\xi')}$. Such weights are present, in many phase space approaches to Gevrey regularity questions. Suitable choices of G are obtained by a procedure of averaging along the boundary geodesic flow. A price for this averaging procedure, is that some of the more refined estimates of [SZ2] appear to be difficult to carry over, because of lack of good control over very long geodesics (the limit $T \longrightarrow \infty$). However, for surfaces of revolution, Zworski (Appendix C of this paper) notices that the full averaging is achieved already at finite time because of certain periodicity properties, and consequently he is able to obtain more precise results in that case.

It might be possible to study in the same spirit general non-trapping analytic obstacles, using classical techniques for studying propagation in the non-diffractive region ([S1], [L]).

The plan of the paper is the following: In Section 1, we adapt a machinery of global FBI-transforms from [HS] to the case of compact analytic manifolds, and define some spaces corresponding to the weights above. In Section 2, we apply such an FBI transform tangentially to the boundary and start the study of the transformed operator $P = -h^2\Delta_{|\Gamma}$, where $0 < h << 1$ is an artificial but convenient small "semi-classical" parameter, and $\Gamma \subset \mathbf{C}^n$ is the deformation of $\mathbf{R}^n \setminus \mathcal{O}$. In Section 3, we use this to get suitable *a priori* estimates on P which are used in Section 4, together with some results of Section 1, to get upper bounds on the number of resonances in certain discs. The remainder of the proof, carried out in Section 5, is then simply a work of translation. In Appendix A, we collect some easy facts about averaging along the trajectories of vectorfields, in Appendix B we explain why the averaging procedure improves the estimates. Appendix C by M. Zworski, establishes improved estimates in the case of surfaces of revolution.

We thank M. Zworski for several stimulating discussions and for reading various versions of this work.

1. **Global FBI-transformation over a compact manifold.** In this section we shall adapt the theory of [HS] to the case when $\mathbf{R}^n$ is replaced by a compact analytic manifold X of dimension n. We equip X with some analytic Riemannian metric, so that we have a distance d and a volume density dy. Let $\phi(\alpha, y)$ be an analytic function on $\{(\alpha, y) \in T^*X \times X\,;\, d(\alpha_x, y) \leq 1/C\}$ (using the notation $\alpha = (\alpha_x, \alpha_\xi)$, $\alpha_x \in X$, $\alpha_\xi \in T^*_{\alpha_x}X$,) with the following two properties (A) and (B):

(A) ϕ has a holomorphic extension to a domain of the form:

$$\left\{(\alpha, y)\,;\, |\Im\alpha_x|, |\Im y| \leq \frac{1}{C}, |\Re\alpha_x - \Re y| \leq \frac{1}{C}, |\Im\alpha_\xi| \leq \frac{1}{C}|\langle\alpha_\xi\rangle|\right\} \tag{1.1}$$

and satisfies $|\phi| \leq O(1)|\langle\alpha_\xi\rangle|$ there.

Here we write $\langle\alpha_\xi\rangle = \sqrt{1 + \alpha_\xi^2}$ and as below, we often give statements in local coordinates whenever convenient and leave to the reader to check that the statements make sense globally. Notice that by the Cauchy inequalities, we get

$$\partial_{\alpha_x}^k \partial_{\alpha_\xi}^\ell \partial_y^m \phi = O_{k,\ell,m}(1)|\langle\alpha_\xi\rangle|^{1-|\ell|} \tag{1.2}$$

in a set of the form (1.1) with an increased constant C.

(B) $\phi(\alpha, \alpha_x) = 0$, $(\partial_y\phi)(\alpha, \alpha_x) = -\alpha_\xi$, $\Im(\partial_y^2\phi)(\alpha, \alpha_x) \sim |\langle\Re\alpha_\xi\rangle| \cdot I$.

By Taylor's formula we then have

$$\phi(\alpha,y) = \alpha_\xi \cdot (\alpha_x - y) + O(1)\langle\alpha_\xi\rangle|\alpha_x - y|^2, \tag{1.3}$$

and on the real domain, for $d(\alpha_x,y) \leq 1/C$ with C sufficiently large:

$$\Im\phi(\alpha,y) \sim \langle\alpha_\xi\rangle(\alpha_x - y)^2. \tag{1.4}$$

EXAMPLE*. Let $\exp_x\colon T_xX \longrightarrow X$ be the geodesic map and define α_ξ^2 invariantly as the dual quadratic form on $T^*_{\alpha_x}X$ applied to the co-vector $\alpha_\xi \cdot dx$. Then we can take

$$\phi(\alpha,y) = -\alpha_\xi \cdot \exp_{\alpha_x}^{-1}(y) + \frac{i}{2}\langle\alpha_\xi\rangle\, d(\alpha_x,y)^2. \tag{1.5}$$

Let $\tilde{X}$ be a suitable complex neighborhood of X and let $\widetilde{T^*X}$ be a complex neighborhood of T^*X of the form $\{(x,\xi)\ ;\ x \in \tilde{X},\ |\Im\xi| \leq \frac{1}{C}|\langle\xi\rangle|\}$. Let $\Lambda \subset \widetilde{T^*X}$ be a closed I-Lagrangian manifold which is close to T^*X in the C^∞ sense and which coincides with this set outside a compact set. Recall that "I-Lagrangian" means Lagrangian for the real symplectic form $-\Im\sigma$, where $\sigma = \sum d\alpha_{\xi_j} \wedge d\alpha_{x_j}$ is the standard complex symplectic form. By being close to T^*X in the C^∞ sense, we mean that for some choice of local coordinates in X, Λ is of the form $\{(y,\eta) + iH_G(y,\eta)\ ;\ (y,\eta) \in T^*X\}$ (in the corresponding canonical coordinates) for some real-valued function G which is sufficiently small in the C^∞-sense. Here H_G denotes the Hamilton field of G. Since Λ is close to T^*X, it is also R-symplectic, *i.e.* the restriction to Λ of $\Re\sigma$ is non-degenerate. It follows that

$$d\alpha_{|\Lambda} = d\alpha_{x_1} \wedge d\alpha_{\xi_1} \cdots \wedge d\alpha_{x_n} \wedge d\alpha_{\xi_n|\Lambda} = \frac{1}{n!}(\sigma \wedge \cdots \wedge \sigma)_{|\Lambda}$$

is a real non-vanishing $2n$ form on Λ, that we view as a positive density.

We also need some symbol classes. A smooth function $a(x,\xi\ ;\ h)$ defined on Λ or on $\widetilde{T^*X}$ is said to be of class $S^{m,k}$, if

$$\partial_x^p\partial_\xi^q a = O(1)h^{-m}\langle\xi\rangle^{k-|q|}. \tag{1.6}$$

In the case when $\widetilde{T^*X}$ is the domain of definition, we require a to be holomorphic in (x,ξ). We extend this definition to symbols of the form $a(\alpha,y\ ;\ h)$ in the natural way, and we say that $a \in S^{m,k}$ is elliptic if $|a| \geq \frac{1}{C}h^{-m}|\langle\xi\rangle|^k$.

A formal classical symbol $a \in S^{m,k}_{\mathrm{cl}}$ is of the form $a \sim h^{-m}(a_0 + ha_1 + \cdots)$, where a_j is independent of h and is of class $S^{0,k-j}$. Here and in the remainder of this work, we let $0 < h \leq h_0$, for some sufficiently small $h_0 > 0$. When the domain of definition is real, we can find a realization of a in $S^{m,k}$, so that $a - h^{-m}\sum_0^N h^j a_j \in S^{-(N+1)+m,k-(N+1)}$. In fact, one verifies that a possible choice is $\sum_0^\infty \chi(\lambda_j h/\langle\xi\rangle)h^j a_j(x,\xi)$, when $\lambda_j \longrightarrow \infty$ sufficiently fast, and $\chi \in C_0^\infty([0,\infty[)$ is equal to 1 near 0. When the domain of definition is complex,

* We owe this example to discussions with M. Zworski.

we say that $a \in S_{\mathrm{cl}}^{m,k}$ is a formal classical analytic symbol ($a \in S_{\mathrm{cla}}^{m,k}$) if a_j (which then by an earlier convention are holomorphic) satisfy:

$$|a_j| \leq C_0 C^j (j!) |\langle \xi \rangle|^{k-j}. \tag{1.7}$$

It is then standard to find a realization a such that

$$\begin{gathered} \partial_x^k \partial_\xi^\ell \bar{\partial}_{(x,\xi)} a = O_{k,\ell}(1) e^{-|\langle \xi \rangle|/Ch} \\ \Big| a - h^{-m} \sum_{0 \leq j \leq |\langle \xi \rangle|/C_0 h} h^j a_j \Big| = O(1) e^{-|\langle \xi \rangle|/C_1 h}, \end{gathered} \tag{1.8}$$

where in the last estimate $C_0 > 0$ is sufficiently large and $C, C_1 > 0$ depend on C_0. We will denote by $S_{\mathrm{cl}}^{m,k}$ and $S_{\mathrm{cla}}^{m,k}$ also the classes of realizations of classical symbols. We say that a classical (analytic) symbol $a \sim h^{-m}(a_0 + ha_1 + \cdots)$ is elliptic, if a_0 is elliptic. Take such an elliptic $a(\alpha, y\,;h) \in S_{\mathrm{cla}}^{\frac{3n}{4},\frac{n}{4}}$ and put

$$Tu(\alpha\,;h) = \int e^{i\phi(\alpha,y)/h} a(\alpha, y\,;h) \chi(\alpha_x, y) u(y)\, dy, \tag{1.9}$$

where χ is smooth with support close to the diagonal and equal to one in a (sufficiently small) neighborhood of the diagonal.

We claim that there is $b(\alpha, x\,;h) \in S_{\mathrm{cla}}^{\frac{3n}{4},\frac{n}{4}}$, such that if

$$Sv(x) = \int_{T^*X} e^{-i\phi^*(\alpha,x)/h} b(\alpha, x\,;h) \chi(\alpha_x, x) v(\alpha)\, d\alpha, \tag{1.10}$$

then

$$STu = u + Ru, \tag{1.11}$$

where R has a distribution kernel $R(x, y\,;h)$ satisfying

$$|\partial_x^k \partial_y^\ell R| \leq C_{k,\ell} e^{-1/C_0 h}. \tag{1.12}$$

Here we denote in general by f^*, the holomorphic extension of the complex conjugate of f: $f^*(z) = \overline{f(\bar{z})}$.

We now prove the claim: Let S_0 be of the form (1.10), where b is replaced by some elliptic b_0 in the same symbol class. Then,

$$S_0 Tu(x) = \int_{T^*X} \int_X e^{\frac{i}{h}(-\phi^*(\alpha,x) + \phi(\alpha,y))} \chi(\alpha_x, x) b_0(\alpha, x\,;h) a(\alpha, y\,;h) \chi(\alpha_x, y) u(y)\, dy\, d\alpha.$$

Since,

$$\Im\big(-\phi^*(\alpha,x) + \phi(\alpha,y)\big) \sim \langle \alpha_\xi \rangle \big((\alpha_x - x)^2 + (\alpha_x - y)^2\big) \geq \langle \alpha_\xi \rangle \frac{1}{C} d(x,y)^2,$$

we see that the distribution kernel of this operator is exponentially decaying with all its derivatives outside any neighborhood of the diagonal. For (x,y) in a small neighborhood of the diagonal, we can apply analytic stationary phase [S2] to the α_x integration and obtain in local coordinates (and using the same symbol for an operator and its distribution kernel):

$$S_0T(x,y\,;h) = \int e^{\frac{i}{h}\psi(x,y,\alpha_\xi)}c(x,y,\alpha_\xi\,;h)\,d\alpha_\xi + \text{ exponentially small error},$$

where $c \in S^{n,0}_{\mathrm{cla}}$ is elliptic and $\psi(x,y,\alpha_\xi) = \mathrm{v.c.}_{\alpha_x}\big(-\phi^*(\alpha,x) + \phi(\alpha,y)\big)$, where $\mathrm{v.c.}_{\alpha_x}$ indicates critical value with respect to α_x, and $\psi(x,y,\alpha_\xi) = (x-y)\cdot\alpha_\xi + O(\langle\alpha_\xi\rangle(x-y)^2)$, $\Im\psi \sim \langle\alpha_\xi\rangle(x-y)^2$. A standard change of variables in α_ξ (a complex version of the Kuranishi trick) reduces ψ to $(x-y)\cdot\alpha_\xi$, and shows that S_0T is formally a h-pseudor (our short word for h-pseudodifferential operator) of the form

$$\frac{1}{(2\pi h)^n}\int e^{\frac{i}{h}(x-y)\cdot\theta}\tilde{c}(x,y,\theta\,;h)\,d\theta,$$

where $\tilde{c} \in S^{0,0}_{\mathrm{cla}}$ is elliptic. At least formally, we can first eliminate the y variable and then write a parametrix D. The formal composition $S = D\circ S_0$ is then of the form (1.10) and one can verify first that $S\circ T$ acts as the identity operator on oscillatory functions, and then by applying $S\circ T$ to a resolution of the identity, that we have (1.11), (1.12). ∎

Put

$$T_\Lambda u = Tu_{|\Lambda}, \tag{1.13}$$

and define $S_\Lambda v$ by (1.10) but with T^*X replaced by Λ. Then,

$$S_\Lambda T_\Lambda u = u + R_\Lambda u, \tag{1.14}$$

where R_Λ satisfies (1.12) (with a slightly larger C_0 and under the assumption that Λ is sufficiently close to T^*X). In fact, using Stokes' formula and the exponential decrease of $\bar\partial$ of the symbols involved, we see that $S_\Lambda T_\Lambda$ coincides up to an exponentially small error with S_0T.

Since Λ is I-Lagrangian, we can find locally a real smooth function $H(\alpha)$ on Λ such that

$$dH = -\Im(\alpha_\xi\cdot d\alpha_x)_{|\Lambda}. \tag{1.15}$$

Indeed, we have $d(\alpha_\xi\cdot d\alpha_x) = \sigma$, so $d\big(-\Im(\alpha_\xi\cdot d\alpha_x)_{|\Lambda}\big) = -\Im\sigma_{|\Lambda} = 0$. We assume:

(1.16) The equation (1.15) has a global solution $H \in C^\infty(\Lambda\,;\mathbf{R})$.

Then H is well defined up to a constant (assuming of course that X is connected).

EXAMPLE. Let $G \in C^\infty(\widetilde{T^*X}\,;\mathbf{R})$ with uniformly compact support in α_ξ. Let $H_G = H_G^{\Im\sigma}$ be the Hamilton field of G with respect to $\Im\sigma$. Then for t real with $|t|$ small enough, we can consider the I-Lagrangian manifold, $\Lambda_t = \exp(tH_G)(\Lambda_0)$, where $\Lambda_0 = T^*X$. On Λ_0 we can take $H_0 = 0$ (satisfying (1.15)) and on Λ_t, we then try:

$$H_t = \int_0^t(\exp(s-t)H_G)^*(\langle H_G,\omega\rangle + G)\,ds + \exp(-tH_G)^*(H_0), \tag{1.17}$$

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where $\omega = -\Im(\alpha_\xi \cdot d\alpha_x)$. More precisely, we first extend H_0 to a neighborhood of Λ_0 and then define H_t near Λ_t by the above expression. We have

$$d(\langle H_G, \omega\rangle) = d(H_G\rfloor\omega) = \mathcal{L}_{H_G}\omega - H_G\rfloor d\omega = \mathcal{L}_{H_G}\omega + H_G\rfloor\Im\sigma = \mathcal{L}_{H_G}\omega - dG,$$

where $H_G\rfloor$ denotes the adjoint of left exterior multiplication, $H_G\wedge$, so $d(\langle H_G, \omega\rangle + G) = \mathcal{L}_{H_G}\omega$, and

$$\begin{aligned} dH_t &= \int_0^t \big(\exp(s-t)H_G\big)^*(\mathcal{L}_{H_G}\omega)\,ds + \big(\exp(-t)H_G\big)^*(dH_0) \\ &= \int_0^t \frac{d}{ds}\big(\exp(s-t)H_G\big)^*\omega\,ds + \big(\exp(-t)H_G\big)^*(dH_0) \\ &= \omega + \big(\exp(-t)H_G\big)^*(-\omega + dH_0). \end{aligned}$$

Restricting the relation to Λ_t, we get,

$$(dH_t)_{|\Lambda_t} = \omega_{|\Lambda_t} + \big(\exp(-t)H_G\big)^*\big((-\omega + dH_0)_{|\Lambda_0}\big) = \omega_{|\Lambda_t}.$$

From (1.17) we get since $H_0 = 0$:

$$H_t = t\big(G - \langle H_G^{\Im\sigma}, \Im(\alpha_\xi \cdot d\alpha_x)\rangle\big) + O(t^2). \tag{1.18}$$

The function H appears naturally in connection with T_Λ. We have $d_\alpha\phi = \alpha_\xi \cdot d\alpha_x + O(|\alpha_x - y|)$, so $(d_\alpha\phi)(\alpha, \alpha_x) = \alpha_\xi \cdot d\alpha_x$ and

$$-\Im(d_\alpha\phi)(\alpha, \alpha_x)_{|\Lambda} = d_\alpha H. \tag{1.19}$$

It may therefore be natural to consider the space of distributions u (depending on h) such that $T_\Lambda u = O(e^{H/h})$ for instance in the L^2 sense.

DEFINITION 1.1. For $m \in \mathbf{R}$, put

$$H(\Lambda\,;\langle\alpha_\xi\rangle^m) = \{u\,;\,T_\Lambda u \in L^2(\Lambda\,;e^{-2H/h}|\langle\alpha_\xi\rangle|^{2m}\,d\alpha)\}.$$

When $\Lambda = T^*X$ we get the usual h-Sobolev spaces with uniformly equivalent norm with respect to h. For general Λ we also get the same spaces but the norm now depends in a more crucial way on h.

We shall next show that the Definition 1.1 does not depend on the choice of the phase and the amplitude in T_Λ. This will also be the opportunity to introduce certain projections on the transform side. Recall first that $S_\Lambda T_\Lambda = I + R_\Lambda$, where the distribution kernel of R_Λ and all its derivatives are $O(e^{-1/C_0h})$ for some fixed $C_0 > 0$. Then for h small enough, $(I+R_\Lambda)^{-1} = I + \tilde{R}_\Lambda$, where $\tilde{R}_\Lambda$ has the same properties. We have the left inverse $T_\Lambda^{-1} = (I+R_\Lambda)^{-1}S_\Lambda$. Let

$$\tilde{T}_\Lambda u(\alpha) = \int e^{\frac{i}{h}\tilde{\phi}(\alpha,y)}\tilde{a}(\alpha, y\,;h)\tilde{\chi}(\alpha_x, y)u(y)\,dy$$

have the same general properties as T_Λ. Consider,

$$\tilde{T}_\Lambda T_\Lambda^{-1} = \tilde{T}_\Lambda S_\Lambda + \tilde{T}_\Lambda\tilde{R}_\Lambda S_\Lambda. \tag{1.20}$$

Write $\tilde{T}_\Lambda T_\Lambda^{-1}u(\alpha) = \int_\Lambda k(\alpha,\beta\ ;\ h)v(\beta)\,d\beta$ and consider first the contribution to k from $\tilde{T}_\Lambda S_\Lambda$:

$$k_1(\alpha,\beta\,;h) = \int e^{\frac{i}{h}(\tilde{\phi}(\alpha,y)-\phi^*(\beta,y))}\tilde{a}(\alpha,y\,;h)b(\beta,y\,;h)\tilde{\chi}(\alpha_x,y)\chi(\beta_x,y)\,dy. \tag{1.21}$$

This expression vanishes when $d(\alpha_x,\beta_x) \geq 1/C$, so we may restrict the attention to $d(\alpha_x,\beta_x) \leq 1/C$. When $|\alpha_\xi - \beta_\xi| \geq \frac{1}{C}|\langle\beta_\xi\rangle|$ and $d(\alpha_x,\beta_x)$ is small enough, we have $|d_y\big(\tilde{\phi}(\alpha,y) - \phi^*(\beta,y)\big)| \sim |\langle\alpha_\xi\rangle| + |\langle\beta_\xi\rangle|$, and we can make integrations by parts plus a contour deformation in y and get

$$k_1(\alpha,\beta\,;h) = O_N(1)e^{-\frac{1}{Ch}\min(|\langle\alpha_\xi\rangle|,|\langle\beta_\xi\rangle|)}\big(\max(|\langle\alpha_\xi\rangle|,|\langle\beta_\xi\rangle|)\big)^{-N}. \tag{1.22}$$

This is valid also without the assumption that $d(\alpha_x,\beta_x)$ is small enough. Since $H(\alpha)$ has compact support and is small, we then have the same estimate for the reduced kernel: $e^{(-H(\alpha)+H(\beta))/h}k_1(\alpha,\beta\ ;\ h)$. Now look at the region: $d(\alpha_x,\beta_x) \leq 1/C$, $|\alpha_\xi - \beta_\xi| \leq |\langle\beta_\xi\rangle|/C$. If $d(\alpha_x,\beta_x) \geq 1/$ Const, we can use the Gaussian behaviour of $\tilde{\phi}$, $-\phi^*$, to see that k_1 satisfies (1.22) in this region also. Here $\min(|\langle\alpha_\xi\rangle|,|\langle\beta_\xi\rangle|) \sim \max(|\langle\alpha_\xi\rangle|,|\langle\beta_\xi\rangle|)$, so we make contour deformation as in the method of steepest descent and obtain:

$$k_1(\alpha,\beta\,;h) = c(\alpha,\beta\,;h)e^{\frac{i}{h}\psi(\alpha,\beta)}, \tag{1.23}$$

where modulo $O(1)h^N\max(|\langle\alpha_\xi\rangle|,|\langle\beta_\xi\rangle|)^{-N}$, c is of class $S^{n,0}_{\mathrm{cla}}$ and where

$$\psi(\alpha,\beta) = \mathrm{v.\,c.}_y\big(\tilde{\phi}(\alpha,y) - \phi^*(\beta,y)\big). \tag{1.24}$$

For $\alpha = \beta$, the critical point is $y = \alpha_x$; $\psi = 0$ and

$$d_\alpha\psi = (d_\alpha\tilde{\phi})(\alpha,\alpha_x) = \alpha_\xi\cdot d\alpha_x, \quad d_\beta\psi = -\beta_\xi\cdot d\beta_x,$$

and taking the imaginary parts, we get for $\alpha = \beta$:

$$d_\alpha(-\Im\psi)_{|\Lambda} = d_\alpha H, \quad d_\beta(-\Im\psi)_{|\Lambda} = -d_\beta H.$$

It follows that

$$-H(\alpha) - \Im\psi(\alpha,\beta) + H(\beta) = O(1)(|\langle\alpha_\xi\rangle|\,|\alpha_x - \beta_x|^2 + |\langle\alpha_\xi\rangle|^{-1}|\alpha_\xi - \beta_\xi|^2).$$

Considering first the case $\Lambda = T^*X$ and making a small perturbation argument, we obtain more precisely:

$$-H(\alpha) - \Im\psi(\alpha,\beta) + H(\beta) \sim -(|\langle\alpha_\xi\rangle|\,|\alpha_x - \beta_x|^2 + |\langle\alpha_\xi\rangle|^{-1}|\alpha_\xi - \beta_\xi|^2). \tag{1.25}$$

The second term of the RHS of (1.20) has a kernel $k_2(\alpha,\beta\,;h)$ satisfying,

$$k_2(\alpha,\beta\,;h) = O_N(1)e^{-\frac{1}{Ch}}\max(|\langle\alpha_\xi\rangle|,|\langle\beta_\xi\rangle|)^{-N}, \tag{1.26}$$

for every N. Again, it is clear that the reduced kernel, $e^{(-H(\alpha)+H(\beta))/h}k_2(\alpha,\beta\ ;\ h)$ satisfies the same estimates.

In conclusion, $\tilde{T}_\Lambda T_\Lambda^{-1}$ has a kernel,

$$\chi(\alpha_x,\beta_x)\chi(\frac{1}{|\langle\alpha_\xi\rangle|}|\alpha_\xi-\beta_\xi|)c(\alpha,\beta\,;h)e^{\frac{i}{h}\psi(\alpha,\beta)}+k_3(\alpha,\beta\,;h), \tag{1.27}$$

where $k_3(\alpha,\beta\,;h)$ and $e^{\frac{1}{h}(-H(\alpha)+H(\beta))}k_3(\alpha,\beta\,;h)$ satisfy (1.26). (Here and in the following we become sloppy with the use of the symbol χ which denotes a standard cut off function, equal to one near the diagonal or near 0, depending on the context.) Using this and (1.25), we see that $\tilde{T}_\Lambda T_\Lambda^{-1} = O(1)$ as an operator: $L^2(\Lambda\,;|\langle\alpha_\xi\rangle|^m e^{-H/h}\,d\alpha)\longrightarrow L^2(\Lambda\,;|\langle\alpha_\xi\rangle|^m e^{-H/h}\,d\alpha)$.

We pause for some general considerations: Let $\phi(x,y)$ be a smooth real (or holomorphic) function, defined near $(x_0,y_0)\in\mathbf{R}^{n+k}$ ($\mathbf{C}^{n+k}$) with $k<n$. Assume that ϕ''_{xy} is of maximal rank, so that $J=\{(x,\phi'_x(x,y))\,;y\in\operatorname{neigh}(y_0)\}$ is a smooth $n+k$-dimensional manifold. J is involutive since it is a union of Lagrangian manifolds, and can be given by $p_1(x,\xi)=\cdots=p_{n-k}(x,\xi)=0$, where dp_j are independent and the Poisson brackets $\{p_j,p_k\}$ vanish on J. Let $a(x,y\,;h)\sim a_0(x,y)+ha_1(x,y)+\cdots$ be a formal classical elliptic symbol, defined near (x_0,y_0). It is then straight forward to construct formal symbols $P_j(x,\xi\,;h)\sim p_j(x,\xi)+hp_j^1(x,\xi)+\cdots$ near (x_0,ξ_0), $\xi_0=\phi'_x(x_0,y_0)$, such that $P_j(x,hD_x\,;h)\big(e^{i\phi(x,y)/h}a(x,y\,;h)\big)\sim 0$ in the sense of formal asymptotic expansions. In general, if $P(x,\xi\,;h)\sim p^0(x,\xi)+hp^1(x,\xi)+\cdots$ is a formal classical symbol defined near (x_0,ξ_0), and $P(x,hD_x\,;h)\big(e^{i\phi(x,y)/h}a(x,y\,;h)\big)\sim 0$, then we see by an easy iteration procedure, that $P=\sum A_\nu P_\nu$ for some h-pseudors A_ν of order 0, (whose symbols are formal classical symbols of order 0). In particular, we have the Lie algebra property: $[P_j,P_k]=h\sum A_{j,k}^\nu P_\nu$.

The manifold $J=\{(\alpha,d_\alpha\phi(\alpha,y))\}$ is a complex involutive manifold of complex codimension n and we can construct a Lie algebra $\mathcal{L}$ of formal analytic h-pseudors on $\widetilde{T^*X}$, which for every choice of local coordinates in X has n generators

$$\zeta_1(\alpha,hD_\alpha\,;h),\ldots,\zeta_n(\alpha,hD_\alpha\,;h),$$

such that

$$\zeta_j(\alpha,hD_\alpha\,;h)T_\Lambda\sim 0 \text{ in the formal asymptotic sense.} \tag{1.28}$$

To be more precise, let $\alpha^*=(\alpha_x^*,\alpha_\xi^*)$ denote the dual variables to α. Then, by expressing y as a function of α, α_ξ^* by the implicit function theorem:

$$\zeta_j(\alpha,\alpha^*\,;h)\sim\zeta_j^0(\alpha,\alpha^*)+h\zeta_j^1(\alpha,\alpha^*)+\cdots$$

and

$$\zeta_j^0=\alpha_{x_j}^*-\alpha_{\xi_j}+O(\langle\alpha_\xi\rangle\alpha_\xi^*)$$

is $O(\langle\alpha_\xi\rangle)$ in polydiscs of the form: $\alpha_x-\alpha_x^0$, $\alpha_\xi^*-\alpha_\xi^{0*}=O(1)$, $\alpha_x^*-\alpha_x^{0*}$, $\alpha_\xi-\alpha_\xi^0=O(\langle\alpha_\xi^0\rangle)$. After a natural dilation in the α_ξ-variables, we see that the effective Planck's constant is $h/\langle\alpha_\xi\rangle$, so $\zeta_j^k = O(\langle\alpha_\xi^0\rangle^{1-k})$ in the same polydiscs. The Lie algebra property is expressed by:

$$[\zeta_j,\zeta_k] = \sum_{\nu=1}^{n} hQ_{j,k}^{\nu}\zeta_\nu, \tag{1.29}$$

where $Q_{j,k}^{\nu}$ is a formal h-pseudor of order 0 in h (and a standard $h/\langle\alpha_\xi^0\rangle$ pseudor in the variables α_x, β_ξ, where $\alpha_\xi - \alpha_\xi^0 = \langle\alpha_\xi^0\rangle\beta_\xi$).

We next recall the construction of an appropriate version B_Λ of the orthogonal projection: $L^2(\Lambda\,;e^{-2H/h}\,d\alpha) \longrightarrow L^2(\Lambda\,;e^{-2H/h}\,d\alpha) \cap$ Image of (T_Λ) (in the spirit of [BoS] as in [HS]). We then want B_Λ to be self-adjoint in $L^2(\Lambda\,;e^{-2H/h}\,d\alpha)$ and to satisfy

$$\zeta_j(\alpha, hD_\alpha\,;h)B_\Lambda \approx 0, \tag{1.30}$$

in a sense to be specified. We then also get

$$B_\Lambda \circ \zeta_j^{\Lambda,H,*} \approx 0, \tag{1.31}$$

where $\zeta_j^{\Lambda,H,*}$ denotes the complex adjoint of ζ_j in $L^2(\Lambda\,;e^{-2H/h}\,d\alpha)$. We get

$$\zeta_j^{\Lambda,H,*} = e^{2H/h}\circ\zeta_j^{\Lambda,*}\circ e^{-2H/h},$$

where $\zeta_j^{\Lambda,*}$ is the adjoint of ζ_j in $L^2(\Lambda, d\alpha)$ and has the leading symbol $\overline{\zeta_j^0(\alpha,\alpha^*)}$, $(\alpha,\alpha^*) \in T^*\Lambda$. (Here $T^*\Lambda$ is viewed as a subspace of the complex cotangent space of $\widetilde{T^*X}$ in the natural way.) These adjoints are not necessarily analytic pseudors, unless Λ is analytic. The symbol of $\zeta_j^{\Lambda,*}$ belongs to $S_{\mathrm{cl}}^{0,1}$.

We now look for B_Λ of the form:

$$B_\Lambda u(\alpha) = \int_\Lambda k(\alpha,\beta\,;h)u(\beta)e^{-2H(\beta)/h}\,d\beta.$$

Then we want

$$\zeta_j(\alpha, hD_\alpha\,;h)k(\alpha,\beta\,;h) \sim 0, \tag{1.32}$$

and

$$\begin{aligned} 0 \sim B_\Lambda\zeta_j^{\Lambda,H,*}u &= \int_\Lambda k(\alpha,\beta\,;h)e^{2H/h}\zeta_j^{\Lambda,*}(e^{-2H/h}u)e^{-2H/h}\,d\beta \\ &= \int_\Lambda k(\alpha,\beta\,;h)\zeta_j^{\Lambda,*}(e^{-2H/h}u)\,d\beta \\ &= \int_\Lambda {}^t\big(\zeta_j^{\Lambda,*}(\beta, hD_\beta)\big)\big(k(\alpha,\beta\,;h)\big)e^{-2H/h}u\,d\beta. \end{aligned}$$

Here ${}^t(\zeta_j^{\Lambda,*}) = \Gamma\circ\zeta_j\circ\Gamma \underset{\mathrm{def}}{=} \tilde\zeta_j^\Lambda(\alpha, hD_\alpha\,;h)$, where Γ is the operator of complex conjugation. The symbol of this operator is $\overline{\zeta_{j|T^*\Lambda}(\alpha,-\alpha^*)} = \tilde\zeta_j^\Lambda(\alpha,\alpha^*)$. Though we finally avoided the problem of studying adjoints of h-pseudors in this setting, we still dropped out of the frame work of analytic h-pseudors, but we can still think of the symbol $\tilde\zeta_j^\Lambda$ as defined near $T^*\Lambda$ by means of almost analytic extensions.

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In [HS], B_Λ was constructed with

$$k(\alpha,\beta\,;h)=e^{\frac{i}{h}\psi(\alpha,\beta)}a(\alpha,\beta\,;h), \tag{1.33}$$

with $k(\beta,\alpha\,;h)=\overline{k(\alpha,\beta\,;h)}$, $\psi(\beta,\alpha)=-\overline{\psi(\alpha,\beta)}$, $a(\beta,\alpha\,;h)=\overline{a(\alpha,\beta\,;h)}$, $a\in S^{n,0}_{\mathrm{cl}}$,

$$\left\{\begin{matrix}\zeta_j(\alpha,hD_\alpha\,;h)\\ \tilde{\zeta}_j(\beta,hD_\beta\,;h)\end{matrix}\right\}\left(e^{\frac{i}{h}\psi(\alpha,\beta)}a(\alpha,\beta\,;h)\right)\sim 0, \tag{1.34}$$

leading to the eikonal equations involving the leading symbols:

$$\zeta_j^0(\alpha,d_\alpha\psi)=0,\quad \tilde{\zeta}_j^0(\beta,d_\beta\psi)=0, \tag{1.35}$$

that should be satisfied to infinite order on $\mathrm{diag}(\Lambda\times\Lambda)$ and a sequence of transport equations. (Actually, this is in essence the construction in [BoS].) From this we get ψ uniquely up to a term vanishing to infinite order on the diagonal, if we add the requirement,

$$\psi(\alpha,\alpha)=-2iH(\alpha),\quad (d_\alpha\psi)(\alpha,\alpha)=\alpha_\xi\cdot d\alpha_{x|\Lambda}. \tag{1.36}$$

Notice that the second equation and the anti Hermitian property of ψ imply $(d_\beta\psi)(\alpha,\alpha)=-\overline{\alpha_\xi\cdot d\alpha_{x|\Lambda}}$, so $d\big(\psi(\alpha,\alpha)\big)=2i\Im\alpha_\xi\cdot d\alpha_{x|\Lambda}=-2idH$, and we see that the first condition in (1.36) only determines an integration constant in ψ. Also $(d_\alpha\Im\psi)(\alpha,\alpha)=\Im(\alpha_\xi\cdot d\alpha_x)_{|\Lambda}=-d_\alpha H$, $(d_\beta\Im\psi)(\alpha,\alpha)=-d_\alpha H$, so if we take into account scaling properties:

$$-\Im\psi(\alpha,\beta)-H(\alpha)-H(\beta)=O\Big(\frac{1}{|\langle\alpha_\xi\rangle|}|\alpha_\xi-\beta_\xi|^2+|\langle\alpha_\xi\rangle|\,|\alpha_x-\beta_x|^2\Big), \tag{1.37}$$

restricting the attention to $|\alpha_x-\beta_x|,|\alpha_\xi-\beta_\xi|/|\langle\alpha_\xi\rangle|\leq 1/C$. By a perturbation argument, starting with the case $\Lambda=T^*X$, and using (1.55) (below) in that case, we see that actually,

$$-\Im\psi(\alpha,\beta)-H(\alpha)-H(\beta)\sim-\frac{1}{|\langle\alpha_\xi\rangle|}|\alpha_\xi-\beta_\xi|^2-|\langle\alpha_\xi\rangle|\,|\alpha_x-\beta_x|^2. \tag{1.38}$$

The amplitude $a=h^{-n}\sum_{j\geq 0}a_j(\alpha,\beta)h^j$ is determined in the following way: We need that $a_0(\alpha,\alpha)>0$. Examining the condition that $B_\Lambda^2=B_\Lambda$ by stationary phase, we get $a_0(\alpha,\alpha)$. Then we get $a_0(\alpha,\beta)$ to infinite order on $\alpha=\beta$, by solving the transport equations (to infinite order on $\alpha=\beta$). The condition $B_\Lambda^2=B_\Lambda$ now gives $a_1(\alpha,\alpha)$ and solving the transport equations, we get $a_1(\alpha,\beta)$ and so on. Notice that $|a_0|\sim 1$. We define B_Λ by introducing a cut off near the diagonal as in (1.27).

PROPOSITION 1.1. *(i)* $B_\Lambda^2=B_\Lambda+R_\Lambda$, *where*

$$R_\Lambda=O(h^\infty)\colon L^2(\Lambda\,;|\langle\alpha_\xi\rangle|^{-m}e^{-2H/h}\,d\alpha)\to L^2(\Lambda\,;|\langle\alpha_\xi\rangle|^{m}e^{-2H/h}\,d\alpha)$$

for all $m\in\mathbf{R}$.

(ii) $\|T_\Lambda u-B_\Lambda T_\Lambda u\|_{L^2(\Lambda,|\langle\alpha_\xi\rangle|^m e^{-2H/h}\,d\alpha)}\leq O(h^\infty)\|u\|_{H(\Lambda,\langle\cdot\rangle^{-m})}$ *for all* m.

PROOF OF (ii). Put $\Pi_\Lambda=T_\Lambda T_\Lambda^{-1}$ (with $T_\Lambda^{-1}=(I+R_\Lambda)^{-1}S_\Lambda$) and notice that Π_Λ is a projection from $L^2(\Lambda\,;e^{-2H/h}\,d\alpha)$ onto the subspace $L=T_\Lambda(L^2)$. Since Π_Λ is bounded, this space is closed. Using this projection we shall construct the corresponding orthogonal projection and show that it is very close to B_Λ. We have already studied the asymptotics of the kernel of Π_Λ. (Notice that the phase appearing there may be different from the one appearing in B_Λ.)

For $f \in S^{0,0}, f \geq 1/C$, consider $\Pi_\Lambda f \Pi_\Lambda^*$, where the adjoint is taken in the L^2 sense, with respect to the measure $e^{-2H/h}\, d\alpha$. For $u \in L$, we have

$$(\Pi_\Lambda f \Pi_\Lambda^* u | u)_{L^2(\dots)} = (f \Pi_\Lambda^* u | \Pi_\Lambda^* u)_{L^2(\dots)} \geq (\inf f) \|\Pi_\Lambda^* u\|^2.$$

Here

$$(\Pi_\Lambda^* u | u) = (u | \Pi_\Lambda u) = \|u\|^2,$$

so $\|u\| \leq \|\Pi_\Lambda^* u\|$. Consequently,

$$(\Pi_\Lambda f \Pi_\Lambda^* u | u) \geq (\inf f) \|\Pi_\Lambda^* u\|^2, \text{ and } (\inf f)\|u\| \leq \|\Pi_\Lambda f \Pi_\Lambda^* u\|, \quad u \in L.$$

Also notice that $\Pi_\Lambda f \Pi_\Lambda^* : L \longrightarrow L$. We conclude that the spectrum $\sigma(\Pi_\Lambda f \Pi_\Lambda^*)$ is contained in $[0, \infty[$ and has 0 as an isolated point. L is the spectral subspace corresponding to the non-vanishing part of the spectrum. Let γ be a closed positively oriented loop around the non-vanishing part of the spectrum which has 0 at its exterior. We then get the orthogonal projection P_Λ onto L by the formula

$$P_\Lambda = \frac{1}{2\pi i} \int_\gamma (z - \Pi_\Lambda f \Pi_\Lambda^*)^{-1}\, dz. \tag{1.39}$$

Using the earlier results on the structure of Π_Λ and the operators ζ_j above plus stationary phase, we see that

$$\Pi_\Lambda f \Pi_\Lambda^* u = \int k_f(\alpha, \beta\,; h) u(\beta) e^{-2H(\beta)/h}\, d\beta + Ru, \tag{1.40}$$

where $R = O(h^\infty): L^2(\Lambda\,; e^{-2H/h} |\langle \alpha_\xi \rangle|^m\, d\alpha) \longrightarrow L^2(\Lambda\,; e^{-2H/h} |\langle \alpha_\xi \rangle|^m\, d\alpha)$, and where

$$k_f(\alpha, \beta\,; h) = e^{\frac{i}{h}\psi(\alpha,\beta)} a_f(\alpha, \beta\,; h) \tag{1.41}$$

with ψ as in (1.33) and with a_f having the same properties as a there (same symbol class and same transport equations). It is then an easy exercise to construct f so that $a_f = a$ in (1.33) and we have then found $A = \Pi_\Lambda f \Pi_\Lambda^*$ so that

$$A = B_\Lambda + R \tag{1.42}$$

with R as in (1.40). In particular

$$A - A^2 = O(h^\infty): L^2(\Lambda\,; e^{-2H/h} |\langle \alpha_\xi \rangle|^{-m}\, d\alpha) \longrightarrow L^2(\Lambda\,; e^{-2H/h} |\langle \alpha_\xi \rangle|^m\, d\alpha) \tag{1.43}$$

We have the same estimates on the derivatives of the kernel of $A - A^2$ so (1.43) also holds in the sense of trace class operators.

Now return to the formula (1.39). As an approximate inverse of $(z-A)$, we try $\frac{1}{z}+\frac{1}{z(z-1)}A$ (for $z\in\gamma$) and get

$$(z-A)\Big(\frac{1}{z}+\frac{1}{z(z-1)}A\Big)=1+\frac{1}{z(z-1)}(A-A^2).$$

Hence

$$(1.44)\qquad (z-A)^{-1}=\Big(\frac{1}{z}+\frac{1}{z(z-1)}A\Big)\Big(1+\frac{1}{z(z-1)}(A-A^2)\Big)^{-1}.$$

For $z\in\gamma$, we have in view of (1.43), that

$$(1.45)\qquad \Big(1+\frac{1}{z(z-1)}(A-A^2)\Big)^{-1}=1+R,$$

where R has the same properties as $A-A^2$ in (1.43). (We let γ be h-independent and choose $h>0$ sufficiently small.) Combining (1.44), (1.45) with (1.39) and (1.42), we get

$$(1.46)\qquad P_\Lambda=A+R_1=B_\Lambda+R_2,$$

where R_1, R_2 are as in (1.43) also in the sense of trace class operators. Since $P_\Lambda T_\Lambda=T_\Lambda$, we get (ii) of Proposition 1.3. ■

We next discuss the action of h-diffors (short word for h-differential operators) in the $H(\Lambda)$-spaces. By definition, an analytic h-diffor of order m is an operator of the form:

$$(1.47)\qquad P(x,hD\,;h)=\sum_{|k|\leq m}a_k(x\,;h)(hD_x)^k,$$

with a_k holomorphic and $O(1)$ in $\tilde{X}$. The corresponding symbol (invariantly defined mod $O(1)h\langle\xi\rangle^{m-1}$) is then $P(x,\xi\,;h)=\sum_{|k|\leq m}a_k(x\,;h)\xi^k$.

PROPOSITION 1.2. *Let P be as above. Then,*

$$\|T_\Lambda Pu-P_{|\Lambda}T_\Lambda u\|_{L^2(\Lambda;e^{-2H/h}d\alpha)}\leq Ch^{1/2}\|u\|_{H(\Lambda;\langle\alpha_\xi\rangle^m)}.$$

A very similar result was obtained in [HS] and we omit the proof. For scalar products, we have an improved result:

PROPOSITION 1.3. *Let P_1, P_2 be analytic h-diffors of order m_1, m_2. Then*

$$(P_1u|P_2v)_{H(\Lambda)}=(P_{1|\Lambda}T_\Lambda u|P_{2|\Lambda}T_\Lambda v)_{L^2(\Lambda;e^{-2H/h}d\alpha)}+O(h)\|u\|_{H(\Lambda,\langle\ \rangle^{\tilde{m}_1})}\|v\|_{H(\Lambda,\langle\ \rangle^{\tilde{m}_2})},$$

if $m_1+m_2=\tilde{m}_1+\tilde{m}_2$.

PROOF. Start with,

$$(P_1u|P_2v)_{H(\Lambda)}=(T_\Lambda P_1T_\Lambda^{-1}T_\Lambda u_1|T_\Lambda P_2T_\Lambda^{-1}T_\Lambda u_2)_{L^2},$$

where the L^2 scalar product is with respect to $e^{-2H/h}\,d\alpha$. Inserting B_Λ here and there, and using (ii) of Proposition 1.1, we get modulo $O(h)\|u\|_{H(\Lambda,\langle\rangle^{\tilde m_1})}\|v\|_{H(\Lambda,\langle\rangle^{\tilde m_2})}$, if $m_1+m_2=\tilde m_1+\tilde m_2$:

$$(1.48)\qquad (P_1u|P_2v)\equiv(B_\Lambda T_\Lambda P_1T_\Lambda^{-1}B_\Lambda T_\Lambda u|B_\Lambda T_\Lambda P_2T_\Lambda^{-1}B_\Lambda T_\Lambda v).$$

Using stationary phase and the general arguments in the construction of B_Λ, we get

$$(1.49)\qquad B_\Lambda T_\Lambda P_jT_\Lambda^{-1}B_\Lambda=B_\Lambda(P_{j|\Lambda})B_\Lambda+O(h)\quad\text{in}$$
$$\mathcal{L}\big(L^2(\Lambda\,;|\langle\alpha_\xi\rangle|^m e^{-2H/h}\,d\alpha),L^2(\Lambda\,;|\langle\alpha_\xi\rangle|^{m-m_j}e^{-2H/h}\,d\alpha)\big),$$

and (1.48) then gives

$$(1.50)\qquad (P_1u|P_2v)\equiv\big(B_\Lambda\overline{P_{2|\Lambda}}B_\Lambda B_\Lambda(P_{1|\Lambda})B_\Lambda T_\Lambda u|T_\Lambda v\big).$$

To complete the proof it then suffices to verify (by stationary phase) the Toeplitz calculus property:

$$(1.51)\qquad B_\Lambda\overline{P_{2|\Lambda}}B_\Lambda B_\Lambda(P_{1|\Lambda})B_\Lambda=B_\Lambda\overline{P_{2|\Lambda}}(P_{1|\Lambda})B_\Lambda+O(h)\quad\text{in}$$
$$\mathcal{L}\big(L^2(\Lambda\,;|\langle\alpha_\xi\rangle|^m e^{-2H/h}\,d\alpha),L^2(\Lambda\,;|\langle\alpha_\xi\rangle|^{m-m_1-m_2}e^{-2H/h}\,d\alpha)\big)$$

In fact, from (1.50) we then get

$$(1.52)\qquad (P_1u|P_2v)\equiv\big((P_{1|\Lambda})B_\Lambda T_\Lambda u_1|(P_{2|\Lambda})B_\Lambda T_\Lambda u_2\big)\equiv\big((P_{1|\Lambda})T_\Lambda u_1|(P_{2|\Lambda})T_\Lambda u_2\big).$$

The arguments above work equally well, if we use the exact orthogonal projection P_Λ instead of B_Λ. ∎

We end this section, by considering finite rank approximations of certain Toeplitzors (short word for Toeplitz operators). Let $q\in C_0^\infty(\Lambda\,;\mathbf{R})$. Then we have

$$(1.53)\qquad (qT_\Lambda u|T_\Lambda u)_{L^2\Lambda;e^{-2H/h}\,d\alpha)}=(P_\Lambda qP_\Lambda T_\Lambda u|T_\Lambda u).$$

The method of stationary phase and the fact that $B_\Lambda qB_\Lambda$ also satisfies (1.30), (1.31) imply that $P_\Lambda qP_\Lambda=B_{\Lambda,\tilde q}+O(h^\infty)$ in $\mathcal{L}\big(L^2(\Lambda\,;e^{-2H/h}\,d\alpha)\big)$, where $B_{\Lambda,\tilde q}$ has the same form as B_Λ except that $a(\alpha,\beta\,;h)$ is now replaced by $\tilde a(\alpha,\beta\,;h)$, satisfying the same transport equations, and with $\tilde a(\alpha,\alpha\,;h)=\tilde q(\alpha\,;h)a(\alpha,\alpha\,;h)$, $\tilde q(\alpha\,;h)\sim\tilde q_0(\alpha)+h\tilde q_1(\alpha)+\cdots$, $\tilde q_0(\alpha)=q(\alpha)$. Then

$$(1.54)\qquad \operatorname{tr}B_{\Lambda,\tilde q}=h^{-n}\int_\Lambda q(\alpha)a_0(\alpha,\alpha)\,d\alpha+O(h^{1-n}).$$

It is now useful to recall a different way of getting P_Λ (*cf.* [HS]): Since

$$T_\Lambda\colon H(\Lambda)\longrightarrow L^2(\Lambda\,;e^{-2H/h}\,d\alpha),$$

and since the natural dual space of $H(\Lambda)$ for the standard L^2 product is $H(\bar\Lambda)$, we have

$$T_\Lambda^*\colon L^2(\Lambda\,;e^{-2H/h}\,d\alpha)\longrightarrow H(\bar\Lambda),$$

when $H(\Lambda)$ is viewed as a Banach space while $L^2(\Lambda\,;\,e^{-2H/h}\,d\alpha)$ is viewed as a Hilbert space. Then $T_\Lambda^* T_\Lambda : H(\Lambda) \longrightarrow H(\bar{\Lambda})$ turns out to be a kind of elliptic Fourior (in short for Fourier integral operator) with inverse $(T_\Lambda^* T_\Lambda)^{-1} : H(\bar{\Lambda}) \longrightarrow H(\Lambda)$. Then we get,

$$P_\Lambda = T_\Lambda (T_\Lambda^* T_\Lambda)^{-1} T_\Lambda^*. \tag{1.55}$$

In the special case when $\Lambda = \Lambda_0 = T^*X$, then $T_\Lambda^* T_\Lambda$ and its inverse are pseudors and we may write $B_{\Lambda_0} = T_{\Lambda_0} U^2 T_{\Lambda_0}^*$, where U is an elliptic pseudor with $U^2 = (T_\Lambda^* T_\Lambda)^{-1}$. The Toeplitzor $B_{\Lambda,\hat{q}}$ is then well approximated mod $O(h^\infty)$ in trace class, by $T_{\Lambda_0} U \hat{q} U T_{\Lambda_0}^*$ where $\hat{q}$ is a h-pseudor with leading symbol q. Now

$$\begin{gathered}\operatorname{tr}(T_{\Lambda_0} U \hat{q} U T_{\Lambda_0}^*) = \operatorname{tr}(U T_{\Lambda_0}^* T_{\Lambda_0} U \hat{q}) = \operatorname{tr}(U U^{-2} U \hat{q}) = \\ \operatorname{tr} \hat{q} = \frac{1}{(2\pi h)^n} \int \hat{q}(\alpha)\,d\alpha + O(h^{1-n}) = \frac{1}{(2\pi h)^n} \int q(\alpha)\,d\alpha + O(h^{1-n}).\end{gathered}$$

Combining this with (1.54), we see that

$$a_0(\alpha,\alpha) = \frac{1}{(2\pi)^n}, \quad \text{when } \Lambda = T^*X. \tag{1.56}$$

If $\Lambda = \Lambda_t$ depends smoothly on a parameter t, with $\Lambda_0 = T^*X$, then for the corresponding Bergman amplitude, we get

$$a_{0,t}(\alpha,\alpha) = \frac{1}{(2\pi)^n} + O(t). \tag{1.57}$$

Now let $q \in C_0^\infty(\Lambda\,;\,[0,1])$ be equal to 1 on $K \subset \tilde{K} \subset\subset \Lambda$ and have support in $\tilde{K}$ with $\operatorname{vol}(\tilde{K} \setminus K) \leq \epsilon$. Put $Q = P_\Lambda q P_\Lambda$. Then $Q^2 = P_\Lambda q^2 P_\Lambda - \tilde{R}$, where $\operatorname{tr} \tilde{R} = O_\epsilon(h^{1-n})$. In view of the special properties of q, we then see that $Q^2 = Q - R$, where $\operatorname{tr} R = \big(O_\epsilon(1)h + O(\epsilon)\big)h^{-n}$. Now $0 \leq Q \leq 1$ in the sense of selfadjoint operators, so $R = Q - Q^2 \geq 0$. Let $\lambda_1, \lambda_2, \ldots$ be the decreasing sequence of non-vanishing eigenvalues of Q, so that $0 \leq \lambda_j \leq 1$. Since $\operatorname{tr} R = \big(O_\epsilon(1)h + O(\epsilon)\big)h^{-n}$, we get

$$\sum \lambda_j - \lambda_j^2 = \big(O_\epsilon(h) + O(\epsilon)\big)h^{-n}. \tag{1.58}$$

The graph of the function $f(\lambda) = \lambda - \lambda^2$ is a vertical parabola crossing the horizontal axis at (0,0), (1,0) and with a strictly positive maximum at $\lambda = 1/2$. It follows from (1.58) that

$$\#\{\lambda_j\,;\,\sqrt{\epsilon} \leq \lambda_j \leq 1 - \sqrt{\epsilon}\} = \Big(\frac{O_\epsilon(h)}{\sqrt{\epsilon}} + O(\sqrt{\epsilon})\Big)h^{-n}.$$

Consequently, using (1.54) and the fact that $\#\{\lambda_j \geq 1 - \sqrt{\epsilon}\} \leq \frac{1}{1-\sqrt{\epsilon}} \operatorname{tr}(Q)$:

$$\#\{\lambda_j\,;\,\lambda_j \geq \sqrt{\epsilon}\} \leq \frac{1}{(2\pi h)^n}\Big(\int_K a_0(\alpha,\alpha)\,d\alpha + O(\sqrt{\epsilon})\Big), \quad h \leq h(\epsilon). \tag{1.59}$$

From the spectral decomposition of Q, it follows that

$$Q \leq K + R, \quad \|R\| \leq \sqrt{\epsilon}, \tag{1.60}$$

where K is of rank

$$\leq \frac{1}{(2\pi h)^n}\Big(\int_K a_0(\alpha,\alpha)\,d\alpha + O(\sqrt{\epsilon})\Big), \quad h \leq h(\epsilon). \tag{1.61}$$

Actually, below we shall need a refinement of this with an h-dependent symbol q (of uniformly compact support) satisfying $\partial^k q = O_\epsilon(h^{-k/3})$ and with $\mathrm{vol}\big(K + B(0,h^{1/3})\big) = O(h^{1/3})$, $\mathrm{vol}(\tilde{K} \setminus K) = O(\epsilon h^{1/3})$. Then stationary phase still works and we get $Q^2 = P_\Lambda q^2 P_\Lambda - \tilde{R}$, where

$$\mathrm{tr}\,\tilde{R} = O_\epsilon(1)h^{1-2/3}h^{-n+1/3} = O_\epsilon(1)h^{1/3}h^{-n+1/3},$$

so $Q^2 = Q - R$ with $\mathrm{tr}(R) = \big(O_\epsilon(1)h^{1/3} + O(\epsilon)\big)h^{-n+1/3}$ (actually for $0 < h \leq h(\epsilon) > 0$). Letting again $\lambda_1, \lambda_2, \ldots \in]0,1]$ be the decreasing sequence of non-vanishing eigenvalues of Q, we get

$$\sum \lambda_j - \lambda_j^2 = O(1)(\epsilon h^{\frac{1}{3}-n}), \quad h \leq h(\epsilon), \tag{1.62}$$

and

$$\#\{\lambda_j\,;\, \sqrt{\epsilon} \leq \lambda_j \leq 1 - \sqrt{\epsilon}\} = O(1)\sqrt{\epsilon}h^{\frac{1}{3}-n}, \quad h \leq h(\epsilon). \tag{1.63}$$

We then have (1.60) with

$$\mathrm{rank}(K) \leq \frac{1}{(2\pi h)^n}\int_K a_0(\alpha,\alpha)\,d\alpha + O(1)\sqrt{\epsilon}h^{\frac{1}{3}-n}. \tag{1.64}$$

2. **Transformation of the operator.** Let $O \subset \mathbf{R}^n$ be strictly convex, $X = \partial O$ analytic. We shall perform the same scaling up to the boundary as in [SZ2] and in addition, we shall apply a global FBI-transform tangentially to the boundary, when we are near the boundary.

To start with, we need some further geometric preliminaries, valid for arbitrary compact analytic manifolds X. Let $G \in C_0^\infty(T^*X\,;\,\mathbf{R})$ and choose a real valued extension to $C_0^\infty(\widetilde{T^*X})$, that we also denote by G. One possibility is to take $G = \Re\tilde{G}$, where $\tilde{G}$ is an almost analytic extension of G. In that case, we have

$$dG(\rho)_{|JT_\rho(T^*X)} = 0, \tag{2.1}$$

for every $\rho \in T^*X$. In general, we assume (2.1). Here $J\colon T_\rho(\widetilde{T^*X}) \longrightarrow T_\rho(\widetilde{T^*X})$ is the map induced by "multiplication by i". In general, if f is a function on $\widetilde{T^*X}$ and H_f^σ, $H_{\Re f}^{\Re\sigma}$, $H_{\Im f}^{\Re\sigma}$, $H_{\Re f}^{\Im\sigma}$, $H_{\Im f}^{\Im\sigma}$, denote the various Hamilton fields that can be formed with respect to σ, $\Re\sigma$, $\Im\sigma$, then at any point where df exists and is $\mathbf{C}$-linear:

$$H_f^\sigma = H_{\Re f}^{\Re\sigma} = H_{\Im f}^{\Im\sigma}, \tag{2.2}$$

$$JH_f^\sigma = H_{if} = -H_{\Im f}^{\Re\sigma} = H_{\Re f}^{\Im\sigma}. \tag{2.3}$$

Here H_f^σ is considered as real vector. See [S2] for more details.

With G as above (satisfying (2.1)), we put for $t \in \mathbf{R}$, $|t|$ small:

$$\Lambda_t = \exp(tH_G^{\Im\sigma})(\Lambda_0), \quad \Lambda_0 = T^*X. \tag{2.4}$$

Then, as we saw in Section 1, Λ_t is an I-Lagrangian manifold such that $-\Im(\xi \cdot dx_{|\Lambda_t})$ admits a global primitive $H = H_t$ and which coincides with Λ_0 outside a compact set. Let p be a holomorphic function on $\widetilde{T^*X}$. For $\rho \in \Lambda_0$, we get,

$$p\big(\exp tH_G^{\Im\sigma}(\rho)\big) = p(\rho) + t\langle dp(\rho), H_G^{\Im\sigma}(\rho)\rangle + O(t^2).$$

Here

$$\begin{aligned}\langle dp, H_G^{\Im\sigma}\rangle &= \langle d\Re p, H_G^{\Im\sigma}\rangle + i\langle d\Im p, H_G^{\Im\sigma}\rangle \\ &= -\langle H_{\Re p}^{\Im\sigma}, dG\rangle - i\langle H_{\Im p}^{\Im\sigma}, dG\rangle = -\langle J(H_p^\sigma), dG\rangle - i\langle H_p^\sigma, dG\rangle.\end{aligned}$$

If p is real-valued, then $\langle J(H_p^\sigma), dG\rangle$ vanishes at real points (by (2.1)) and we get for $\rho \in \Lambda_0$:

$$p\big(\exp tH_G^{\Im\sigma}(\rho)\big) = p(\rho) - it\langle H_p^\sigma, dG\rangle(\rho) + O(t^2). \tag{2.5}$$

We now return to the boundary problem case and work in geodesic coordinates $x = (x', x_n)$, $x' \in X = \partial O$, $x_n \geq 0$ (*cf.* [SZ2], Section 2). Let $t(x_n)$ be a smooth non-increasing function with support close to $x_n = 0$, equal to $h^{2/3}$ in a small neighborhood of that point, and such that $h^{-2/3}t(x_n)$ is independent of h. Choose G as above and let H_t be the corresponding primitive for $-\Im(\xi' \cdot dx'_{|\Lambda_t})$, chosen according to the recipe (1.17). For u with support close to the boundary, we define the norm:

$$|||u|||^2 = \int_0^\infty \|T_{\Lambda_{t(x_n)}} u(\cdot, x_n)\|^2_{L^2(\Lambda_{t(x_n)}, e^{-2H_{t(x_n)}/h}\, d\alpha)}\, dx_n = \int_0^\infty \|u(\cdot, x_n)\|^2_{H(\Lambda_{t(x_n)})} dx_n. \tag{2.6}$$

The function $T_{\Lambda_{t(x_n)}} u$ on $\{(\alpha, x_n)\ ;\ \alpha \in \Lambda_{t(x_n)}\}$ will be considered as a function on $\Lambda_0 \times [0, \infty[$, by means of the parametrization,

$$\Lambda_0 \ni \alpha_0 \longmapsto \alpha_{t(x_n)} = \exp\big(t(x_n)H_G^{\Im\sigma}\big)(\alpha_0). \tag{2.7}$$

Using that $T_{\Lambda_t} u = Tu_{|\Lambda_t}$ with T independent of t, we then obtain,

$$\begin{aligned} hD_{x_n}\big(T_{\Lambda_{t(x_n)}} u(\cdot, x_n)\big)(\alpha_{t(x_n)}) &= T_{\Lambda_{t(x_n)}}(hD_{x_n} u)(\alpha_t) \\ &\quad + \frac{\partial t}{\partial x_n}\frac{\partial \alpha_t}{\partial t} \cdot hD_\alpha(T)u(\cdot, x_n)(\alpha_t), \end{aligned} \tag{2.8}$$

where (as well in the following), we sometimes write t instead of $t(x_n)$. Denote the last term by $S^{(1)}_{\Lambda_{t(x_n)}} u$. $\frac{\partial \alpha_t}{\partial t}$ has compact support in α and $\frac{\partial t}{x_n}$ is $O(h^{2/3})$ and has compact support in x_n, away from the boundary. It follows as in Section 1 that

$$\|S^{(1)}_{\Lambda_{t(x_n)}} u\|_{L^2(\Lambda_{t(x_n)}, e^{-2H_{t(x_n)}/h}\, d\alpha)} = O(h^{2/3})\|u(\cdot, x_n)\|_{H(\Lambda_{t(x_n)})}. \tag{2.9}$$

We rewrite (2.8) as

$$T_{\Lambda_{t(x_n)}}(hD_{x_n} u) = hD_{x_n}(T_{\Lambda_{t(x_n)}} u) - S^{(1)}_{\Lambda_{t(x_n)}} u, \tag{2.10}$$

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and recall that our transformed functions are considered on $\Lambda_0 \times [0,\infty[$ by means of (2.7).

Differentiating (2.8) once more, we get

$$(2.11)\qquad (hD_{x_n})^2(T_{\Lambda_{t(x_n)}}u) = T_{\Lambda_{t(x_n)}}\big((hD_{x_n})^2u\big) + 2S^{(1)}_{\Lambda_{t(x_n)}}(hD_{x_n}u) + S^{(2)}_{\Lambda_{t(x_n)}}u,$$

where $S^{(2)}$ satisfies (2.9) with $h^{2/3}$ replaced by $h^{4/3}$, and has the same support properties as $S^{(1)}$, and we rewrite this as

$$(2.12)\qquad T_{\Lambda_{t(x_n)}}\big((hD_{x_n})^2u\big) = (hD_{x_n})^2(T_{\Lambda_{t(x_n)}}u) - 2S^{(1)}_{\Lambda_{t(x_n)}}(hD_{x_n}u) - S^{(2)}_{\Lambda_{t(x_n)}}u.$$

Let $P = \sum_{|k|\leq 2} a_k(x)(hD_x)^k$ be a second order h-diffor and consider for u with support close to $x_n = 0$:

$$(2.13)\qquad |||Pu|||^2 = \int_0^\infty \|T_{\Lambda_{t(x_n)}}Pu\|^2_{L^2(\Lambda_{t(x_n)},e^{-2H_{t(x_n)}/h}\,d\alpha)}\,dx_n.$$

From the results of Section 1 and the above discussion about transforms of $hD_{x_n}u$ and $(hD_{x_n})^2u$, we get:

(2.14)

$$\begin{aligned}
&\|T_{\Lambda_{t(x_n)}}Pu\|^2_{L^2(\Lambda_{t(x_n)};e^{-2H_t/h}\,d\alpha)}\\
&\quad= \|P(x,\xi',hD_{x_n})T_{\Lambda_{t(x_n)}}u\|^2_{L^2(\cdots)}\\
&\qquad+ O(h)\sum_{j+k\leq 2}\|\langle\xi'\rangle^j(hD_{x_n})^kT_{\Lambda_{t(x_n)}}u\|^2_{L^2(\cdots)}\\
&\qquad+ O(h^{2/3})\sum_{j\leq 2,k\leq 2,j+k\leq 3}\|(hD_{x_n})^jT_{\Lambda_{t(x_n)}}u\|_{L^2(\cdots)}\,\|(hD_{x_n})^kT_{\Lambda_{t(x_n)}}u\|_{L^2(\cdots)},
\end{aligned}$$

where the last term has to be present only on the support of $\frac{\partial t}{\partial x_n}$. In fact,

(2.15)

$$\begin{aligned}
&2\Re(T_{\Lambda_{t(x_n)}}a_\alpha(hD)^\alpha u\,|\,T_{\Lambda_{t(x_n)}}a_\beta(hD)^\beta u)_{L^2(\Lambda_{t(x_n)},\dots)}\\
&\quad= 2\Re(a_\alpha(x',x_n)\xi'^{\alpha'}T_{\Lambda_{t(x_n)}}(hD_{x_n})^{\alpha_n}u\,|\,(a_\beta(x',x_n)\xi'^{\beta'}T_{\Lambda_{t(x_n)}}(hD_{x_n})^{\beta_n}u)_{L^2(\Lambda\dots,\dots)}\\
&\qquad+ O(h)\|\langle\xi'\rangle^{|\alpha'|}T_{\Lambda_{t(x_n)}}(hD_{x_n})^{\alpha_n}u\|_{L^2(\dots)}\|\langle\xi'\rangle^{|\beta'|}T_{\Lambda_{t(x_n)}}(hD_{x_n})^{\beta_n}u\|_{L^2(\dots)}
\end{aligned}$$

Here we use (2.10), (2.12) which imply that for $k \leq 2$:

$$\begin{aligned}
&\|\langle\xi'\rangle^jT_{\Lambda_{t(x_n)}}(hD_{x_n})^ku\|_{L^2(\cdots)}\\
&\quad= \|\langle\xi'\rangle^j(hD_{x_n})^kT_{\Lambda_{t(x_n)}}u\|_{L^2(\cdots)} + \sum_{\nu\leq k-1}O(h^{\frac{2}{3}(k-\nu)})\|T_{\Lambda_{t(x_n)}}(hD_{x_n})^\nu u\|_{L^2(\cdots)},
\end{aligned}$$

and using this identity once more on its own last term, we see that the last sum here can be replaced by

$$\sum_{\nu\leq k-1}O(h^{\frac{2}{3}(k-\nu)})\|(hD_{x_n})^\nu T_{\Lambda_{t(x_n)}}u\|_{L^2(\cdots)}.$$

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using this in (2.15), we get

(2.16)

$$
\begin{aligned}
2\Re(T_{\Lambda_{t(x_n)}} & a_\alpha(hD)^\alpha u | T_{\Lambda_{t(x_n)}} a_\beta(hD)^\beta u)_{L^2(\cdots)} \\
&= 2\Re\big(a_\alpha(x',x_n)\xi'^{\alpha'}(hD_{x_n})^{\alpha_n} T_{\Lambda_{t(x_n)}} u | a_\beta(x',x_n)\xi'^{\beta'}(hD_{x_n})^{\beta_n} T_{\Lambda_{t(x_n)}} u\big) \\
&\quad + \sum_{\substack{j\leq\alpha_n,k\leq\beta_n,\\ j+k\leq\alpha_n+\beta_n-1}} O(h^{\frac{2}{3}(\alpha_n+\beta_n-j-k)}) \|(hD_{x_n})^j T_{\Lambda_{t(x_n)}} u\| \, \|(hD_{x_n})^k T_{\Lambda_{t(x_n)}} u\| \\
&\quad + O(h)\Big(\|\langle\xi'\rangle^{|\alpha'|}(hD_{x_n})^{\alpha_n} T_{\Lambda_{t(x_n)}} u\| + \sum_{j<\alpha_n} O(h^{\frac{2}{3}(\alpha_n-j)}) \|(hD_{x_n})^j T_{\Lambda_{t(x_n)}} u\|\Big) \\
&\quad \times \Big(\|\langle\xi'\rangle^{|\beta'|}(hD_{x_n})^{\beta_n} T_{\Lambda_{t(x_n)}} u\| + \sum_{k<\beta_n} O(h^{\frac{2}{3}(\beta_n-j)}) \|(hD_{x_n})^k T_{\Lambda_{t(x_n)}} u\|\Big),
\end{aligned}
$$

which can also be written as,

$$
\begin{aligned}
2\Re\big(a_\alpha(x',x_n)\xi'^{\alpha'} & (hD_{x_n})^{\alpha_n} T_{\Lambda_{t(x_n)}} u | a_\beta(x',x_n)\xi'^{\beta'}(hD_{x_n})^{\beta_n} T_{\Lambda_{t(x_n)}} u\big) \\
&+ O(h) \sum_{j\leq\alpha_n,k\leq\beta_n} \|\langle\xi'\rangle^{|\alpha'|}(hD_{x_n})^j T_{\Lambda_{t(x_n)}} u\| \, \|\langle\xi'\rangle^{|\beta'|}(hD_{x_n})^k T_{\Lambda_{t(x_n)}} u\| \\
&+ O(h^{2/3}) \sum_{\substack{j\leq\alpha_n,k\leq\beta_n,\\ j+k\leq\alpha_n+\beta_n-1}} \|(hD_{x_n})^j T_{\Lambda_{t(x_n)}} u\| \, \|(hD_{x_n})^k \cdots\|.
\end{aligned}
$$

Here the last term, as well as all other terms containing powers of $h^{2/3}$ can be suppressed outside the support of $\frac{\partial t}{\partial x_n}$. From this (2.14) follows.

Writing

$$\Lambda_{t(x_n)} \ni (x',\xi') = \exp\big(t(x_n)H_G\big)(y',\eta'),$$

we introduce the Jacobian:

$$J = \frac{dx'd\xi'}{dy'd\eta'},$$

so that $J-1 = O(h^{2/3})$, and similarly for all derivatives of $J-1$. Clearly, $\frac{\partial J}{\partial x_n}$ vanishes outside the support of $\frac{\partial t}{\partial x_n}$. In (2.14), we want to make a modification, namely in the RHS, we want to express everything in terms of operators acting on $J^{1/2}e^{-H_t/h}T_{\Lambda_t}u$, $(t = t(x_n))$ and then take the corresponding ordinary L^2 norm over Λ_0 for the standard measure $dy'd\eta'$. The only problem (essentially similar to one already treated) is then the action of hD_{x_n}. Continuing to use the parametrization (2.7), (so that $\frac{\partial}{\partial x_n}$ denotes the partial derivative for the coordinates (y',η',x_n)) we notice that (with $t = t(x_n)$):

$$J^{1/2}e^{-H_t/h}hD_{x_n}(T_{\Lambda_t}u) = (J^{1/2}e^{-H_t/h}hD_{x_n}J^{-1/2}e^{H_t/h})(J^{1/2}e^{-H_t/h}T_{\Lambda_t}u),$$

and that

$$J^{1/2}e^{-H_t/h}hD_{x_n}J^{-1/2}e^{H_t/h} = hD_{x_n} + S^{(3)}, \tag{2.17}$$

where $S^{(3)}$ and all its derivatives are $O(h^{2/3})$. Moreover $S^{(3)}$ has uniformly compact support in (y',η') and vanishes outside the support of $\frac{\partial t}{\partial x_n}$. It is then clear that (2.14) can be

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rewritten as

(2.18)

$$\|T_{\Lambda_{t(x_n)}}Pu\|^2_{L^2(\Lambda_{t(x_n)},e^{-2H_{t(x_n)}/h}\,dx'\,d\xi')}$$

$$= \|P(x,\xi',hD_{x_n})(J^{1/2}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u)\|^2_{L^2(\Lambda_0,dy'd\eta')}$$
$$+ O(h)\sum_{j+k\le 2}\|\langle\xi'\rangle^j(hD_{x_n})^kJ^{1/2}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u\|^2_{L^2(\Lambda_0,dy'd\eta')}$$
$$+ O(h^{2/3})\sum_{\substack{j\le 2,k\le 2,\\ j+k\le 3}}\|(hD_{x_n})^j(J^{1/2}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u)\|_{L^2}\,\|(hD_{x_n})^k(\mathrm{idem}\cdots)\|_{L^2},$$

where the last term can be suppressed for x_n outside the support of $\frac{\partial t}{\partial x_n}$.

3. **A priori estimates near the boundary.** Let $O\subset\mathbf{R}^n$, $n\ge 2$, be bounded, strictly convex with smooth boundary. In [SZ2] we constructed and injective smooth map with injective differential, $\gamma\colon\mathbf{R}^n\setminus O\longrightarrow\mathbf{C}^n$, such that:

If $x=x'+x_nn(x')$, is in a neighborhood of ∂O, where $x'\in\partial O$, $x_n\ge 0$, and $n(x')$ is the exterior unit normal at x', then $\gamma(x)=x'+e^{i\pi/3}x_nn(x')$.

If $|x|\gg 1$, then $\gamma(x)=(1+i\theta)x$, where $\theta>0$ is small.

The image $\Gamma=\gamma(\mathbf{R}^n\setminus O)$ is totally real and the scaled operator $-\Delta_{|\Gamma}$ has a principal symbol, whose values avoid some conic neighborhood of $]0,\infty[$, when we restrict the attention to points away from some given neighborhood of ∂O.

The resonances of $-\Delta$ in a conic neighborhood of $[0,\infty[$ are then the values k_j, where k_j^2 is an eigenvalue of $-\Delta_{|\Gamma}$. (The multiplicities can also be properly identified.)

From now on, we identify Γ with $\mathbf{R}^n\setminus O$, by means of γ. We recall from [SZ2] that the scaled operator near the boundary is of the form,

$$P\underset{\mathrm{def}}{=}-h^2\Delta_{|\Gamma}=e^{-\frac{2\pi i}{3}}\big((hD_{x_n})^2+2x_nQ(x',hD_{x'})\big)+R(x',hD_{x'})$$
$$+O\big(x_n^2(hD_{x'})^2\big)+O(h)hD_x+O(h^2), \tag{3.1}$$

where all the diffors have analytic coefficients. Here Q and R are positive elliptic operators of degree 2, homogeneous in the sense that $Q(x',hD_{x'})=h^2Q(x',D_{x'})$ and similarly for R. We have $R=-h^2\Delta_{|\partial O}$ and the ellipticity of Q results from the strict convexity of the boundary. Here Γ is the deformed contour, which in the following will be identified with $\mathbf{R}^n\setminus O$.

From (2.5) it follows that the operator $P(x,\xi',hD_{x_n}\,;\,h)$ appearing in (2.18) can be represented in the following form, where we write $(x',\xi')=\exp\big(t(x_n)H_G^{\Im\sigma}\big)(y',\eta')$:

$$P(x,\xi',hD_{x_n}\,;\,h)=e^{-2\pi i/3}\big((hD_{x_n})^2+2x_nQ(y',\eta')\big)+R(y',\eta')$$
$$-it(x_n)H_RG+O(x_n^2\langle\eta'\rangle^2)+O(h\langle\eta'\rangle)+O(h)hD_{x_n}$$
$$+O(x_nh^{2/3}+O(h^{4/3}). \tag{3.2}$$

Here the last two O terms can be absorbed into the first two, when applying to a function and taking L^2 norms. We do not want to consider the x_n dependence in $t(x_n)$, so we write this as

$$
\begin{aligned}
(3.3)\qquad P(x,\xi',hD_{x_n};h) = e^{-\frac{2\pi i}{3}}\left((hD_{x_n})^2 + 2x_nQ(y',\eta')\right) + R(y',\eta') \\
- ih^{2/3}H_RG(y',\eta') + O(x_n^2\langle\eta'\rangle^2 + h\langle\eta'\rangle) + O(h)hD_{x_n} \\
+ O(x_nh^{2/3}) + O(h^{4/3}).
\end{aligned}
$$

The third term of the RHS represents the effect of the FBI-distortion. We decompose it into a real term and one of argument $-2\pi/3$:
(3.4)

$$
\begin{aligned}
P(x,\xi',hD_{x_n};h) = e^{-\frac{2\pi i}{3}}\left((hD_{x_n})^2 + 2x_nQ(y',\eta') + \frac{h^{2/3}}{\cos\frac{\pi}{6}}H_RG(y',\eta')\right) \\
+ \left(R(y',\eta') + \left(\tan\frac{\pi}{6}\right)h^{2/3}H_RG(y',\eta')\right) + O(x_n^2\langle\eta'\rangle^2 + h\langle\eta'\rangle) \\
+ O(h)hD_{x_n} + O(x_nh^{2/3}) + O(h^{4/3}).
\end{aligned}
$$

Let $-\zeta_j, j = 1, 2, \ldots$ with $0 < \zeta_1 < \zeta_2 < \cdots$ be the zeros of the Airy function, and put

$$
(3.5)\qquad \zeta_j(y',\eta') = \left(2Q(y',\eta')\right)^{\frac{2}{3}}\zeta_j.
$$

Then the eigenvalues of

$$
(hD_{x_n})^2 + 2x_nQ(y',\eta') + \frac{h^{2/3}}{\cos\frac{\pi}{6}}H_RG(y',\eta')
$$

on $[0,\infty[$ with Dirichlet boundary condition at 0, are of the form $h^{2/3}\tilde{\zeta}_j(y',\eta')$, where

$$
(3.6)\qquad \tilde{\zeta}_j(y',\eta') = \zeta_j(y',\eta') + \frac{1}{\cos\frac{\pi}{6}}H_RG(y',\eta').
$$

When passing to $(P(x,\xi',hD_{x_n}\ ;\ h) - \omega_0)$ as in Section 5 in [SZ2], we may consider these modifications as a small modification in ω_0. More precisely, for our new operator P, we have the estimate (5.2) of [SZ2] (when $\Im\omega_0 > 0$, $\Re\omega_0$ close to $R(y',\eta')$), provided that we replace ω_0 to the right by $\tilde{\omega}_0 = \omega_0 + ih^{2/3}H_RG(y',\eta')$. Then, if we choose $\tilde{\mu} = \zeta_2(y',\eta')h^{2/3}$ in that lemma, we get for $v \in C_0^\infty([0,\frac{1}{CL}[)$, $v(0) = 0$:
(3.7)

$$
\begin{aligned}
&\left\|\left(P(x',t,\xi',hD_t) - \omega_0\right)v\right\|^2 \\
&\quad\geq \left(|\tilde{\omega}_0 - R(y',\eta') - e^{-\frac{2\pi i}{3}}\zeta_2(y',\eta')h^{\frac{2}{3}}|^2 - O(1)\sqrt{L}h\right)\|v\|^2 \\
&\qquad - 2\Re\left(e^{-\frac{2\pi i}{3}}\left(R(y',\eta') - \tilde{\omega}_0\right)\right)\left(\zeta_2(y',\eta') - \zeta_1(y',\eta')\right)h^{\frac{2}{3}}|\gamma(y',\eta')v|^2 \\
&\qquad + \frac{1}{2}\|\left((hD_t)^2 + 2tQ(y',\eta')\right)v\|^2 + \frac{L}{2}\|tv\|^2,
\end{aligned}
$$

where (*cf.* [SZ2], Section 6):

(3.8)
$$\gamma(y',\eta')v = \int_0^\infty v(x_n)\overline{e_{y',\eta'}(x_n)}\,dx_n,$$

with $e_{y',\eta'}$ being the first normalized (Dirichlet) eigenfunction of $(hD_t)^2 + 2tQ(y',\eta')$. In (3.7), we can get rid of $\tilde{\omega}_0$, by using $\tilde{R}, \tilde{\zeta}_j, j = 1,2$:

(3.9)
$$\begin{aligned}
&\left\|\left(P(x',t,\xi',hD_t) - \omega_0\right)v\right\|^2 \\
&\quad \geq \left(|\omega_0 - \tilde{R}(y',\eta') - e^{-\frac{2\pi i}{3}}\tilde{\zeta}_2(y',\eta')h^{\frac{2}{3}}|^2 - O(1)\sqrt{L}h\right)\|v\|^2 \\
&\qquad - 2\Re\left(e^{-\frac{2\pi i}{3}}\left(\tilde{R}(y',\eta') - \bar{\omega}_0\right)\right)\left(\tilde{\zeta}_2(y',\eta') - \tilde{\zeta}_1(y',\eta')\right)h^{\frac{2}{3}}|\gamma(y',\eta')v|^2 \\
&\qquad + \frac{1}{2}\left\|\left((hD_t)^2 + 2tQ(y',\eta')\right)v\right\|^2 + \frac{L}{2}\|tv\|^2,
\end{aligned}$$

where we put $\tilde{R}(y',\eta') = R(y',\eta') + (\tan\frac{\pi}{6})h^{2/3}H_R G(y',\eta')$, and where we also notice that a term $O(h^{4/3})|\gamma(y',\eta')v|^2$, which appears, can be absorbed into $-O(1)\sqrt{L}h\|v\|^2$.

We recall that

$$\begin{aligned}
|\omega_0 - \tilde{R} - e^{-\frac{2\pi i}{3}}\tilde{\zeta}_2 h^{\frac{2}{3}}|^2 &= |\omega_0 - \tilde{R}|^2 + 2\Re\left(e^{-\frac{2\pi i}{3}}\tilde{\zeta}_2(\tilde{R} - \bar{\omega}_0)\right)h^{\frac{2}{3}} + O(h^{\frac{4}{3}}) \\
&= (\Re\omega_0 - \tilde{R})^2 + r_0^2 + 2\tilde{\zeta}_2\Re\left(e^{\frac{-2\pi i}{3}}(\tilde{R} - \bar{\omega}_0)\right)h^{\frac{2}{3}} + O(h^{\frac{4}{3}}),
\end{aligned}$$

where $r_0 = \Im\omega_0$, and similarly with $\tilde{\zeta}_2$ replaced by $\tilde{\zeta}_1$. (3.9) can therefore be rewritten as:

(3.10)
$$\begin{aligned}
&\left\|\left(P(x',t,\xi',hD_t) - \omega_0\right)v\right\|^2 \\
&\quad \geq \left(|\Re\omega_0 - \tilde{R}|^2 + r_0^2 + 2\tilde{\zeta}_2\Re\left(e^{-\frac{2\pi i}{3}}(\tilde{R} - \bar{\omega}_0)\right)h^{\frac{2}{3}} - O(1)\sqrt{L}h\right)\|v\|^2 \\
&\qquad - 2(\tilde{\zeta}_2 - \tilde{\zeta}_1)h^{\frac{2}{3}}\Re\left(e^{-\frac{2\pi i}{3}}(\tilde{R} - \bar{\omega}_0)\right)|\gamma(y',\eta')v|^2 \\
&\qquad + \frac{1}{2}\left\|\left((hD_t)^2 + 2tQ(y',\eta')\right)v\right\|^2 + \frac{L}{2}\|tv\|^2 \\
&\quad = \left(|\Re\omega_0 - \tilde{R}|^2 + r_0^2 + 2h^{\frac{2}{3}}\tilde{\zeta}_2\Re\left(e^{-\frac{2\pi i}{3}}(\tilde{R} - \bar{\omega}_0)\right)\right)\left(\|v\|^2 - |\gamma(y',\eta')v|^2\right) \\
&\qquad + \left(|\Re\omega_0 - \tilde{R}|^2 + r_0^2 + 2h^{\frac{2}{3}}\tilde{\zeta}_1\Re\left(e^{-\frac{2\pi i}{3}}(\tilde{R} - \bar{\omega}_0)\right)\right)|\gamma(y',\eta')v|^2 \\
&\qquad + \frac{1}{2}\left\|\left((hD_t)^2 + 2tQ(y',\eta')\right)v\right\|^2 + \frac{L}{2}\|tv\|^2 - O(1)\sqrt{L}h\|v\|^2.
\end{aligned}$$

Here we use that $\tilde{R} = R + O(h^{2/3})$ and that

$$\begin{aligned}
&\Re\left(e^{-\frac{2\pi i}{3}}(\tilde{R} - \bar{\omega}_0)\right) \\
&\quad = O\left(h^{\frac{2}{3}} + \Re\left(e^{-\frac{2\pi i}{3}}(R - \bar{\omega}_0)\right)\right) = O(h^{\frac{2}{3}}) + O(|R - \Re\omega_0|) + \Re(e^{-\frac{2\pi i}{3}} i r_0)\Big) \\
&\quad = O(h^{\frac{2}{3}}) + O(|R - \Re\omega_0|) + \left(\cos\frac{\pi}{6}\right)r_0,
\end{aligned}$$

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and we get from (3.10):

(3.11)

$$\begin{aligned}\Big\|\big(P(x',t,\xi',hD_t)-\omega_0\big)v\Big\|^2 &\geq \Big(|R-\Re\omega_0|^2+r_0^2+2h^{\frac{2}{3}}\Big(\cos\frac{\pi}{6}\Big)r_0\tilde{\zeta}_2\Big)\big(\|v\|^2-|\gamma(y',\eta')v|^2\big)\\ &+\Big(|R-\Re\omega_0|^2+r_0^2+2h^{\frac{2}{3}}\Big(\cos\frac{\pi}{6}\Big)r_0\tilde{\zeta}_1\Big)|\gamma(y',\eta')v|^2\\ &+\frac{1}{2}\Big\|\big((hD_t)^2+2tQ(y',\eta')\big)v\Big\|^2+\frac{L}{2}\|tv\|^2\\ &-O(1)(\sqrt{L}h+h^{\frac{2}{3}}|R-\Re\omega_0|)\|v\|^2.\end{aligned}$$

Notice here that $|\gamma(y',\eta')v|\leq\|v\|$.

For (y',η') outside any fixed neighborhood of the energy surface $R(y',\eta')=\Re\omega_0$, we obtain from Lemma 5.2 of [SZ2]:

(3.12)

$$\begin{aligned}\Big\|\big(P(x',t,\xi',hD_t)-\omega_0\big)v\Big\|^2 &\geq \Big(r_0+\frac{1}{O(1)}\Big)^2\|v\|^2\\ &+\frac{1}{O(1)}\big(\|(hD_t)^2v\|^2+\langle\eta'\rangle^2\|hD_tv\|^2+\langle\eta'\rangle^4\|v\|^2\big),\end{aligned}$$

when $v\in C_0^\infty([0,1[),\ v(0)=0$.

We now combine the ODE-estimates above with the results of Section 2 (more precisely (2.18)) and obtain for $u\in(H^2\cap H_0^1)(\mathbf{R}^n\setminus O)$ with support within the distance $1/L$ from the boundary:

$$\begin{aligned}&\int_0^\infty\|T_{\Lambda_{t(x_n)}}(P-\omega_0)u\|^2_{L^2(\Lambda_{t(x_n)};e^{-2H_{t(x_n)}/h}\,dx'\,d\xi')}\,dx_n\\ &\geq\iint 1_V(y',\eta')\Big(|R(y',\eta')-\Re\omega_0|^2+r_0^2+2h^{\frac{2}{3}}\Big(\cos\frac{\pi}{6}\Big)r_0\tilde{\zeta}_2(y',\eta')\Big)\\ &\quad\times\Big(\int_0^\infty|J^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dx_n-|\gamma(y',\eta')J^{1/2}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\Big)\,dy'\,d\eta'\\ &+\iint 1_V(y',\eta')\Big(|R(y',\eta')-\Re\omega_0|^2+r_0^2+2h^{\frac{2}{3}}\Big(\cos\frac{\pi}{6}\Big)r_0\tilde{\zeta}_1(y',\eta')\Big)\\ &\quad\times|\gamma(y',\eta')J^{1/2}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dy'\,d\eta'\\ &+\Big(r_0^2+\frac{1}{O_V(1)}\Big)\iint\big(1-1_V(y',\eta')\big)\int_0^\infty|J^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'\\ &+\frac{1}{O(1)}\iiint_0^\infty|(hD_{x_n})^2(J^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u)|^2\,dx_n\,dy'\,d\eta'\\ &+\frac{1}{O(1)}\iint\big(1-1_V(y',\eta')\big)\langle\eta'\rangle^4\int_0^\infty|J^{\frac{1}{2}}e^{-H_{\dots}/h}T_{\dots}u|^2\,dx_n\,dy'\,d\eta'\\ &+\frac{1}{O(1)}\iiint_0^\infty|x_ne^{-H_{t(x_n)}/h}J^{\frac{1}{2}}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'\\ &-O_L(h)\sum_{j+k=2}\iiint_0^\infty|\langle\eta'\rangle^j(hD_{x_n})^kJ^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'\end{aligned}\tag{3.13}$$

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$$
\begin{aligned}
&-\tilde{O}(h^{\frac{2}{3}})\sum_{\substack{j\le 2,k\le 2\\ j+k\le 3}}\Big(\iiint_0^\infty |(hD_{x_n})^j(J^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u)|^2\,dx_n\,dy'\,d\eta'\Big)^{\frac{1}{2}}\\
&\qquad\qquad\times\big(\cdots(hD_{x_n})^k\cdots\big)^{\frac{1}{2}}\\
&-O(h^{\frac{2}{3}})\iiint_0^\infty 1_V(y',\eta')|R(y',\eta')-\Re\omega_0|\,|J^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'.
\end{aligned}
$$

Here V is some arbitrarily small but fixed neighborhood of the energy surface. Moreover in the $\tilde{O}$ term we can restrict the x_n-integrals to the region (away from $x_n = 0$) where $t \neq h^{2/3}$, so this term can clearly be estimated by the 4-th, 5-th and 6-th terms of the RHS. We have also used Lemma 4.3 of [SZ2]. We next try to eliminate as much as possible from the third term from the end. First of all the contribution with $k = 1$ ($j = 1$), can be eliminated by interpolation. Secondly, the contribution with $k = 2$ can be absorbed by the fourth term. It then only remains the contribution with $k = 0$. From this contribution, we can absorb the integral over the complement of V. We then get the slightly simplified version of (3.13):

$$
\begin{aligned}
&\int_0^\infty \|T_{\Lambda_{t(x_n)}}(P-\omega_0)u\|^2_{L^2(\Lambda_{t(x_n)};e^{-2H_{t(x_n)}/h}\,dx'\,d\xi')}\,dx_n\\
&\quad\ge\iint 1_V(y',\eta')\Big(|R(y',\eta')-\Re\omega_0|^2+r_0^2+2h^{\frac{2}{3}}\Big(\cos\frac{\pi}{6}\Big)r_0\tilde{\zeta}_2(y',\eta')\Big)\\
&\qquad\qquad\times\Big(\int_0^\infty |J^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2dx_n-|\gamma(y',\eta')J^{1/2}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\Big)\,dy'\,d\eta'\\
&\qquad+\iint 1_V(y',\eta')\Big(|R(y',\eta')-\Re\omega_0|^2+r_0^2+2h^{\frac{2}{3}}\Big(\cos\frac{\pi}{6}\Big)r_0\tilde{\zeta}_1(y',\eta')\Big)\\
&\qquad\qquad\times|\gamma(y',\eta')J^{1/2}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dy'\,d\eta'\\
&\qquad+\Big(r_0^2+\frac{1}{O_V(1)}\Big)\iint\big(1-1_V(y',\eta')\big)\int_0^\infty |J^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'\\
&\qquad+\frac{1}{O(1)}\iiint_0^\infty |(hD_{x_n})^2(J^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u)|^2dx_ndy'd\eta'\\
&\qquad+\frac{1}{O(1)}\iint\big(1-1_V(y',\eta')\big)\langle\eta'\rangle^4\int_0^\infty |J^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'\\
&\qquad+\frac{1}{O(1)}\iiint_0^\infty |x_ne^{-H_{t(x_n)}/h}J^{\frac{1}{2}}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'\\
&\qquad-O_L(h)\iint 1_V(y',\eta')|J^{\frac{1}{2}}e^{-H_{t(x_n}/h}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'\\
&\qquad-O(h^{\frac{2}{3}})\iiint_0^\infty 1_V(y',\eta')|R(y',\eta')-\Re\omega_0|\,|J^{\frac{1}{2}}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'.
\end{aligned}
\tag{3.14}
$$

In this estimate we may replace $\tilde{\zeta}_j$ by $\zeta_j\circ\Pi$, where $\Pi\colon V\longrightarrow\Sigma$ is a smooth map with $\Pi_{|\Sigma}=\mathrm{id}$. Here $\Sigma=\{(y',\eta')\ ;\ R(y',\eta')=\Re\omega_0\}$. The two last terms of the RHS will be incorporated into the first two terms.

For $\mu\ge 0$ bounded, it will be of interest to estimate from above the volume of the set of all (y',η') such that

$$(3.15)\qquad \begin{aligned}\big(R(y',\eta')-\Re\omega_0\big)^2+r_0^2+2h^{\frac{2}{3}}\Big(\cos\frac{\pi}{6}\Big)r_0(\tilde\zeta_1\circ\Pi)(y',\eta')\\ -O(h)-O(h^{\frac{2}{3}})|R-\Re\omega_0|\le r_0^2+2r_0\mu h^{\frac{2}{3}}.\end{aligned}$$

In other words, we wish to estimate the volume of

$$\big(R(y',\eta')-\Re\omega_0\big)^2-O(h^{\frac{2}{3}})|R(y',\eta')-\Re\omega_0|\le 2r_0h^{\frac{2}{3}}\Big(\mu+O(h^{\frac{1}{3}})-\Big(\cos\frac{\pi}{6}\Big)\tilde\zeta_1\circ\Pi\Big).$$

Here with $t=R(y',\eta')-\Re\omega_0$:

$$t^2-2Ch^{\frac{2}{3}}|t|\ge(1-h^{\frac{1}{3}})t^2-C^2h,$$

so it is enough to estimate the volume of the larger set:

$$(1-h^{\frac{1}{3}})\big(R(y',\eta')-\Re\omega_0\big)^2\le 2r_0h^{\frac{2}{3}}\Big(\mu+O(h^{\frac{1}{3}})-\Big(\cos\frac{\pi}{6}\Big)\tilde\zeta_1\circ\Pi\Big),$$

which is contained in a set of the form

$$\big(R(y',\eta')-\Re\omega_0\big)^2\le 2r_0h^{\frac{2}{3}}(\mu+O(h^{\frac{1}{3}})-(\cos\frac{\pi}{6})\tilde\zeta_1\circ\Pi),$$

or equivalently

$$(3.16)\qquad |R(y',\eta')-\Re\omega_0|\le\sqrt{2r_0}h^{\frac{1}{3}}\Big(\mu+O(h^{\frac{1}{3}})-\Big(\cos\frac{\pi}{6}\Big)\tilde\zeta_1\circ\Pi\Big)_+^{\frac{1}{2}}.$$

Let $S_{R,\Sigma}$ be the Liouville measure on Σ with respect to R, so that $dy'd\eta'=\big(1+O(|R-\Re\omega_0|)\big)\,dRS_{R,\Sigma}\big(d\Pi(y',\eta')\big)$ near Σ, where we parametrize points by $\big(R(y',\eta'),\Pi(y',\eta')\big)\in\mathbf{R}\times\Sigma$. Then the volume of the set (3.16) is of the form

$$(3.17)\qquad 2\big(1+O(h^{\frac{1}{3}})\big)\sqrt{2r_0}h^{\frac{1}{3}}\int_\Sigma\Big(\mu+O(h^{\frac{1}{3}})-\Big(\cos\frac{\pi}{6}\Big)\tilde\zeta_1\Big)_+^{\frac{1}{2}}S_{R,\Sigma}\big(d(y',\eta')\big).$$

In particular, (3.17) is an upper bound for the volume of the set defined by (3.15). For every $\epsilon>0$, we can now find $q\in S^0_{2/3}$ (in the sense that $\partial_{y'}^{\alpha'}\partial_{\eta'}^{\beta'}q=O(h^{-\frac{1}{3}(|\alpha'|+|\beta'|)})$) with values in $[0,1]$, equal to 1 on the set (3.16) and such that the volume of $\operatorname{supp} q$ minus this set is $\le\epsilon h^{1/3}$.

Now choose $0\le\mu\le\tilde\zeta_{2,\min}\cos\frac{\pi}{6}-\frac{1}{O(1)}$, where $\tilde\zeta_{j,\min}=\inf_\Sigma\tilde\zeta_j$. Then the sum of the first two and the last two terms of the RHS of (3.14) can be bounded from below by:

$$
\begin{aligned}
&\iint 1_V(y',\eta')\\
&\quad\Big(|R(y',\eta')-\Re\omega_0|^2+r_0^2+2h^{\frac{2}{3}}\Big(\cos\frac{\pi}{6}\Big)r_0(\tilde{\zeta}_2\circ\Pi)-O(h^{\frac{2}{3}})|R-\Re\omega_0|-O(h)\Big)\\
&\quad\times\Big(\int_0^\infty |J^{\frac12}e^{-H/h}T_\Lambda u|^2\,dx_n-|\gamma(y',\eta')J^{\frac12}e^{-H/h}T_\Lambda u|^2\Big)\,dy'\,d\eta'\\
&\quad+\iint 1_V(y',\eta')\\
&\quad\Big(|R(y',\eta')-\Re\omega_0|^2+r_0^2+2h^{\frac{2}{3}}\Big(\cos\frac{\pi}{6}\Big)r_0(\tilde{\zeta}_1\circ\Pi)-O(h^{\frac{2}{3}})|R-\Re\omega_0|-O(h)\Big)\\
&\quad\times|\gamma(y',\eta')J^{\frac12}e^{-H/h}T_\Lambda u|^2\,dy'\,d\eta'\\
&\geq\iint 1_V(y',\eta')(r_0^2+2h^{\frac23}r_0\mu)\\
&\quad\Big(\int_0^\infty |J^{\frac12}e^{-H/h}T_\Lambda u|^2\,dx_n-|\gamma(y',\eta')J^{\frac12}e^{-H/h}T_\Lambda u|^2\Big)\,dy'\,d\eta'\\
&\quad+\iint 1_V(y',\eta')(r_0^2+2h^{\frac23}r_0\mu)|\gamma J^{\frac12}e^{-H/h}T_\Lambda u|^2\,dy'\,d\eta'\\
&\quad-\iint O(h^{\frac23})q(y',\eta')|\gamma J^{\frac12}e^{-H/h}T_\Lambda u|^2\,dy'\,d\eta'\\
&=\iint 1_V(r_0^2+2h^{\frac23}r_0\mu)\int_0^\infty |J^{\frac12}e^{-H/h}T_\Lambda u|^2\,dx_n\,dy'\,d\eta'\\
&\quad-\iint O(h^{\frac23})q(y',\eta')|\gamma(y',\eta')J^{\frac12}e^{-H/h}T_\Lambda u|^2\,dy'\,d\eta'.
\end{aligned}
$$

Let $\gamma(x',hD_{x'})\colon L^2(\mathbf{R}^n\setminus O)\longrightarrow L^2(\partial O)$ be a suitable realization of the operator valued symbol $\chi(y',\eta')\gamma(y',\eta')$, where $\chi\in C_0^\infty$ has its support near Σ and is equal to 1 near that set. Then (*cf.* Proposition 1.2 and [HS]):

$$
\begin{aligned}
&\|\chi\gamma(y',\eta')J^{\frac12}e^{-H/h}T_\Lambda u-(J^{\frac12}e^{-H/h})_{|x_n=0}T_\Lambda\gamma(x',hD_{x'})u\|_{L^2(\Lambda_0;\,dy'\,d\eta')}\\
&\qquad=O(h^{\frac12})\Big(\iiint_0^\infty |J^{\frac12}e^{-H/h}T_\Lambda u|^2\,dx_n\,dy'\,d\eta'\Big)^{\frac12}.
\end{aligned}
$$

From (3.14) and the estimates above, we get:

(3.18)
$$
\begin{aligned}
&\int_0^\infty \|T_{\Lambda_{t(x_n)}}(P-\omega_0)u\|^2_{L^2(\Lambda_{t(x_n)};e^{-2H_{t(x_n)}/h}\,dx'\,d\xi')}\,dx_n\\
&\quad\geq(r_0^2+2r_0h^{\frac23}\mu)\int_0^\infty \|T_{\Lambda_{t(x_n)}}u\|^2_{L^2(\dots)}\,dx_n\\
&\qquad-O(h^{\frac23})\iint q(y',\eta')|J^{\frac12}_{|x_n=0}e^{-H_{t(0)}/h}T_{\Lambda_{t(0)}}\gamma(x',hD_{x'})u|^2\,dy'\,d\eta'\\
&\qquad+\frac{1}{O(1)}\iiint_0^\infty |(hD_{x_n})^2(J^{\frac12}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u)|^2dx_ndy'd\eta'\\
&\qquad+\frac{1}{O(1)}\iiint_0^\infty\big(1-1_V(y',\eta')\big)\langle\eta'\rangle^4|J^{\frac12}e^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'\\
&\qquad+\frac{1}{O(1)}\iiint_0^\infty |x_ne^{-H_{t(x_n)}/h}T_{\Lambda_{t(x_n)}}u|^2\,dx_n\,dy'\,d\eta'.
\end{aligned}
$$

According to Section 1, in particular (1.64), and (3.17), we have for $0 < h \leq h(\epsilon) > 0$:

$$\begin{aligned}&\iint q(y',\eta')|J^{\frac{1}{2}}_{|x_n=0}e^{-H_{t(0)}/h}T_{\Lambda_{t(0)}}\gamma(x',hD_{x'})u|^2\,dy'\,d\eta'\\ &\leq \sqrt{\epsilon}\iint |J^{\frac{1}{2}}_{|x_n=0}e^{-H_{t(0)}/h}T_{\Lambda_{t(0)}}\gamma(x',hD_{x'})u|^2\,dy'\,d\eta'\\ &\quad+\left(KT_{\Lambda_{t(0)}}\gamma(x',hD'_x)u|T_{\Lambda_{t(0)}}\gamma(x',hD'_x)u\right)_{L^2(e^{-2H_{t(0)}/h}\,dy'\,d\eta')},\end{aligned}\tag{3.19}$$

where K is of rank $\leq$

$$\frac{1+O(h^{\frac{1}{3}})}{(2\pi h)^{n-1}}2\sqrt{2r_0}h^{\frac{1}{3}}\int_\Sigma\left(\mu+O(h^{\frac{1}{3}})-\left(\cos\frac{\pi}{6}\right)\tilde{\zeta}_1\right)_+^{\frac{1}{2}}S_{R,\Sigma}\big(d(y',\eta')\big)+O(1)\sqrt{\epsilon}h^{\frac{1}{3}-n}.\tag{3.20}$$

So far, we have assumed that u is supported near the boundary, and we have to remove that assumption. We notice that if the supports of u and of $t(x_n)$ are disjoint, then the (squared) norm

$$\int_0^\infty \|T_{\Lambda_{t(x_n)}}u\|^2_{L^2(\dots)}\,dx_n$$

is equivalent (uniformly with respect to h) to $\|u\|^2_{L^2}$. Let $\psi(x_n)\in C_0^\infty([0,\infty[\,;[0,1])$ have small support and be equal to 1 near the support of $t(x_n)$. The global norm (equivalent to the standard L^2 norm for every fixed h, but not uniformly with respect to h) is then defined by

$$|||u|||^2=\int_0^\infty \psi(x_n)\|T_{\Lambda_{t(x_n)}}u\|^2_{L^2(\Lambda_{t(x_n)};e^{-2H_{t(x_n)}/h}\,dx'\,d\xi')}\,dx_n+\int\big(1-\psi(x_n)\big)|u(x)|^2\,dx.\tag{3.21}$$

For u supported away from some fixed neighborhood of the boundary, we have (by the very idea of complex scaling):

$$\|(P-\omega_0)u\|^2\geq\left(r_0+\frac{1}{O(1)}\right)^2\|u\|^2+\frac{1}{O(1)}\sum_{|\alpha|\leq 2}\|(hD)^\alpha u\|^2,\tag{3.22}$$

where all norms are the standard ones in L^2. We notice that the sum of the last three terms in (3.18) is also equivalent (uniformly with respect to h) to $\sum_{|\alpha|\leq 2}\|(hD)^\alpha u\|^2$, for u supported away from supp$t(x_n)$. It is easy to absorb the cut-off errors due to $\psi(x_n)$ and get from (3.18), (3.22):

$$\begin{aligned}&|||(P-\omega_0)u|||^2\geq(r_0^2+2\mu h^{\frac{2}{3}}r_0)|||u|||^2\\ &\qquad-O(h^{\frac{2}{3}})\iint q(y',\eta')|J^{\frac{1}{2}}_{|x_n=0}e^{-H_{t(0)}/h}T_{\Lambda_{t(0)}}\gamma(x',hD_{x'})u|^2\,dy'\,d\eta'\end{aligned}\tag{3.23}$$

Here we use (3.19), with K satisfying (3.20) and get for $0<h\leq h(\epsilon)>0$:

$$|||(P-\omega_0)|||^2\geq\big(r_0+(\mu-\sqrt{\epsilon})h^{\frac{2}{3}}\big)^2|||u|||^2-O(h^{\frac{2}{3}})\big(KT_{\Lambda_{t(0)}}\gamma(x',hD_{x'})u|T_{\Lambda_{t(0)}}\gamma(x',hD_{x'})u\big).\tag{3.24}$$

REMARK. When $\mu<\tilde{\zeta}_{1,\min}(\cos\frac{\pi}{6})$, we get by the same proof:

$$|||(P-\omega_0)u|||^2\geq(r_0+\mu h^{\frac{2}{3}})^2|||u|||^2,\tag{3.25}$$

for h small enough.

4. **Estimates on the resonances.** We shall proceed very much as in [SZ2] (and in earlier works cited there), but we shall pay more attention to the choice of some constants. In (3.24) we are free to choose $\epsilon > 0$ arbitrarily small and independent of h. From (3.25), we get

$$\|\!|(P-\omega_0)u\|\!|^2 \geq \left(r_0 + \left(\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} - o(1)\right)h^{\frac{2}{3}}\right)^2 \|\!|u\|\!|^2, \quad h \longrightarrow 0. \tag{4.1}$$

For $r \leq r_0 + (\tilde{\zeta}_{2,\min}\cos\frac{\pi}{6} - \frac{1}{O(1)})h^{\frac{2}{3}}$, we define:

$$N(r) = \#\text{ of eigenvalues of } P-\omega_0 \text{ of modulus } \leq r, \tag{4.2}$$

$$M(r) = \#\text{ of characteristic values of } P-\omega_0 \text{ of modulus } \leq r. \tag{4.3}$$

(4.1) shows that $N(r) = M(r) = 0$ for $r \leq r_0 + \left(\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} - o(1)\right)h^{\frac{2}{3}}$, and (3.24) shows that

$$M\left(r_0 + (\mu - \sqrt{\epsilon})h^{\frac{2}{3}}\right) \leq \operatorname{rank}(K),$$

where rank(K) is bounded as in (3.20). Since we can let $\epsilon \longrightarrow 0$, we get the following uniform estimate, for $0 \leq \mu \leq \tilde{\zeta}_{2,\min}\cos\frac{\pi}{6} - \frac{1}{O(1)}$:

$$M(r_0 + \mu h^{\frac{2}{3}}) \leq V(\mu) + o(1)h^{\frac{1}{3}-(n-1)}, \quad h \longrightarrow 0, \tag{4.4}$$

where

$$V(\mu) = \frac{2\sqrt{2r_0}h^{\frac{1}{3}}}{(2\pi h)^{n-1}} \int_\Sigma \left(\mu - \left(\cos\frac{\pi}{6}\right)\tilde{\zeta}_1\right)_+^{\frac{1}{2}} S_{R,\Sigma}\big(d(y',\eta')\big). \tag{4.5}$$

Put $r_j = r_0 + (\cos\frac{\pi}{6})\tilde{\zeta}_{j,\min}h^{\frac{2}{3}}$ and let $r_1 \leq r \leq R \leq r_2 - \frac{1}{O(1)}h^{\frac{2}{3}}$. If $N(r) \leq M(R)$ we do nothing. If $N(r) > M(R)$, let $\lambda_1, \dots, \lambda_N$, $N > N(r)$, be the eigenvalues of $P-\omega_0$ with $|\lambda_j| \leq |\lambda_{j+1}|$ (counted with their algebraic multiplicity) and let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$ be the first N characteristic values of $P - \omega_0$ *i.e.* eigenvalues of $\sqrt{(P-\omega_0)^*(P-\omega_0)}$.

As noticed above, $\mu_j \geq r_1 - o(1)h^{\frac{2}{3}}$, and also by definition, $\mu_j \geq R$, for $j \geq M(R)+1$. The Weyl inequality,

$$\mu_1 \cdot \cdots \cdot \mu_N \leq |\lambda_1| \cdot \cdots \cdot |\lambda_N|, \tag{4.6}$$

then implies that

$$\left(r_1 - o(1)h^{\frac{2}{3}}\right)^M R^{N-M} \leq r^N, \quad N = N(r),\ M = M(R),$$

i.e. with $\tilde{r}_1 = r_1 - o(1)h^{\frac{2}{3}}$:

$$\left(\frac{R}{r}\right)^N \leq \left(\frac{R}{\tilde{r}_1}\right)^M,$$

or:

$$N(r) \leq \frac{\log(\frac{R}{\tilde{r}_1})}{\log(\frac{R}{r})} M(R). \tag{4.7}$$

It is not clear in general what is the optimal choice of R for a given $r > \tilde{r}_1$, but the situation can be clarified a little bit: Write

$$\frac{\log(\frac{R}{\tilde{r}_1})}{\log(\frac{R}{r})} = C > 1,$$

and assume

$$1 + \frac{1}{O(1)} \leq C \leq O(1). \tag{4.8}$$

Then $R = (\frac{r}{\tilde{r}_1})^{\frac{1}{C-1}} r$, and (4.7) can be rewritten as,

$$N(r) \leq CM\left(\left(\frac{r}{\tilde{r}_1}\right)^{\frac{1}{C-1}} r\right). \tag{4.9}$$

Write $r = r_0 + \mu h^{\frac{2}{3}}$ and recall that $\tilde{r}_1 = r_0 + \left(\tilde{\zeta}_{1,\min}(\cos\frac{\pi}{6}) - o(1)\right)h^{\frac{2}{3}}$. Then

$$\left(\frac{r}{\tilde{r}_1}\right)^{\frac{1}{C-1}} r = r_0 + \left(\mu + \frac{1}{C-1}\left(\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\right) + o(1)\right)h^{\frac{2}{3}},$$

so (4.9) reads,

$$N(r_0 + \mu h^{\frac{2}{3}}) \leq CM\left(r_0 + \left(\mu + \frac{1}{C-1}\left(\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\right) + o(1)\right)h^{\frac{2}{3}}\right), \tag{4.10}$$

uniformly in μ, C, as long as

$$\begin{gathered} \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} \leq \mu \leq \tilde{\zeta}_{2,\min}\cos\frac{\pi}{6} - \frac{1}{O(1)}, \quad 1 + \frac{1}{O(1)} \leq C \leq O(1), \\ \mu + \frac{1}{C-1}\left(\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\right) \leq \tilde{\zeta}_{2,\min}\cos\frac{\pi}{6} - \frac{1}{O(1)}. \end{gathered} \tag{4.11}$$

Combining this with (4.4), we get

$$N(r_0 + \mu h^{\frac{2}{3}}) \leq CV\left(\mu + \frac{1}{C-1}\left(\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\right)\right) + o(1)h^{\frac{1}{3}-(n-1)}, \quad h \to 0. \tag{4.12}$$

Notice that (4.12) still holds, if we drop the lower bound on μ in (4.11). Later we shall sometimes use that

$$\mu + \frac{1}{C-1}\left(\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\right) = \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} + \frac{C}{C-1}\left(\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\right).$$

H_R preserves the Liouville measure on Σ, so Appendix A shows that G can be chosen in (3.6), so that

$$\tilde{\zeta}_1 = \frac{1}{T}\int_0^T \zeta_1 \circ \exp(tH_R)\,dt \underset{\text{def}}{=} \zeta_1^T \text{ on } \Sigma, \tag{4.13}$$

for any $T > 0$. Let $\zeta_2^T = \zeta_1^T + \left(2Q(y',\eta')\right)^{\frac{2}{3}}(\zeta_2 - \zeta_1)$ be the corresponding $\tilde{\zeta}_2$ (not obtained by averaging ζ_2 as in (4.13)) and notice that $\zeta_2^T - \zeta_1^T$ is independent of T. For every fixed $T > 0$, (4.12) reads:

$$(4.14)\qquad N(r_0 + \mu h^{\frac{2}{3}}) \leq CV_T\Big(\mu + \frac{1}{C-1}\Big(\mu - \zeta_{1,\min}^T \cos\frac{\pi}{6}\Big)\Big) + o(1)h^{\frac{1}{3}-(n-1)},$$

under the now T-dependent restriction (4.11), where the lower bound on μ may be dropped, and with V_T defined by (4.5), with $\tilde{\zeta}_1 = \zeta_1^T$.

Consider

$$(4.15)\qquad I(\mu, T) = \int_\Sigma \Big(\mu - \zeta_1^T(\rho)\cos\frac{\pi}{6}\Big)_+^{\frac{1}{2}} S_{\Sigma,R}(d\rho),$$

which appears in the definition of V_T. The functions $\mu \longmapsto (\mu - \zeta_1^T(\rho)\cos\frac{\pi}{6})_+^{\frac{1}{2}}$ defined on any fixed compact interval, are uniformly continuous for $\rho \in \Sigma$, $T \in [0,\infty[$, so it follows that the functions $\mu \longmapsto I(\mu, T)$, (defined on some fixed compact interval) are uniformly continuous for $T \in [0,\infty[$. On the other hand, by Birkhoff's ergodic theorem, we know that $\zeta_1^T(\rho) \longrightarrow \zeta_1^\infty(\rho)$ a.e., so by dominated convergence,

$$(4.16)\qquad I(\mu, T) \longrightarrow I(\mu, \infty) \underset{\text{def}}{=} \int_\Sigma \Big(\mu - \zeta_1^\infty(\rho)\cos\frac{\pi}{6}\Big)_+^{\frac{1}{2}} S_{\Sigma,R}(d\rho), \quad T \longrightarrow \infty.$$

Because of the uniform continuity of the functions $I(\cdot, T)$, it follows that the convergence in (4.16) is uniform on any bounded interval and that the limiting function $I(\mu,\infty)$ is continuous. In Appendix A, we see that

$$(4.17)\qquad \lim_{T\to\infty} \zeta_{1,\min}^T = \sup_{T>0} \zeta_{1,\min}^T \leq \operatorname{ess\,inf} \zeta_1^\infty.$$

We want to pass to the limit $T \longrightarrow \infty$ in (4.14) so we replace the T-dependent assumption (4.11) (without the lower bound on μ), by

$$(4.18)\qquad 1 + \frac{1}{O(1)} \leq C \leq O(1), \quad 0 \leq \mu,$$
$$\mu + \frac{1}{C-1}\Big(\mu - \Big(\cos\frac{\pi}{6}\Big)\lim_{T\to\infty}\zeta_{1,\min}^T\Big) \leq \Big(\cos\frac{\pi}{6}\Big)\liminf_{T\to\infty}\zeta_{2,\min}^T - \frac{1}{O(1)}.$$

In view of (3.5), (3.6), we have

$$\liminf_{T\to\infty}\zeta_{2,\min}^T - \lim_{T\to\infty}\zeta_{1,\min}^T > 0,$$

so (4.18) is non-empty. If μ satisfies (4.18), then it also satisfies (4.11) (without the lower bound) for T large enough. Let

$$(4.19)\qquad V_\infty(\mu) = \frac{2\sqrt{2r_0}h^{\frac{1}{3}}}{(2\pi h)^{n-1}} \int \Big(\mu - \zeta_1^\infty \cos\frac{\pi}{6}\Big)_+^{\frac{1}{2}} S_{R,\Sigma}(d\rho) = \frac{2\sqrt{2r_0}h^{\frac{1}{3}}}{(2\pi h)^{n-1}} I(\mu,\infty),$$

so that according to the earlier discussion, $V_\infty(\mu) - V_T(\mu) = o(1)h^{\frac{1}{3}-(n-1)}$, $T \longrightarrow \infty$, uniformly with respect to μ for μ in any compact interval. We then get,

$$
\begin{aligned}
N(r_0 + \mu h^{\frac{2}{3}}) &\leq CV_\infty\left(\mu + \frac{1}{C-1}\left(\mu - \left(\cos\frac{\pi}{6}\right)\lim_{T\to\infty}\zeta^T_{1,\min}\right)\right) + o(1)h^{\frac{1}{3}-(n-1)} \\
&= CV_\infty\left(\left(\cos\frac{\pi}{6}\right)\lim_{T\to\infty}\zeta^T_{1,\min} + \frac{C}{C-1}\left(\mu - (\cos\frac{\pi}{6})\lim_{T\to\infty}\zeta^T_{1,\min}\right)\right) \\
&\quad + o(1)h^{\frac{1}{3}-(n-1)},
\end{aligned} \tag{4.20}
$$

when $h \longrightarrow 0$.

A SLIGHTLY MORE SOPHISTICATED APPROACH. We shall make a more systematic use of (4.6) and as before, we will only work in a region where $\mu_j = r_0 + O(h^{\frac{2}{3}})$, $|\lambda_j| = r_0 + O(h^{\frac{2}{3}})$. Then,

$$
\log \mu_j = \log r_0 + \frac{\mu_j - r_0}{r_0} + O(h^{\frac{4}{3}}), \quad \log|\lambda_j| = \log r_0 + \frac{|\lambda_j| - r_0}{r_0} + O(h^{\frac{4}{3}})
$$

Taking the logarithm of (4.6), we then get,

$$
\mu_1 + \cdots + \mu_N \leq |\lambda_1| + \cdots + |\lambda_N| + O(Nh^{\frac{4}{3}}) \leq \left(|\lambda_N| + O(h^{\frac{4}{3}})\right)N. \tag{4.21}
$$

Recalling that $M(r)$ is the number of μ_j's that are $\leq r$, we put

$$
\tilde{M}(r) = \int_{-\infty}^{r} \rho\, dM(\rho).
$$

Then from (4.21), we get

$$
\tilde{M}(r) \leq \left(\lambda + O(h^{\frac{4}{3}})\right)N(\lambda), \text{ whenever } M(r) \leq N(\lambda), \tag{4.22}
$$

and still under the assumption, $r, \lambda = r_0 + O(h^{\frac{2}{3}})$.

Let $W(\mu) = V(\mu) + o(h^{\frac{1}{3}-(n-1)})$ be a non-negative continuous function vanishing for $\mu \leq \tilde{\zeta}_{1,\min}(\cos\frac{\pi}{6}) - o(1)$, strictly increasing when positive, and with the property that

$$
M(r_0 + \mu h^{\frac{2}{3}}) \leq W(\mu), \quad \mu \leq \tilde{\zeta}_{2,\min}\left(\cos\frac{\pi}{6}\right) - \frac{1}{O(1)}. \tag{4.23}
$$

Since $M(\mu_j) \geq j$, we have $W(\frac{\mu_j - r_0}{h^{\frac{2}{3}}}) \geq j$, $\mu_j \geq \left(r_0 + h^{\frac{2}{3}}W^{-1}(j)\right)$, provided that μ_j satisfies the last estimate in (4.23). Hence (4.21) implies,

$$
\sum_{j=1}^{N}\left(r_0 + h^{\frac{2}{3}}W^{-1}(j)\right) \leq N\cdot\left(|\lambda_N| + O(h^{\frac{4}{3}})\right). \tag{4.24}
$$

Comparing the LHS with an integral, we get,

$$
\int_0^N \left(r_0 + h^{\frac{2}{3}}W^{-1}(T)\right) dT \leq N\cdot\left(|\lambda_N| + O(h^{\frac{4}{3}})\right). \tag{4.25}
$$

Introduce $\nu = W^{-1}(T) \leq \tilde{\zeta}_{2,\min} \cos\frac{\pi}{6} - \frac{1}{O(1)}$ as a new integration variable, so that $T = W(\nu)$:

$$\int_{-\infty}^{\nu(N)} (r_0 + h^{\frac{2}{3}}\nu)\, dW(\nu) \leq N \cdot \left(|\lambda_N| + O(h^{\frac{4}{3}})\right), \quad W\big(\nu(N)\big) = N.$$

Here we may replace $|\lambda_N|$by any larger number $\lambda = O(1)$, so we get

$$\int_{-\infty}^{\nu(N)} (r_0 + h^{\frac{2}{3}}\nu)\, dW(\nu) \leq N\big(\lambda + O(h^{\frac{4}{3}})\big), \quad W\big(\nu(N)\big) = N,\ N \leq N(\lambda), \tag{4.26}$$

as long as $W^{-1}(N) \leq \tilde{\zeta}_{2,\min} \cos\frac{\pi}{6} - \frac{1}{O(1)}$. We now want to return to the more explicit function V. We have $V\big(\nu(N)\big) = N + o(1)h^{\frac{1}{3}-(n-1)}$, so

$$\nu(N) = V^{-1}\big(N + o(1)h^{\frac{1}{3}-(n-1)}\big). \tag{4.27}$$

After an integration by parts, the LHS of (4.26) becomes,

$$\begin{aligned} &[(r_0 + h^{\frac{2}{3}}\nu)W(\nu)]_{-\infty}^{\nu(N)} - h^{\frac{2}{3}} \int_{-\infty}^{\nu(N)} W(\nu)\, d\nu \\ &\quad = \big(r_0 + h^{\frac{2}{3}}\nu(N)\big)N - h^{\frac{2}{3}} \int_{-\infty}^{\nu(N)} V(\nu)\, d\nu + o(1)h^{1-(n-1)}. \end{aligned} \tag{4.28}$$

Using this in (4.26), we get for $N \leq N(\lambda)$:

$$\big(r_0 + h^{\frac{2}{3}}\nu(N)\big)N - h^{\frac{2}{3}} \int_{-\infty}^{\nu(N)} V(\nu)\, d\nu \leq N \cdot \lambda + o(1)h^{1-(n-1)}. \tag{4.29}$$

Here we also used that $N = O(1)h^{\frac{1}{3}-(n-1)}$ ([SZ2]). To get explicit bounds on N from this, we need some convenient upper bounds on the integral in (4.29). From the definition of V, we see that for $0 \leq k \leq 1, t \geq 0$:

$$V\Big(\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} + kt\Big) \leq k^{\frac{1}{2}} V\Big(\tilde{\zeta}_{1,\min} \cos\frac{\pi}{6} + t\Big). \tag{4.30}$$

Hence,

$$\begin{aligned} \int_{-\infty}^{\nu(N)} V(\nu)\, d\nu &= \int_0^{\nu(N)-\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}} V\Big(\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} + t\Big)\, dt \\ &\leq V\big(\nu(N)\big) \int_0^{\nu(N)-\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}} \frac{t^{\frac{1}{2}}}{\big(\nu(N) - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\big)^{\frac{1}{2}}}\, dt \\ &= \frac{2}{3} V\big(\nu(N)\big)\Big(\nu(N) - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\Big) \\ &= \frac{2}{3} N\Big(\nu(N) - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\Big) + o(1)h^{\frac{1}{3}-(n-1)}. \end{aligned} \tag{4.31}$$

Using this in (4.29), we get

$$r_0 + h^{\frac{2}{3}}\nu(N) - h^{\frac{2}{3}}\frac{2}{3}\Big(\nu(N) - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\Big) \leq \lambda + \frac{o(1)h^{1-(n-1)}}{N}. \tag{4.31}$$

If we write $\lambda = r_0 + \mu h^{2/3}$ with

$$0 \le \mu \le \tilde{\zeta}_{2,\min} \cos\frac{\pi}{6} - \frac{1}{O(1)}, \tag{4.32}$$

we get:

$$\nu(N) - \frac{2}{3}\Big(\nu(N) - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\Big) \le \mu + \frac{o(1)h^{\frac{1}{3}-(n-1)}}{N},$$

$$\nu(N) \le 3\mu - 2\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} + \frac{o(1)h^{\frac{1}{3}-(n-1)}}{N}.$$

Applying V and using (4.27), we get

$$N \le V\Big(\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} + 3\Big(\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\Big) + \omega(h)\frac{h^{\frac{1}{3}-(n-1)}}{N}\Big) + o(1)h^{\frac{1}{3}-(n-1)},$$

where $0 < \omega(h) \longrightarrow 0$, $h \longrightarrow 0$. Distinguishing the two cases $N \le \sqrt{\omega(h)}h^{\frac{1}{3}-(n-1)}$ and $N > \sqrt{\omega(h)}h^{\frac{1}{3}-(n-1)}$, we get in both cases:

$$N \le V\Big(\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} + 3\Big(\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\Big)\Big) + o(1)h^{\frac{1}{3}-(n-1)}. \tag{4.34}$$

By induction over N, we see that $N(r_0 + \mu h^{\frac{2}{3}})$ is bounded by the RHS of (4.34), provided that

$$\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} + 3\Big(\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\Big) \le \tilde{\zeta}_{2,\min} - \frac{1}{O(1)}.$$

As before, we can take $\tilde{\zeta}_1 = \zeta_1^T, \tilde{\zeta}_2 = \zeta_2^T = \zeta_1^T + \big(2Q(y',\eta')\big)^{\frac{2}{3}}(\zeta_2 - \zeta_1)$ and let T tend to infinity. Then under the assumption:

$$\mu \ge 0, \quad \lim_{T\to\infty}\zeta_{1,\min}^T\cos\frac{\pi}{6} + 3\Big(\mu - \lim_{T\to\infty}\zeta_{1,\min}^T\cos\frac{\pi}{6}\Big) \le \Big(\cos\frac{\pi}{6}\Big)\liminf_{T\to\infty}\zeta_{2,\min}^T - \frac{1}{O(1)}, \tag{4.35}$$

we get

$$N(r_0 + \mu h^{\frac{2}{3}}) \le V_\infty\Big(\Big(\cos\frac{\pi}{6}\Big)\lim_{T\to\infty}\zeta_{1,\min}^T + 3\Big(\mu - \Big(\cos\frac{\pi}{6}\Big)\lim_{T\to\infty}\zeta_{1,\min}^T\Big)\Big) + o(1)h^{\frac{1}{3}-(n-1)}. \tag{4.36}$$

Let us now compare (4.12) with (4.34) (or the corresponding limiting estimates for $T = +\infty$). Of particular interest is the case $\frac{C}{C-1} = 3$, *i.e.* when $C = \frac{3}{2}$. For $1 < C \le 3/2$, we have $C/(C-1) \ge 3$, and since there is no prefactor, it is clear that (4.34) is sharper than (4.12) for these values of C. It cannot be excluded however, that (4.12) is sometimes sharper, when $C > 3/2$.

Let us compare the two estimates (or the corresponding analogues for $T = +\infty$) in the possibly theoretical case, when $V(\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} + t) = C_0 t^\alpha$, for some $\alpha > 0$. For simplicity, we skip the remainder terms in the following discussion. With t= $\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}$, the RHS of (4.12) is $C(\frac{C}{C-1})^\alpha C_0 t^\alpha$, so the loss factor is $C(\frac{C}{C-1})^\alpha$. It attains its infimum for

$C = 1+\alpha$: $(1+\alpha)(1+\frac{1}{\alpha})^\alpha \leq (\alpha+1)e$. For α large enough, this is clearly smaller than the corresponding loss factor 3^α resulting from (4.34), so in this case (4.12) is sharper than (4.34). However, if we examine the proof of (4.34) in this special case, we see that (4.31) improves to

$$\int_{-\infty}^{\nu(N)} V(\nu)\,d\nu \leq \frac{1}{\alpha+1}N\Big(\nu(N) - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6}\Big) + o(1)h^{\frac{1}{3}-(n-1)},$$

and then the factor 3 in (4.34) can be replaced by $(1+\frac{1}{\alpha})$. We then get a loss factor in (4.34) of the form $(1+\frac{1}{\alpha})^\alpha$ which is smaller than $(1+\alpha)(1+\frac{1}{\alpha})^\alpha$.

In Appendix B, explain why (4.29) gives improved estimates on N, when we replace ζ_1, by ζ_1^T and let $T\longrightarrow\infty$.

5. **End of the proof.** We start from (4.36), where $N(r_0+\mu h^{\frac{2}{3}})$ is the number of eigenvalues of $-h^2\Delta_{|\Gamma}$ in the disc $|z-\omega_0| \leq r_0+\mu h^{\frac{2}{3}}$, $\omega_0 = \Re\omega_0 + ir_0$, $\Re\omega_0, r_0 > 0$. Choose $\Re\omega_0 = 1$ from now on. For $k_0 \gg 1$, put $k_0 = 1/h$, $z = k^2h^2 = \zeta^2$, $\zeta = kh$, $\Re k > 0$. Then $-h^2\Delta_{|\Gamma}u = zu$ is equivalent to $-\Delta_{|\Gamma}u = k^2u$, so z is an eigenvalue of $-h^2\Delta_{|\Gamma}$ iff k is a resonance, and there is no problem to identify the multiplicities (see [SZ1,2] and the references cited there). Consider the image of $|z-\omega_0| \leq r_0+\mu h^{\frac{2}{3}}$, $\Im z \leq 0$ in the ζ-plane. Since $\Re z = (\Re\zeta)^2-(\Im\zeta)^2$, $\Im z = 2\Re\zeta\Im\zeta$, this image is given by $\Im\zeta\leq 0$ and

$$\Big((\Re\zeta)^2-(\Im\zeta)^2-1\Big)^2 + (2\Re\zeta\Im\zeta - r_0)^2 \leq r_0^2 + 2\mu r_0 h^{\frac{2}{3}} + O(h^{\frac{4}{3}}).$$

Using that $\Im\zeta = O(h^{\frac{2}{3}})$, $\Re\zeta - 1 = O(h^{\frac{1}{3}})$, we get

$$\Big((\Re\zeta)^2-1\Big)^2 + (2\Im\zeta - r_0)^2 \leq r_0^2 + 2\mu r_0 h^{\frac{2}{3}} + O(h),$$
$$4(\Re\zeta-1)^2 - 4r_0\Im\zeta \leq 2\mu r_0 h^{\frac{2}{3}} + O(h),$$

so the image in the ζ-plane becomes

$$\frac{1}{r_0}(\Re\zeta-1)^2 - \frac{1}{2}\Big(\mu + O(h^{\frac{1}{3}})\Big)h^{\frac{2}{3}} \leq \Im\zeta \leq 0, \tag{5.1}$$

and modulo the O-term this is the region below the real axis and above the parabola, symmetric around $\Re\zeta = 1$, which passes through $1 - i\frac{\mu}{2}h^{\frac{2}{3}}$, $1 \pm \sqrt{\frac{r_0\mu}{2}}h^{\frac{1}{3}}$. The image in the k-plane then becomes

$$\frac{(\Re k - k_0)^2}{k_0 r_0} - \frac{\Big(\mu + O(k_0^{-\frac{1}{3}})\Big)}{2}k_0^{\frac{1}{3}} \leq \Im k \leq 0, \tag{5.2}$$

which modulo the O-term is precisely the region described in the theorem. It is also clear that the region in the theorem has an image in the z-plane which is contained in

$$|z-\omega_0| \leq r_0 + \Big(\mu + O(h^{\frac{1}{3}})\Big)h^{\frac{2}{3}}, \quad \Im z \leq 0.$$

We can then apply (4.36) since a substitution, $\mu \longmapsto \mu + O(h^{\frac{1}{3}})$ does not change the right hand side there, nor the validity of the condition (4.35).

To identify the other quantities, we make explicit computations at some fixed point of the boundary (as in [SZ2]). After a Euclidean change of coordinates, we may assume that the fixed point is $y = 0$ and that the exterior unit normal is $(0, 1)$. Then ∂O is of the form $y_n + f(y') = 0$, where $f(y') = \frac{1}{2}\langle Fy', y'\rangle + O(y'^3)$, and where $\langle Fy', y'\rangle$ is the second fundamental form at 0, if we identify the y'-plane with $T_0\partial O$. The exterior unit normal at a neighboring point $\big(x', -f(x')\big) \in \partial O$ is given by

$$n(x') = \frac{\big(\nabla f(x'), 1\big)}{\sqrt{\big(\nabla f(x')\big)^2 + 1}} = (Fx', 1) + O(x'^2),$$

so the geodesic coordinates (x', x_n) (with x' parametrizing the boundary) are given by

$$y' = x' + x_n Fx' + O(x'^2), \quad y_n = x_n + O(x'^2).$$

Hence

$$\frac{\partial y}{\partial x} = \begin{pmatrix} I + x_n F & 0 \\ 0 & 1 \end{pmatrix} + O(|x'|),$$
$$\frac{\partial x}{\partial y} = \begin{pmatrix} I - x_n F & 0 \\ 0 & 1 \end{pmatrix} + O(|x'| + |x_n|^2),$$

and the principal symbol of $-\Delta_y$ becomes

$$\eta^2 = \left({}^t\!\left(\frac{\partial x}{\partial y}\right)\xi \right)^2 = \xi_n^2 + \xi'^2 - 2x_n\langle F\xi', \xi'\rangle + O(|x'| + |x_n|^2)|\xi|^2.$$

At the point $x' = 0$ we recognize $\xi'^2 = R(0, \xi')$ as the symbol of the tangential Laplacian, and $Q(0, \xi') = \langle F\xi', \xi'\rangle$ as the second fundamental form, after identifying $T_0^*\partial O$ with $T_0\partial O$ by means of the induced Riemannian metric. Finally the Liouville measure $S_{\Sigma,R}$ associated to Σ becomes under the same identification, the Liouville measure on $S\partial O \simeq \{g = 1\}$ associated to the metric g and the natural volume density on $T\partial O$, obtained by pulling over the symplectic volume. This is $\frac{1}{2}$ times the natural measure on $S\partial O$. The theorem follows.

Appendix A. We collect here some simple facts concerning averaging along integral curves of a vector field.

Consider first on $\mathbf{R}$ the problem, for a given $v \in L^\infty(\mathbf{R})$, to find a bounded function u, such that $\frac{du}{dt} = v+$ a slowly varying function. We solve this by putting $u = k_T * v$, where $k_T(t) = k(\frac{t}{T})$ and where k is a fixed function with compact support and uniformly Lipschitz, except for a jump discontinuity of height 1 at 0. More precisely:

$$\text{(A. 1)} \qquad k_{|\pm t>0} \in \mathrm{Lip}_{\mathrm{comp}}([0, \pm\infty[), \quad k(+0) - k(-0) = 1.$$

Then $\frac{dk}{dt} = \delta_0 - \ell$, where $\ell \in L^\infty_{\mathrm{comp}}$, $\int \ell \, dt = 1$, and with $u = k_T * v$, we get,

$$\frac{du}{dt} = v - \frac{1}{T}\ell_T * v, \quad \ell_T(t) = \ell\Big(\frac{t}{T}\Big). \tag{A.2}$$

If we choose $k(t) = -1 - t$ for $-1 \leq t \leq 0$ and 0 elsewhere, we get $\ell = 1_{[-1,0]}$ and (A.2) becomes,

$$\frac{du}{dt} = v - \frac{1}{T}1_{[-T,0]} * v. \tag{A.3}$$

Now let Σ be a compact smooth manifold, equipped with a strictly positive smooth density ω, and let ν be a smooth real vectorfield on Σ which preserves ω: $\mathcal{L}_\nu \omega = 0$, where $\mathcal{L}_\nu$ denotes the Lie derivation with respect to ν. Let q be a realvalued smooth function on Σ. Consider

$$G_T = -\int k_T(-s) q \circ \exp(s\nu)\, ds \in C^\infty(\Sigma). \tag{A.4}$$

By a change of variables, we see that this is a convolution of q and $-k_T$ along the trajectories:

$$G_T \circ \exp(t\nu) = -\int k_T(t-s) q \circ \exp(s\nu)\, ds,$$

and consequently,

$$\nu(G_T) = -q + \frac{1}{T}\int \ell_T(-s) q \circ \exp(s\nu)\, ds. \tag{A.5}$$

Choose k as prior to (A.3). Then

$$q + \nu(G_T) = \frac{1}{T}\int_0^T q \circ \exp(s\nu)\, ds \underset{\mathrm{def}}{=} q^T, \tag{A.6}$$

so up to an element of the image of the diffor ν, we may replace q by its time average q^T.

Put

$$\underline{q}^T = \inf_\Sigma q^T. \tag{A.7}$$

LEMMA A.1. $\sup_{T>0} \underline{q}^T = \lim_{T\to\infty} \underline{q}^T$.

PROOF. Adding a constant to q, we may assume that $q \geq 0$. For $T > 0$, let $k \in \{1, 2, \ldots\}$, $\theta \in [0, 1[$, and consider

$$\begin{aligned} q^{(k+\theta)T} &= \frac{1}{(k+\theta)T}\int_0^{(k+\theta)T} q \circ \exp(s\nu)\, ds \\ &= \left(\sum_{j=0}^{k-1} \frac{1}{(k+\theta)T}\int_{jT}^{(j+1)T} q \circ \exp(s\nu)\, ds\right) + \frac{1}{(k+1)T}\int_{kT}^{(k+\theta)T} q \circ \exp(s\nu)\, ds \\ &\geq \frac{k}{k+\theta}\underline{q}^T. \end{aligned}$$

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Taking the infimum over Σ, we get

$$\underline{q}^{(k+\theta)T} \geq \frac{k}{k+\theta}\underline{q}^{T}.$$

Letting $S = (k+\theta)T$ tend to infinity, we see that

$$\liminf_{S\to\infty} \underline{q}^{S} \geq \underline{q}^{T},$$

for every $T > 0$. It follows that

$$\sup_{T} \underline{q}^{T} \leq \liminf_{S\to\infty} \underline{q}^{S} \leq \limsup_{S\to\infty} \underline{q}^{S} \leq \sup_{T} \underline{q}^{T},$$

where the last two inequalities are obvious. Hence, we have equality everywhere, and the lemma follows. ∎

According to Birkhoff's ergodic theorem, (and here is where we use that ν is measure preserving), the limit

$$q^{\infty} = \lim_{T\to\infty} q^{T} \tag{A.8}$$

exists almost everywhere. We have clearly,

$$\lim_{T\to\infty} \underline{q}^{T} \leq q^{\infty}, \text{ a.e.} \tag{A.9}$$

Appendix B. Let M be a smooth compact manifold equipped with some smooth density $dm > 0$, which is ν-invariant, where ν is some smooth vectorfield. It will be more convenient to average by means of the heat kernel along the trajectories. Put

$$E(T,s) = \frac{1}{\sqrt{2\pi T}} e^{-t^2/2T}, \quad T > 0,\ t \in \mathbf{R},$$

so that $E(T+S) = E(T) * E(S)$, where $*$ indicates ordinary convolution. Let u be a smooth realvalued function on M, and put

$$u^T = \int E(T,-s) u \circ \exp(s\nu)\, ds.$$

As in Appendix A, this is a convolution along trajectories:

$$u^T \circ \exp(t\nu) = \int E(T, t-s) u \circ \exp(s\nu)\, ds.$$

Noticing that $E(1,s)$ can be approximated in L^1 by linear combinations with positive coefficients of functions of the form $1_{[-R,R]}$, we see that $u^T \longrightarrow u^{\infty}$ a.e. where u^{∞} is the same Birkhoff limit as in the preceding section. For simplicity, we shall assume,

$$\int_{u^T = C} dm = 0, \quad \forall T, C \tag{B.1}$$

or in other words that all the level surfaces of all the functions u^T are of zero measure. Put

$$M_T(\mu) = M(T,\mu) = \int_{u^T \leq \mu} dm, \quad \tilde{M}_T(\mu) = \tilde{M}(T,\mu) = \int_{u^T \leq \mu} u^T\, dm.$$

Thanks to (B.1), the inverse M_T^{-1} is defined: $[0, m(M)] \longrightarrow [\inf u^T, \sup u^T]$.

PROPOSITION B.1. *The function $\tilde{M}_T \circ M_T^{-1}$ increases when T increases.*

PROOF. We first examine the properties of $\mu(T,N) = M_T^{-1}(N)$. Let $N \in [0, m(M)]$, and fix T_0. Then for T close to T_0, we have the implications

$$u^T(\rho) \leq \mu(T_0,N) - \|\frac{\partial u^{T_0}}{\partial T}\|_{L^\infty}|T-T_0| - C|T-T_0|^2 \Rightarrow u^{T_0}(\rho)$$
$$\leq \mu(T_0,N) \Rightarrow u^T(\rho) \leq \mu(T_0,N) + \|\frac{\partial u^{T_0}}{\partial T}\|_{L^\infty}|T-T_0| + C|T-T_0|^2.$$

It follows that,

$$\mu(T_0,N) - \left\|\frac{\partial u^{T_0}}{\partial T}\right\|_{L^\infty}|T-T_0| - C|T-T_0|^2 \leq \mu(T,N) \leq \mu(T_0,N)$$
$$+ \|\frac{\partial u^{T_0}}{\partial T}\|_{L^\infty}|T-T_0| + C|T-T_0|^2.$$

Hence $T \longmapsto \mu(T,N)$ is locally Lipschitz and the corresponding a.e. defined derivative satisfies:

$$\left|\frac{\partial \mu}{\partial T}(T,N)\right| \leq \left\|\frac{\partial u^T}{\partial T}\right\|_{L^\infty}.$$

We shall next see that $T \longmapsto \tilde{M}_T \circ M_T^{-1} \underset{\text{def}}{=} F(T,N)$ is locally Lipschitz in T, and that the corresponding (a.e. defined) derivative is ≥ 0. Indeed, we have $F(T,N) = \int_{\Omega_T} u^T dm$, where $\Omega_T = \Omega\big(T, \mu(T,N)\big)$ (with N fixed) is the set defined by $u^T \leq \mu(T,N)$. With $\mu_T = \mu(T,N)$, we get

$$\begin{aligned} F(T+\delta,N) - F(T,N) &= \int_{\Omega_{T+\delta}} u^{T+\delta}\, dm - \int_{\Omega_T} u^T\, dm \\ &= \int_{\Omega_T} (u^{T+\delta} - u^T)\, dm + \int u^{T+\delta}(1_{\Omega_{T+\delta}} - 1_{\Omega_T})\, dm \\ &= \int_{\Omega_T} (u^{T+\delta} - u^T)\, dm + \int (u^{T+\delta} - \mu_T)(1_{\Omega_{T+\delta}} - 1_{\Omega_T})\, dm, \end{aligned} \tag{B.2}$$

where we used that $\int(1_{\Omega_{T+\delta}} - 1_{\Omega_T})\, dm = 0$. On the support of $1_{\Omega_{T+\delta}} - 1_{\Omega_T}$ there are two possibilities:

1) $u^{T+\delta}(\rho) \leq \mu_{T+\delta}, u^T(\rho) > \mu_T$.
2) $u^{T+\delta}(\rho) > \mu_{T+\delta}, u^T(\rho) \leq \mu_T$.

Using that $\mu_{T+\delta} = \mu_T + O(\delta)$, $u^{T+\delta} = u^T + O(\delta)$, we see in each case that $u^{T+\delta}(\rho) - \mu_T = O(\delta)$. Hence

$$\int (u^{T+\delta} - \mu_T)(1_{\Omega_{T+\delta}} - 1_{\Omega_T})\, dm = O(\delta)\|1_{\Omega_{T+\delta}} - 1_{\Omega_T}\|_{L^1} = O(\delta),$$

locally uniformly in T. Hence $F(\cdot, N)$ is locally Lipschitz. For every fixed T, we also have

$$\|1_{\Omega_{T+\delta}} - 1_{\Omega_T}\|_{L^1} \longrightarrow 0, \quad \delta \longrightarrow 0, \tag{B.3}$$

by (B.1), and the fact that $\mu_{T+\delta} \longrightarrow \mu_T$. Then,

$$\int (u^{T+\delta} - \mu_T)(1_{\Omega_{T+\delta}} - 1_{\Omega_T})\, dm = o(\delta), \quad \delta \longrightarrow 0, \tag{$B.4$}$$

for every fixed T. We now look at the sign of $\int (u^{T+\delta} - u^T)\,dm$ when $\delta > 0$. Since the heat kernels form a convolution semi-group, we have $u^{T+\delta} = (u^T)^\delta$. Put

$$\tilde{u}(\rho) = \begin{cases} u^T(\rho) & \text{if } \rho \in \Omega_T \\ \mu_T & \text{if } \rho \in M \setminus \Omega_T. \end{cases}$$

Using that $\int_M f^\delta dm = \int_M f dm$ and that $\tilde{u}^\delta - \mu_T = (\tilde{u} - \mu_T)^\delta \leq 0$, we get

$$\begin{aligned} \int_{\Omega_T} u^{T+\delta}\,dm \geq \int_{\Omega_T} \tilde{u}^\delta\,dm &= \int_M \tilde{u}^\delta\,dm - \int_{M\setminus\Omega_T} \tilde{u}^\delta\,dm \\ &= \int_M \tilde{u}\,dm - \int_{M\setminus\Omega_T} \tilde{u}^\delta\,dm \\ &= \int_{\Omega_T} u^T dm - \int_{M\setminus\Omega_T} (\tilde{u}^\delta - \mu_T)\,dm \geq \int_{\Omega_T} u^T\,dm. \end{aligned}$$

Combining this with (B.2) and (B.4), we get

$$F(T+\delta, N) - F(T,N) \geq o_T(\delta), \quad \delta \searrow 0.$$

Combining this with the local Lipschitz property, we see that $T \longmapsto F(T,N)$ is an increasing function. ∎

We now return to (4.27), (4.29) (leading to (4.36)). From these equations, we get

$$\int_{-\infty}^{\nu(N)} (r_0 + h^{\frac{2}{3}}\nu)\,dV(\nu) \leq N|\lambda_N| + o(1)h^{1-(n-1)}, \tag{B.5}$$

$$V\big(\nu(N)\big) = N, \tag{B.6}$$

where we have changed the definition of $\nu(N)$ by a term $o(1)$. In these relations, we take $\tilde{\zeta}_1 = \zeta_1^{T_0,T}$, the heat kernel regularization along the H_R-trajectories with parameter T, of $\zeta_1^{T_0}$, the latter being defined as in Section 4. Notice that if we pass to the limit $T = \infty$, we get $\zeta_1^{T_0,\infty} = \zeta_1^\infty$ a.e. on Σ. Put $\mu = r_0 + h^{\frac{2}{3}}\nu$, $M(\mu, T) = V(\nu)$, with $\tilde{\zeta}_1 = \zeta_1^{T_0,T}$,

$$\tilde{M}(\mu, T) = \int_{-\infty}^{\nu} (r_0 + h^{\frac{2}{3}}\tilde{\nu})\,dV(\tilde{\nu}) = \int_{-\infty}^{\mu} \tilde{\mu}\,dM(\tilde{\mu}, T).$$

We shall check that $M(\mu, T)$, $\tilde{M}(\mu, T)$ can also be defined as in this appendix, so that Proposition B.1 applies. Then for a fixed N the integral to the left in (B.5) increases with T, and consequently (B.5,6) becomes a more severe restriction on N when we keep $|\lambda_N|$ fixed and increase T. This then justifies the averaging procedure.

To see that the earlier discussion applies, we work on $M \underset{\text{def}}{=} \Sigma \times \mathbf{R}$ equipped with the measure

$$m\big(d(\rho, s)\big) = \frac{\sqrt{2r_0}}{(2\pi h)^{n-1}} S_{R,\Sigma}(d\rho)\,ds.$$

On $\Sigma \times \mathbf{R}$, we consider

$$u(\rho, s) = r_0 + h^{\frac{2}{3}}\zeta_1^{T_0} \cos\frac{\pi}{6} + s^2, \quad \nu = H_R,$$

so that s is constant along the integral curves of ν and with the notation of the present appendix:

$$u^T = r_0 + h^{\frac{2}{3}}\zeta_1^{T_0,T} \cos\frac{\pi}{6} + s^2.$$

Then

$$M(\mu, T) = \int_{u^T \leq \mu} m\big(d(\rho, s)\big),$$
$$\tilde{M}(\mu, T) = \int_{u^T \leq \mu} u^T m\big(d(\rho, s)\big),$$

as in Proposition B.1. The assumption (B.1) is obviously satisfied. M is not compact but Σ is and it is easy to see that Proposition B.1 applies.

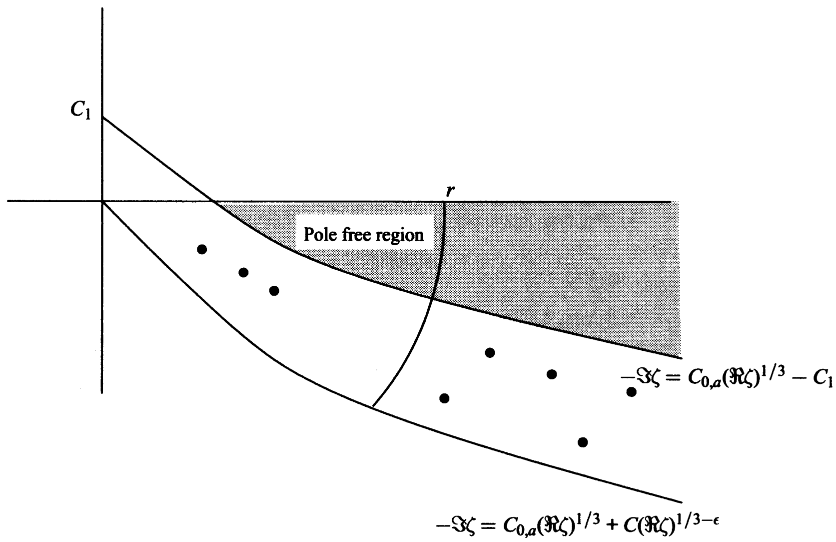

FIGURE C.1. Counting poles in a neighbourhood of the pole free region.

Appendix C (by Maciej Zworski). The purpose of this appendix is to present a simple application of the proof of the main theorem stated in the introduction to scattering by obstacles of revolution in $\mathbb{R}^3$. Thus we let ∂O be a strictly convex analytic surface of revolution in $\mathbb{R}^3$ which we normalize so that the x_3-axis is its axis of revolution and the maximal radius of an orbit of the rotation action is one. With that normalization ∂O can be parametrized as follows

(C.1)
$$\partial O = \{(x_1, x_2, x_3) : x_1 = \sin r \cos\theta, x_2 = \sin r \sin\theta, x_3 = z(r), 0 \leq r \leq \pi, 0 \leq \theta < 2\pi\}.$$

THEOREM C. *Let $O \subset \mathbb{R}^3$ be strictly convex and such that ∂O is an analytic surface of revolution. If for the parametrization (C.1) of the normalized ∂O, $|z'(\pi/2)| \neq 1$, then for any $0 < \epsilon < \frac{1}{3}$ and $C > 0$ there exists $\alpha > 0$ such that*

(C.2)
$$\#\{\zeta : \zeta \textit{ is a scattering pole of } O, |\Re\zeta| \leq r, -\Im\zeta \leq C_{0,a}|\Re\zeta|^{\frac{1}{3}} + C|\Re\zeta|^{\frac{1}{3}-\epsilon}\} = O(r^{2-\alpha\epsilon}),$$

where $C_{0,a}$ is as (0.3).

The geometric definition of α is given by (C.21) below and in some cases α can be easily computed—see Example C.2.

Before proceeding with the proof we shall make a few remarks. For an arbitrary C^∞ strictly convex obstacle an upper bound for the left hand side in (C.2) is $O(r^2)$ (see Corollary 1.1 of [SZ2]) and it cannot be improved for the sphere. It is very likely that the improved estimate (C.2) holds for any non-spherical analytic surface of revolution and its validity depends only on a simple geometric statement which we suspect to be true for any such surface (see Proposition C.1). In particular, as we will point out in the discussion of the geometry, (C.2) also holds when the meridians ($\theta = \text{const}$) are assumed to have the same length as the equator ($r = \pi/2$) (so in particular for any strictly convex analytic Zoll surface of revolution). That provides many examples where $z'(\pi/2) = 1$. However it seems unlikely that (C.2) holds for *any* non-spherical strictly convex analytic surface.

We start the proof of Theorem C by recalling some simple facts about convex surfaces of revolution (see [Be], Section 4B from where we borrow some notation). If we put $h(\cos r)^2 = \cos^2 r + z'(r)^2$ then ∂O is strictly convex and analytic if and only if

(C.3)
$$h \in C^\omega([-1,1]), \quad h(\pm 1) = 1, \quad h(x)^2 \geq x^2, \quad h(x) - xh'(x) > 0, \quad x \in [-1,1].$$

The first inequality for h guarantees embeddability in $\mathbb{R}^3$ and the second one the strict convexity. In the coordinates (r,θ), the metric on ∂O is given by $g = h(\cos r)^2\, dr^2 + \sin^2 r d\theta^2$ and the second fundamental form is

(C.4)
$$l = \begin{pmatrix} \frac{h(\cos r)\sin r - h'(\cos r)\cos r \sin r}{(h(\cos r)^2 - \cos^2 r)^{\frac{1}{2}}} & 0 \\ 0 & \frac{(h(\cos r)^2 - \cos^2 r)^{\frac{1}{2}} \sin r}{h(\cos r)} \end{pmatrix}$$

To study the geodesics flow we use (C.1) to write

(C.5)
$$T^*\Big(\partial O \setminus \big\{(0,0,z(0)),(0,0,z(\pi))\big\}\Big) \simeq T^*\big((0,\pi)_r \times S^1_\theta\big).$$

The dual form to the metric is $R(r,\theta\,;\rho,t) = h(\cos r)^{-2}\rho^2 + (\sin r)^{-2}t^2$ and the geodesics are the integral curves of the Hamilton vector field $H_{\frac{1}{2}R}$. Thus on $S^*\partial O = \{R = 1\}$ we obtain, with $\epsilon = \pm 1$,

(C.6)
$$t = \text{const}, \quad \rho = \epsilon h(\cos r)(1 - (\sin r)^{-2}t^2)^{\frac{1}{2}}$$
$$\dot\theta = (\sin r)^{-2}t, \quad \dot r = h(\cos r)^{-2}\rho = \epsilon h(\cos r)^{-1}(1 - (\sin r)^{-2}t^2)^{\frac{1}{2}}.$$

This is of course a well known example of a completely integrable system with the invariant tori given by $R = \text{const}$ and $t = \text{const}$. Geometrically, the torus for a given t (and $R = 1$, say) consists of geodesics contained between and intersecting tangentially the parallels $\sin r = |t|$, $t \in (-1,1) \setminus 0$ with the sign of t determining the orientation. In (C.6) the sign of ϵ changes at each contact with a limiting parallel. For $t = 0$ the parallels degenerate into the two poles and we obtain a family of meridians. The cases of $t = \pm 1$ are degenerate and correspond to the equator $r = \pi/2$ with two different orientations.

We will be interested in functions which are invariant under the flow and the action of rotations. For that case the discussion above is summarized in

LEMMA C.1. *Let $f \in C^\infty(T^*\partial O \backslash 0)$ be homogeneous of degree 0 and invariant under the geodesic flow and the S^1 rotational action. Then f can be identified with a smooth function on the circle which is the fibre of $S^*\partial O$ over a point on the equator. If that circle is parametrized by $t/\sqrt{R}$ in the coordinates (C.5) then*

$$f(r,\theta\,;\rho,t) = g\Big(\frac{t}{\sqrt{R}}\Big), \quad g \in C^\infty([-1,1]). \tag{C.7}$$

If f does not depend on the orientation of geodesics then g is even and if f is analytic then so is g.

The specific function we want to study is discussed in greater generality in Appendix A:

$$S^*\partial O \ni m \longmapsto f_T(m) = \frac{1}{T}\int_0^T Q^{\frac{2}{3}} \circ \exp sH_R(m)\, ds. \tag{C.8}$$

Here Q is a quadratic form on $T^*\partial O$ dual to the second fundamental form. From the expression for the second fundamental form (C.4) (or simply from the obvious rotational invariance properties) and from the discussion of the flow we note that the behaviour as $T \longrightarrow \infty$ is very simple in this case:

$$f_T(m) = f(m) + O\Big(\frac{1}{T}\Big), \quad f(m) = \frac{1}{d(m)}\int_0^{d(m)} Q^{\frac{2}{3}} \circ \exp\big(s + s(m)\big)H_{\frac{1}{2}R}(m)\, ds, \tag{C.9}$$

where $d(m)$ is the distance covered by a geodesic through m as it moves between the two limiting parallels (or poles when $t(m) = 0$) and $s(m)$ is chosen so that $\pi\big(\exp s(m)H_{\frac{1}{2}R}(m)\big)$ lies on a limiting parallel. We should note here that this is an explicit description of $f(m)$ which could also be defined as the integral of $Q^{\frac{2}{3}}$ over the invariant torus containing m. When $t(m) = \pm 1$ then any choice of $d(m)$ in (C.9) gives the same answer $f(m) = 1$ (∂O is normalized as in (C.1)).

We will now describe $Q \circ \exp sH_{\frac{1}{2}R}(m)$ explicitly. To do that we observe that if we put $y(s) = \pi\big(\exp sH_{\frac{1}{2}R}(m)\big)$ then $Q \circ \exp sH_{\frac{1}{2}R}(m) = l_{y(s)}\big(\dot y(s), \dot y(s)\big)$. In fact, under the identification of cotangent and tangent bundles using the metric, the fibre variable, η, of $\exp sH_{\frac{1}{2}R}$ goes to $\dot y(s)$: $\dot y(s) = \frac{1}{2}\partial_\eta R$, $\langle \eta, \dot y(s)\rangle = R(y,\eta)$. Hence using (C.4) and (C.6) we obtain at $\exp sH_{\frac{1}{2}R} = (r,\theta\,;\rho,t)$

$$Q \circ \exp sH_{\frac{1}{2}R}(m)$$
$$= \frac{\big(h(\cos r)\sin r - h'(\cos r)\cos r \sin r\big)(\sin^2 r - t^2)}{\big(h(\cos r)^2 - \cos^2 r\big)^{\frac{1}{2}} h(\cos r)^2 \sin^2 r} + \frac{\big(h(\cos r)^2 - \cos^2 r\big)^{\frac{1}{2}} \sin r t^2}{h(\cos r)\sin^4 r}.$$

When $t \in (-1,1)$ we parametrize the geodesic between the limiting parallels by

$$r \in (\sin^{-1}|t|, \pi - \sin^{-1}|t|), \quad \sin^{-1}\colon [-1,1] \longrightarrow [-\pi/2, \pi/2],$$

with $ds = \dot{r}^{-1}\,dr = \sin r h(\cos r)(\sin^2 r - t^2)^{-\frac{1}{2}}\,dr$. Hence with $t = t(m)$, $m \in S^*\partial O$

$$\begin{aligned} d(m) &= \int_{\sin^{-1}|t|}^{\pi-\sin^{-1}|t|} \frac{\sin r h(\cos r)}{(\sin^2 r - t^2)^{\frac{1}{2}}}\,dr = \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \frac{h(x)}{(1-t^2-x^2)^{\frac{1}{2}}}\,dx \\ &\stackrel{\text{def}}{=} I_1(h, 1-t^2), \end{aligned} \tag{C. 10}$$

where $I_1(h,\delta)$ is analytic (respectively C^∞) in δ for analytic (respectively C^∞) h. We conclude that d is an analytic function on $S^*\partial O$. We note that the geometric definition after (C.9) did not specify the value of $d(m)$ for $t(m) = \pm 1$ which is now given by (C.10) and is not related in general to the length of the equator.

Similarly we obtain

$$\begin{aligned} &\int_0^{d(m)} Q^{\frac{2}{3}} \circ \exp\big(s + s(m)\big) H_{\frac{1}{2}R}(m)\,ds \\ &= \int_{-\sqrt{1-t^2}}^{\sqrt{1-t^2}} \left[\frac{\big(h(x) - xh'(x)\big)(1-t^2-x^2)(1-x^2) + \big(h(x)^2 - x^2\big)h(x)t^2}{\big(h(x)^2 - x^2\big)^{\frac{1}{2}}(1-x^2)^{\frac{3}{2}} h(x)^2}\right]^{\frac{2}{3}} \\ &\quad \cdot \frac{h(x)}{(1-t^2-x^2)^{\frac{1}{2}}}\,dx \\ &\stackrel{\text{def}}{=} I_2(h, 1-t^2), \end{aligned} \tag{C. 11}$$

where again for analytic (respectively C^∞) h, $I_2(h,\delta)$ is analytic (respectively C^∞) in δ. A straightforward but tedious computation gives

LEMMA C.2. *If for x near 0, $h = h_0 + h_1 x + h_2 x^2 + O(x^3)$ then for δ near 0*

$$\begin{aligned} I_1(h,\delta) &= \pi\Big(h_0 + \frac{1}{2}h_2\delta + O(\delta^2)\Big) \\ I_2(h,\delta) &= I_1(h,\delta) + \frac{\pi}{6}\Big(h_0^{-1} - h_0\Big)\delta + O(\delta^2). \end{aligned} \tag{C.12}$$

This lemma gives one half of the following

PROPOSITION C.1. *Let ∂O be a strictly convex C^∞ surface of revolution and $f \in C^\infty(S^*\partial O)$ be defined by (C.9). If, in the parametrization (C.1), $|z'(\pi/2)| \neq 1$* or *if the equator has the same length as the meridians and ∂O is* not *a sphere, then f is* not *identically constant.*

PROOF. The first part is immediate from Lemma C.2: $f(m) = I_2\big(h, 1 - t(m)^2\big)/I_1\big(h, 1 - t(m)^2\big)$ and $|z'(\pi/2)| = h_0$. For the second one we use the following elementary observation†: if γ is a closed convex planar curve of length d and $\kappa(s)$ is the curvature of γ with s the length parameter then

$$\frac{1}{d}\int_0^d \kappa(s)^{\frac{2}{3}}\,ds \leq \Big(\frac{2\pi}{d}\Big)^{\frac{2}{3}} \tag{C. 13}$$

† I owe it to a conversation with Bernard Helffer and Bernard Shiffman.

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with equality only if γ is a circle. To see this we parametrize γ by the angle θ made by the tangent to γ with a fixed axis. Then $\kappa = d\theta/ds$, $0 \leq \theta < 2\pi$ and (C.13) follows from Hölder's inequality, with equality only if κ = const. The meridians are planar curves and if they have the same length as the equator and if f is constant then the value of the left hand side of (C.13) for them is the same as for the circle of the same length. Hence they have to be circles and the surface a sphere. ∎

In view of Lemma C.1 f can be considered as a function on a circle and if it is analytic then its minima cannot be flat. A nice case where the minima are actually non-degenerate and can be easily described is given in the following

EXAMPLE C.1. Put $z(r) = \beta \cos r$ in (C.1). This gives a family of spheroids (ellipsoids of revolution): for $\beta = 1$ we get the sphere, for $\beta < 1$ oblate spheroids and for $\beta > 1$ prolate spheroids—see Figure C.2. As suggested by the picture the minimum of f is achieved on the meridians (a codimension one submanifold of $S^*\partial O$) for $\beta > 1$ and on the equator (a codimension two submanifold of $S^*\partial O$) for $\beta < 1$. This is supported by an explicit computation based on (C.10) and (C.11):

$$f(m) = \beta^{-\frac{4}{3}}[(1-\beta^2)(1-t^2)+\beta^2]^{\frac{2}{3}}\frac{K\big((1-t^2)(1-\beta^{-2})\big)}{E\big((1-t^2)(1-\beta^{-2})\big)},$$

where K and E are the elliptic integrals of the first and second type:

$$E(\xi) = \int_0^1 (1-s^2)^{-\frac{1}{2}}(1-\xi s^2)^{\frac{1}{2}}\,ds, \quad K(\xi) = \int_0^1 (1-s^2)^{-\frac{1}{2}}(1-\xi s^2)^{-\frac{1}{2}}\,ds.$$

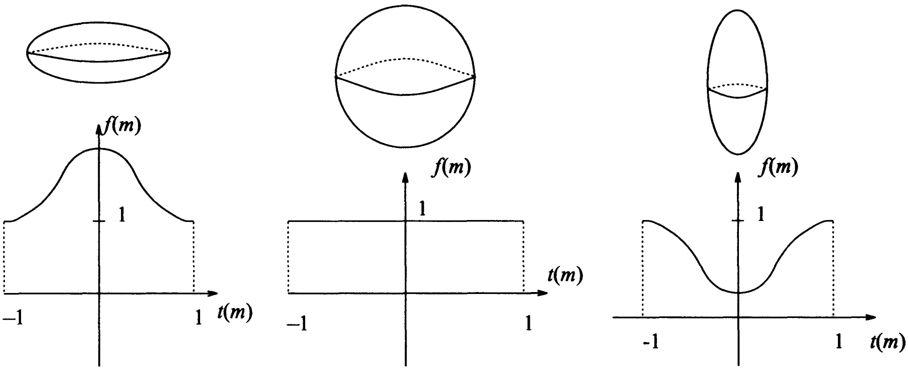

FIGURE C.2. The function f in a simple case; for G given by (C.14), $\tilde{\zeta}_1(m) = 2^{\frac{2}{3}}\zeta_1 f(m)$.

We can now apply this geometric discussion to the results of Section 4. The main observation is that by choosing G in a more specific way than in Appendix A we can obtain $C_{0,a}$ as a minimum of a fixed function of $S^*\partial O$ (without taking the supremum over T for T dependent functions as is needed in the general case). To construct G we use the notation of Appendix A and in (A.4) we put $\Sigma = S^*\partial O$, $\nu = (\cos\pi/6)^{-1}H_R$ and $q = (2Q)^{\frac{2}{3}}$. We then define

$$G(m) \stackrel{\text{def}}{=} G_{\cos\frac{\pi}{6}\,d(m)/2}(m), \tag{C.14}$$

where, since $d(m)$ is constant on geodesics, the definition makes sense and $G \in C^\infty(S^*\partial O)$. Using the invariance of $d(m)$ again we apply (A.5) and (A.6) to obtain

$$(2Q)^{\frac{2}{3}}(m) + \frac{1}{\cos\frac{\pi}{6}} H_R G_{\cos\frac{\pi}{6}T/2} = 2^{\frac{2}{3}} f_T(m),$$
$$(2Q)^{\frac{2}{3}}(m) + \frac{1}{\cos\frac{\pi}{6}} H_R G = 2^{\frac{2}{3}} f(m),$$

where G_T is defined as in (A.4) and f_T and f are given by (C.8) and (C.9) respectively. Hence in the notation of Section 4, $\zeta_1^T(m) = 2^{\frac{2}{3}}\zeta_1 f_{2T/\cos\frac{\pi}{6}}$ and in the notation of Section 3 (with G given by (C.14)), $\tilde{\zeta}_1(m) = 2^{\frac{2}{3}}\zeta_1 f(m)$. By (C.9)

$$\lim_{T\to\infty} \zeta_1^T(m) = \tilde{\zeta}_1(m),$$

$$\text{(C.15)} \quad \begin{aligned} C_{0,a} &= 2^{-\frac{1}{3}} \cos\frac{\pi}{6} \lim_{T\to\infty} \min_{m\in S^*\partial O} \frac{1}{T}\int_0^T Q^{\frac{2}{3}} \circ \exp sH_R(m)\, ds \\ &= 2^{-\frac{1}{3}} \cos\frac{\pi}{6} \min_{m\in S^*\partial O} \tilde{\zeta}_1(m) = 2^{-\frac{1}{3}} \cos\frac{\pi}{6} \tilde{\zeta}_{1,\min}. \end{aligned}$$

The semi-classical and local version of Theorem C is given by

PROPOSITION C.2. *Let O be a strictly convex analytic surface of revolution for which f given by (C.9) is* not *identically constant. If $N(r)$ is defined by (4.2) and $\tilde{\zeta}_1$ and $\tilde{\zeta}_{1,\min}$ are given by (C.15) then for any* $0 < \epsilon < 1/3$ *and* $C > 0$ *there exists* $\alpha > 0$ *(given by (C.21) below) such that*

$$\text{(C. 16)} \qquad N\Big(r_0 + \Big(\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} + Ch^\epsilon\Big)h^{\frac{2}{3}}\Big) = O(1) h^{\frac{\epsilon}{2}+\frac{1}{3}} h^{\epsilon\alpha-2}, \quad \text{as } h \to 0.$$

PROOF. We will apply the estimates of Section 3 in a way similar to that in Section 6 of [SZ2]. We start by observing that (3.14) and the discussion following (3.15) give a slightly stronger version of (3.23):

$$\text{(C. 17)} \quad \begin{aligned} |||(P-\omega_0)u|||^2 \geq{}& (r_0 + \mu h^{2/3})^2 |||u|||^2 \\ &- C_0 h^{2/3}\Big(\mu + Ah^{1/3} - \cos\frac{\pi}{6}\tilde{\zeta}_{1,\min}\Big)\int_{\Lambda_0} q_0(y',\eta') \\ &|J^{\frac{1}{2}}|_{x_n=0} e^{-H_{t(0)}/h} T_{\Lambda_{t(0)}}\gamma(x', hD_{x'})u|^2\, dy'\, d\eta', \end{aligned}$$

where $q_0 \in L^\infty_{\mathrm{comp}}(\Lambda_0)$ is the characteristic function of the set (3.16) and $C_0 > 0$.

The last term in (C.17) can be written as $-\langle QT_{\Lambda_{t(0)}}\gamma(x', hD_{x'}), T_{\Lambda_{t(0)}}\gamma(x', hD_{x'})\rangle$ where

$$Q = C_0 h^{2/3}\Big(\mu + Ah^{1/3} - \cos\frac{\pi}{6}\tilde{\zeta}_{1,\min}\Big) P_{\Lambda_{t(0)}} q_0 P_{\Lambda_{t(0)}}$$

with $P_{\Lambda_{t(0)}}$ defined by (1.55). We now apply a variant of Lemma 6.1 of [SZ2]: if $Q \geq 0$ is a trace class operator then for every $\epsilon > 0$ there exists a finite rank operator $\tilde{K}_\epsilon$ such that $\|Q - \tilde{K}_\epsilon\| \leq \epsilon$ and $\operatorname{rank}\tilde{K}_\epsilon \leq \operatorname{tr} Q/\epsilon$. As in the proof of Lemma 6.2 of [SZ2] this gives

$$\text{(C. 18)} \quad |||(P-\omega_0)u|||^2 \geq ((r_0+\mu h^{2/3})^2 - \delta h^{2/3}\Big(\mu - \tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} - O(h)\Big)|||u|||^2 - \langle K_\delta u, u\rangle,$$

with

$$\operatorname{rank} K_\delta \leq \frac{C}{\delta\big(\mu + Ah^{1/3} - \tilde{\zeta}_{1,\min}\cos(\pi/6)\big)} \operatorname{tr} Q.$$

From (1.46),(1.33) and (1.38) we conclude that

$$\operatorname{tr} Q \leq Ch^{-(n-1)} C_0 h^{2/3}\Big(\mu + Ah^{1/3} - \cos\frac{\pi}{6}\tilde{\zeta}_{1,\min}\Big)\int_{\Lambda_0} q_0(\alpha)\, d\alpha,$$

which by (3.17) then gives

$$(C.19) \qquad \operatorname{rank} K_\delta \leq \frac{C}{\delta} h^{-(n-1)+\frac{1}{3}} \int_\Sigma \big(\mu + O(h^{1/3}) - \tilde{\zeta}_1 \cos(\pi/6)\big)_+^{\frac{1}{2}} S_{R,\Sigma}\big(d(y',\eta')\big).$$

We now specialize to the case $n = 3$ and use (C.19) to obtain (see Section 7 of [SZ2] or Section 4 above):

(C.20)

$$N\Big(r_0 + \Big(\tilde{\zeta}_{1,\min}\cos\frac{\pi}{6} + Ch^\epsilon\Big)h^{\frac{2}{3}}\Big) = O(1)h^{\frac{1}{3}-2}\int_\Sigma \Big(C'h^\epsilon - \big(\tilde{\zeta}_1(m) - \tilde{\zeta}_{1,\min}\big)\cos\frac{\pi}{6}\Big)_+^{\frac{1}{2}} dS_{R,\Sigma}(m),$$

where, as in (C.19), $dS_{R,\Sigma}(m)$ is the Liouville measure on $\Sigma = \{m : R(m) = \Re\omega_0\} \simeq S^*\partial O$.

The function $\tilde{\zeta}_1(m) = 2^{\frac{2}{3}}\zeta_1 f(m)$ is not constant and by Lemma C.1 it can be identified with an analytic function, g, on $S^1 \simeq S_z^*\partial O$, $r(z) = \pi/2$. We will denote the finite set of the necessarily non-flat absolute minima of g by $\phi_1, \ldots, \phi_J$. If $\cos\phi_j \neq 0$ then the minimum is achieved on an embedded submanifold of codimension one (corresponding to the non-degenerate invariant torus) and if $\cos\phi_j = 0$ then on an embedded submanifold of codimension two (corresponding to the equator). These submanifolds are clearly isolated. Let us denote them by $M_j \subset \Sigma$. Then near M_j we can parametrize Σ by (z', z''), $z' \in M_j$, $z'' \in \mathbb{R}^k$, $k = 1, 2$ (depending on codimension) and with M_j given by $z'' = 0$. Since $\tilde{\zeta}_1$ is not flat, we can use Lemma C.1 to see that $\tilde{\zeta}_1(z', z'') - \tilde{\zeta}_{1,\min} \sim |z''|^{2N}$ for some N.

In fact, when $k = 1$ this is clear from rotational symmetry. When $k = 2$ we consider the coordinates $(r, \theta\,; \rho, t)$ on $T^*\partial O \setminus 0$ near the homogeneous extension of M_j which is given by $r = \pi/2$, $\rho = 0$ and $t/\sqrt{R} = \pm 1$, say +1. Then

$$g(t/\sqrt{R}) - g(1) \sim \Big(\frac{t}{\sqrt{R}} - 1\Big)^N \sim \frac{t^N}{R^N}\Big(\frac{h(0)}{t^2}\rho^2 + \Big(r - \frac{\pi}{2}\Big)^2\Big)^N \sim |z''|^{2N}.$$

Hence the first term on the right hand side of (C.20) is bounded by $C_1(h^\epsilon)^{\frac{1}{2}+\frac{k}{2N}} h^{\frac{1}{3}} h^{-2}$. We can now put

(C.21)

$$\alpha = \min\Big\{\frac{\operatorname{codim}(Y)}{2N} : Y \text{ a submanifold of } S^*\partial O, \tilde{\zeta}_1(m) - \tilde{\zeta}_{1,\min} = O\big(\operatorname{dist}(Y, m)^{2N}\big)\Big\},$$

which gives (C.16). ∎

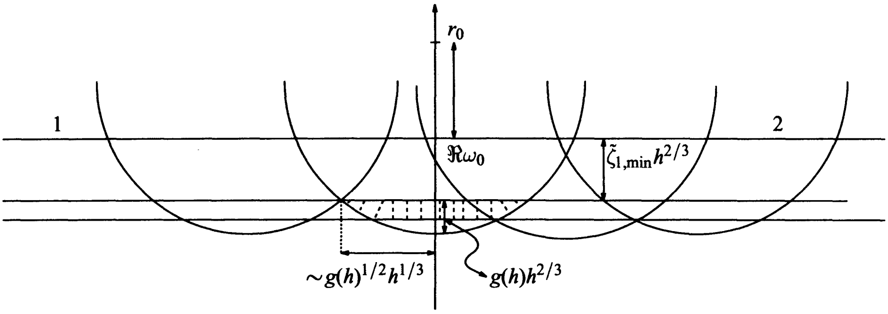

FIGURE C.3. The covering argument with $g(h) = Ch^\epsilon$.

The estimate (C.16) is uniform with respect to ω_0 if $\Im\omega_0 = r_0$ is fixed and $1/2 \leq \Re\omega_0 \leq 5/2$. Hence by covering the rectangle $1 \leq \Re z \leq 2$, $-\Im z \leq \tilde\zeta_{1,\min}\cos(\pi/6) + Ch^\epsilon$ by $C_2h^{-\frac{1}{3}+\frac{\epsilon}{2}}$ discs (see Figure C.3) we obtain from (C.16)

$$\#\{z : z \text{ is an eigenvalue of } P, 1 \leq \Re z \leq 2, -\Im z \leq \tilde\zeta_{1,\min}\cos(\pi/6) + Ch^\epsilon\} = O(1)h^{\alpha\epsilon-2}.$$

The scaling and covering argument as in the proof of Theorem 2 in [SZ2] give Theorem C.

EXAMPLE C.2. We can apply Theorem C with α given by (C.21) to the surfaces in Example C.1 (see Figure C.2). Hence for oblate spheroids, $\beta < 1$, $\alpha = 1$ and for the prolate ones, $\beta > 1$, $\alpha = 1/2$. Hence in $O\big(\Re(\zeta)^{\frac{1}{3}-\epsilon}\big)$ neighbourhoods of the critical curve (as in (C.2)) we obtain the bounds $O(r^{2-\epsilon})$ and $O(r^{2-\frac{\epsilon}{2}})$ respectively.

Appendix D (with M. Zworski). In this appendix we will apply some of the methods of Sections 1 and 4 to restate and slightly improve the results of [SZ2]. We start by recalling briefly the notation used in that paper (some of which appeared already in Section 3). We put $\zeta_j(x',\xi') = \big(2Q(x',\xi')\big)^{2/3}\zeta_j$, where $-\zeta_2 < -\zeta_1 < 0$ are the first two zeros of the Airy function and Q is the dual second fundamental form on $T^*\partial O$. For a coordinate patch Ω of ∂O, identified with a subset of $\mathbb{R}^{n-1}$, we let $\gamma(x', hD_{x'}) : L^2\big(\Omega\times[0,\infty)\big) \longrightarrow L^2(\mathbb{R}^{n-1})$ be an h-pseudodifferential operator (of class $S^0_{0,0}$) corresponding to the projection onto the first eigenspace of $(hD_t)^2 + 2tQ(x',\xi')$ with the fibre variables, ξ', cut-off to a compact region. We denote by T the standard FBI transform from $L^2(\mathbb{R}^{n-1}) \longrightarrow L^2_\Phi(\mathbb{C}^{n-1})$, with the phase $i(z-x)^2/2$, $\Phi(z) = |\Im z|^2/2$ (in Section 6 of [SZ2], by a slight abuse of notation, T is used also for $I\otimes T : L^2\big([0,\infty)\times\mathbb{R}^{n-1}\big) \longrightarrow L^2\big([0,\infty), L^2_\Phi(\mathbb{C}^{n-1})\big)\big)$, $z = x' - i\xi'$, $(x',\xi') \in T^*\partial O$.

The bound (6.3) of [SZ2] gave the local C^∞ analogue of (3.23) above: for $u \in C_0^\infty([0,\infty)\times\Omega)$, $u|_{\partial O} = 0$ and $\mu < \min_{\Sigma_{\omega_0}} \zeta_2 \cos(\pi/6) - 1/C$,

$$
\begin{aligned}
\|(P-\omega_0)u\|^2 \geq \Big((r_0 + \mu h^{2/3})^2 - O(h)\Big)\|u\|^2 \\
- \int_{\Omega\times\mathbb{R}^{n-1}} \tilde q(x',\xi')|T\gamma(x',hD_{x'})u|^2 e^{-|\xi'|^2/h}\,dx'\,d\xi',
\end{aligned}
\tag{D.1}
$$

$$
\tilde q(x',\xi') = 2\chi_2(\xi')^2\Re\Big(\big(R(x',\xi')-\overline{\omega_0}\big)e^{-2\pi i/3}\Big)
$$

$$
\left(\frac{\mu h^{\frac23}}{\cos(\pi/6)} - \frac{\big(R(x',\xi')-\Re\omega_0\big)^2}{\big(2r_0\cos(\pi/6)\big)} - \zeta_1(x',\xi')h^{2/3} + O(h)\right)_+,
$$

where $\Sigma_0 = \{\alpha \in T^*\partial O : R(\alpha) = \Re\omega_0\}$, $\chi_2 \in C_0^\infty(\mathbb{R}^{n-1},[0,1])$, with support in $1/(2C) < |\xi'| < 2C$ and equal to 1 in $1/C < |\xi'| < C$, and where we used (6.7) of [SZ2].

Replacing T by a global FBI-transformation given by (1.9) with the phase (1.5), mapping $L^2(\partial O)$ to $L^2(T^*\partial O)$, and γ by an h-pseudodifferential operator in $\Psi^0_{0,0}(\partial O)$ obtained from using the globally defined symbol $\gamma(x',\xi')$, produces an error $O(h)\|u\|^2$. Thus we obtain a global version of (D.1) which is the needed analogue of (3.23):

$$
\|(P-\omega_0)u\|^2 \geq \Big((r_0+\mu h^{2/3})^2 - O(h)\Big)\|u\|^2 - \int_{T^*\partial O} \tilde q(\alpha)\Big|T\big(\gamma(x',hD_{x'})u\big)(\alpha)\Big|^2\,d\alpha. \tag{D.2}
$$

Proceeding in the spirit of Section 4 above but still by the methods of [SZ2] we obtain a more precise version of the estimate (7.8) there:

(D.3)

$$
\begin{aligned}
\#\Big\{z : z \text{ an eigenvalue of } P, \frac12 \leq \Re z \leq \frac32, -\Im z \leq 2C_{0,\infty}(\Re z)^{2/3}h^{2/3} + \frac12\mu(\Re z)^{2/3}h^{2/3}\Big\} \\
\leq Ch^{-(n-1)}\int_{S^*\partial O}\Big(1-\cos(\pi/6)\mu^{-1}(\zeta_1-\zeta_{1,\min})\Big)_+^{\frac12}\,dS,
\end{aligned}
$$

$h^{1/3} \ll \mu < \cos(\pi/6)(\zeta_{2,\min} - \zeta_{1,\min}) - 1/C$, and where $\zeta_{j,\min} = \min_{S^*\partial O}\zeta_j$ and dS the Liouville measure on $S^*\partial O$. To see (D.3) we use Lemma 6.1 and (D.1) (formula (6.3) of that paper) as in the proof of Lemma 6.2 of [SZ2] to obtain, for $h^{1/3} \ll \tilde\mu - \zeta_{1,\min}\cos(\pi/6)(\Re\omega_0)^{2/3}$, $\tilde\mu \leq \zeta_{2,\min}\cos(\pi/6)(\Re\omega_0)^{2/3} - 1/C$,

$$
\begin{aligned}
\|(P-\omega_0)u\|^2 \geq \Big((r_0+\tilde\mu h^{2/3})^2 + \delta\big(\tilde\mu - \zeta_{1,\min}(\Re\omega_0)^{2/3}\cos(\pi/6)\big)h^{2/3} - O(h)\Big) \\
\|u\|^2 - \langle Q_\delta u, u\rangle,
\end{aligned}
\tag{D.4}
$$

$$
\operatorname{rank}(Q_\delta) \leq \frac{C}{\delta}h^{-(n-1)+1/3}\int_{\Sigma_{\omega_0}}\big(\tilde\mu - \zeta_1\cos(\pi/6)\big)_+^{1/2}\,dS_{R,\Sigma_{\omega_0}},
$$

where $dS_{R,\Sigma_{\omega_0}}$ is the Liouville measure on Σ_{ω_0} as defined before (3.17). This gives an estimate for the number of eigenvalues of P in the region

$$
\begin{aligned}
|z-\omega_0| < r_0 + \zeta_{1,\min}(\Re\omega_0)^{2/3}\cos(\pi/6)h^{2/3} + \frac12\big(\tilde\mu \\
- \zeta_{1,\min}(\Re\omega_0)^{2/3}\cos(\pi/6)\big)h^{2/3}, \quad r_0 < \frac12\Re\omega_0,
\end{aligned}
$$

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$$\tilde{\mu} - \zeta_{1,\min}(\Re\omega_0)^{2/3}\cos(\pi/6) \gg h^{1/3}, \quad \tilde{\mu} \le \zeta_{2,\min}(\Re\omega_0)^{2/3}\cos(\pi/6) - \frac{1}{C},$$

by

(D.5)
$$Ch^{-(n-1)+\frac{1}{3}}\left(\tilde{\mu} - \zeta_{1,\min}(\Re\omega_0)^{2/3}\cos(\pi/6)\right)^{1/2}\int_{\Sigma_{\omega_0}} \left(1 - \frac{\zeta_1 - \zeta_{1,\min}(\Re\omega_0)^{2/3}}{\tilde{\mu}/\cos(\pi/6) - \zeta_{1,\min}(\Re\omega_0)^{2/3}}\right)_+^{\frac{1}{2}} dS_{R,\Sigma_{\omega_0}}.$$

The estimate (D.3) follows by putting $\tilde{\mu} = \zeta_{1,\min}(\Re\omega_0)^{2/3}\cos(\pi/6)+(\Re\omega_0)^{2/3}\mu$, a covering argument (see Figure 2 of [SZ2] and Figure C.3 above) and noticing that $\Re\omega_0$ scales out of the integral in (D.5). We can now state a generalization of Theorem 2 of [SZ2]:

THEOREM D.1. *The number of scattering poles of ∂O in*

(D.6)
$$\begin{cases} 1 \le \Re\zeta \le r \\ -\Im\zeta \le C_{0,\infty}(\Re\zeta)^{1/3}\left(1 + c(\Re\zeta)^{-\beta}\right), \quad 0 \le \beta \le \frac{1}{3} \end{cases}$$

is bounded by

(D.7)
$$C\int_0^{Cr}\int_{S^*\partial O}\left(1 - x^\beta(Q - \min_{S^*\partial O} Q)\right)_+^{\frac{1}{2}} dS x^{n-2}\,dx,$$

where Q is the dual second fundamental form on $T^*\partial O$ and C depends only on O.

PROOF. We apply the dyadic decomposition argument of the proof of Theorem 2 in [SZ2] and the estimate (D.5). Then (D.7) follows as for $m \ge 0$, $K = [\log_2 r]$ we have $\sum_{k=0}^{K}(2^{-k}r)^{n-1}\left(1 - (2^{-k}r)^\beta m\right)_+^{\frac{1}{2}} \le C\int_0^r(1 - x^\beta m)_+^{\frac{1}{2}} x^{n-2}\,dx$ and $Q^{2/3} - \min_{S^*\partial O} Q^{2/3} \sim Q - \min_{S^*\partial O} Q$. ∎

We can also use (D.2) exactly as in Section 4 above and that gives

THEOREM D.2. *Let $C_0 > 0$, $r_0 > 0$. For $\mu \ge 0$ with*

$$\cos\frac{\pi}{6}\zeta_{1,\min} + 3\left(\mu - \cos\frac{\pi}{6}\zeta_{1,\min}\right) \le \cos\frac{\pi}{6}\zeta_{2,\min} - 1/C_0,$$

and $k_0 \ge k_0(C, r_0) > 0$, the number of resonances k with $\Im k \ge f_{k_0,r_0,\mu}(\Re k)$, is less than or equal to

(D.8)
$$\frac{\sqrt{2r_0}}{(2\pi)^{n-1}}k_0^{n-1-\frac{1}{3}}\left(\int_{S^*\partial O}\left(3\left(\mu - \cos\frac{\pi}{6}\zeta_{1,\min}\right) - \left(\cos\frac{\pi}{6}\zeta_1 - \cos\frac{\pi}{6}\zeta_{1,\min}\right)\right)_+^{\frac{1}{2}} dS + o(1)\right),$$

as $k_0 \to \infty$, uniformly with respect to μ and $f_{k_0,r_0,\mu}$ is the unique quadratic polynomial, such that the parabola $\Im k = f_{k_0,r_0,\mu}(\Re k)$ passes through the three points $k_0 \pm \sqrt{\frac{\mu r_0}{2}}k_0^{\frac{2}{3}}$, $k_0 - i\frac{\mu}{2}k_0^{\frac{1}{3}}$.

The estimate (D.8) is much more accurate than (D.7) when μ is independent of k_0.

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