Canad. Math. Bull. Vol. **44** (3), 2001 pp. 355–369

Hilbert Bimodules with Involution

Nik Weaver

Abstract. We examine Hilbert bimodules which possess a (generally unbounded) involution. Topics considered include a linking algebra representation, duality, locality, and the role of these bimodules in noncommutative differential geometry

Many Hilbert modules which arise in practice carry natural involutions, typically deriving from the involutions of C^*-algebras involved in their construction. Usually these Hilbert module involutions are not only non-isometric, they are unbounded—possibly even if the module is finitely generated.

The same examples generally also have a bimodule structure which interacts with, and may be recovered from, the involution via the equation $(ax)^* = x^*a^*$, where a is an element of the C^*-algebra and x is an element of the Hilbert module. Thus, involutions are closely related to bimodule structure. (But these are very different from the sort of bimodules that arise in the context of Morita equivalence [16].)

Philosophically, if one regards Hilbert modules as "noncommutative complex Hilbert bundles" ([17], [21], [22]) then Hilbert bimodules with involution may be seen as "noncommutative real Hilbert bundles." Indeed, assuming a locality condition, in the commutative case the extra structure provided by the involution corresponds precisely to a real structure on the corresponding bundle (Theorem 11).

The motivating commutative example is the tangent bundle TX of a Riemannian manifold X. Since the tangent space at each point carries an inner product, the space $S_0(TX)$ of continuous sections of TX which vanish at infinity has a $C_0(X)$-valued inner product given by $\langle \phi, \psi \rangle(x) = \langle \phi(x), \psi(x) \rangle$. If we are using complex scalars, we must complexify $S_0(TX)$ to make it a module over $C_0(X)$, and we must also extend the inner product to the complexification. But in addition the complexified module now has an involution given by $\phi_0 + i\phi_1 \mapsto \phi_0 - i\phi_1$. Thus, in the noncommutative setting ([5], [18], [24]) we expect a description of "noncommutative Riemannian structure" to involve involutive Hilbert bimodules.

This description should also include an "exterior derivative" realized as an unbounded self-adjoint derivation from the C^*-algebra into the bimodule. Therefore both the $*$-operation and the bimodule structure are important.

It was argued in [14] that operator modules are the correct noncommutative version of complex Banach bundles. Thus self-adjoint operator bimodules may be seen as a noncommutative version of real Banach bundles. But we do not pursue this issue here.

The proof of Theorem 8 was supplied by Charles Akemann, using the powerful excision technique of [2].

Received by the editors August 31, 1999.
AMS subject classification: Primary: 46L08; secondary: 46L57, 46L87.

1 Preliminaries

We will use basic facts about Hilbert modules without comment; see [11], [12], or [15] for background.

In the following definition we require the involution to be defined everywhere. Thus, we get around the unboundedness problem mentioned in the introduction by eliminating that portion of the module on which the involution is not defined. This means that what we call the left and right seminorms, $\|\cdot\|_l$ and $\|\cdot\|_r$, in general cannot be complete. However, there is no obstruction to completeness of the max norm $\|\cdot\|_m$ and in general this seems to be the appropriate requirement.

Definition 1 Let A be a pre-C^*-algebra. A *pre-Hilbert $*$-bimodule over A* is an A-A-bimodule E together with an A-bilinear map $\langle\cdot\,,\,\cdot\rangle\colon E \times E \to A$ and an antilinear involutive map ${}^*\colon E \to E$ such that

(a) $\langle x, y\rangle^* = \langle y^*, x^*\rangle$,
(b) $(ax)^* = x^*a^*$, and
(c) $\langle x, x^*\rangle \geq 0$

hold for all $a \in A$ and $x, y \in E$. We define sesquilinear A-valued inner products $\langle x, y\rangle_l = \langle x, y^*\rangle$ and $\langle x, y\rangle_r = \langle x^*, y\rangle$ and seminorms $\|x\|_l^2 = \|\langle x, x\rangle_l\|$, $\|x\|_r^2 = \|\langle x, x\rangle_r\|$, and $\|x\|_m = \max(\|x\|_l, \|x\|_r)$. If A is a C^*-algebra, we say E is a *Hilbert $*$-bimodule over A* provided $\|\cdot\|_m$ is a complete norm. ■

By A-bilinearity we mean that $\langle ax, y\rangle = a\langle x, y\rangle$, $\langle xa, y\rangle = \langle x, ay\rangle$, and $\langle x, ya\rangle = \langle x, y\rangle a$ for $a \in A$ and $x, y \in E$.

In most cases A will be complete from the start, but in one or two places we will want to allow it to be incomplete. (We never need to allow nonzero elements in A to have zero norm, however.) To make sense of axiom (c), we take positivity in an incomplete algebra to mean positivity in its completion.

If A is complete then E is a left pre-Hilbert A-module with respect to the inner product $\langle\cdot\,,\,\cdot\rangle_l$ and a right pre-Hilbert A-module with respect to $\langle\cdot\,,\,\cdot\rangle_r$. In any case $\|\cdot\|_l$ and $\|\cdot\|_r$ are seminorms by the same argument which shows this for ordinary Hilbert modules.

Note also that $\|x\|_l = \|x^*\|_r$ for any $x \in E$; in particular, if x is self-adjoint then $\|x\|_l = \|x\|_r = \|x\|_m$.

Our first order of business is to show that pre-Hilbert $*$-bimodules can always be completed to Hilbert $*$-bimodules.

Lemma 2 *Let A be a pre-C^*-algebra and E a pre-Hilbert $*$-bimodule over A. Then for any $a \in A$ and $x \in E$ we have*

$$\|ax\|_l, \|xa\|_l \leq \|a\|\,\|x\|_l \quad \textit{and} \quad \|ax\|_r, \|xa\|_r \leq \|a\|\,\|x\|_r.$$

Proof First, we have

$$\|ax\|_l^2 = \|\langle ax, (ax)^*\rangle\| = \|a\langle x, x^*\rangle a^*\| \leq \|a\|^2\|x\|_l^2,$$

so $\|ax\|_l \le \|a\|\,\|x\|_l$. To see that $\|xa\|_l \le \|a\|\,\|x\|_l$, observe that $b \le c$ implies $\langle xb, x\rangle_l \le \langle xc, x\rangle_l$ since

$$\langle x(c-b), x\rangle_l = \langle x(c-b)^{1/2}, x(c-b)^{1/2}\rangle_l \ge 0.$$

Without loss of generality suppose $\|a\| \le 1$; then letting $b = aa^*$ we have $b^2 \le b$ and

$$\begin{aligned} 0 &\le \langle x - xb, x - xb\rangle_l \\ &= \langle x, x\rangle_l - 2\langle xb, x\rangle_l + \langle xb^2, x\rangle_l \\ &\le \langle x, x\rangle_l - \langle xb, x\rangle_l \end{aligned}$$

so that $\|xa\|_l^2 = \|\langle xb, x\rangle_l\| \le \|x\|_l^2$ as desired. Taking adjoints yields the same inequalities for the norm $\|\cdot\|_r$. ■

Proposition 3 *Let E be a pre-Hilbert $*$-bimodule over a pre-C^*-algebra A and let $N = \{x \in E : \|x\|_m = 0\}$. Then N is a sub-bimodule of E and the inner product and involution on E descend to E/N and extend to the completion of E/N. The completion of E/N is a Hilbert $*$-bimodule over the completion of A.*

Proof By Lemma 2, N is a sub-bimodule of E and the left and right actions of A on E/N extend continuously to its completion for $\|\cdot\|_m$; applying Lemma 2 again allows us to extend the module actions to the completion of A. The inner product descends to E/N and then extends to its completion by the Cauchy-Schwarz inequality for ordinary Hilbert modules, and the corresponding assertions for the involution are trivial. Axioms (a) to (c) of Definition 1 all hold in the completion by continuity. ■

Next, we briefly consider bounded module maps on Hilbert $*$-bimodules.

Definition 4 Let E be a Hilbert $*$-bimodule over a C^*-algebra A. We let E_l and E_r denote the set E considered respectively as a left or right pre-Hilbert A module with the left or right inner product described in Definition 1. If F is another Hilbert $*$-bimodule over A then we define $B_l(E, F)$ to be the set of all left A-linear adjointable maps from E_l into F_l such that

$$\|T\| = \sup_{x\in E}\{\|Tx\|_l/\|x\|_l, \|Tx\|_r/\|x\|_r\} < \infty,$$

assuming the convention $0/0 = 0$.

We define $B_r(E, F)$ similarly as the right A-linear adjointable maps from E_r to F_r and write $B_l(E) = B_l(E, E)$ and $B_r(E) = B_r(E, E)$. ■

For $a \in A$ define $L_a, R_a \colon E \to E$ by $L_a x = ax$ and $R_a x = xa$.

Proposition 5 *Let A be a C^*-algebra and let E and F be Hilbert $*$-bimodules over A. Then $B_l(E, F)$ is naturally isometrically anti-isomorphic to $B_r(E, F)$. Furthermore, $B_l(E)$ and $B_r(E)$ are C^*-algebras, and the left and right representations $a \mapsto L_a$, $a \mapsto R_a$ are $*$-homomorphisms from A into $B_r(E)$ and $B_l(E)$, respectively.*

https://doi.org/10.4153/CMB-2001-036-8 Published online by Cambridge University Press

Proof Given $T\colon E \to F$ define $\widehat{T}\colon E \to F$ by $\widehat{T}(x) = (Tx^*)^*$. If $T \in B_l(E,F)$ then $\widehat{T}(xa) = (\widehat{T}x)a$, so $\widehat{T}$ is right A-linear, and

$$\langle \widehat{T}x, y\rangle_r = \langle T(x^*), y\rangle = \langle T(x^*), y^*\rangle_l = \langle x^*, T^*y^*\rangle_l = \langle x, \widehat{T^*}y\rangle_r,$$

so $\widehat{T}$ is adjointable. Thus the map $T \mapsto \widehat{T}$ is a linear anti-isomorphism from $B_l(E,F)$ into $B_r(E,F)$, and since norms are computed the same way in each case the correspondence is isometric.

It is trivial to check that $B_l(E)$ is complete. Let $B_l^0(E)$ be the pre-C^*-algebra of left A-linear adjointable maps bounded only for the seminorm $\|\cdot\|_l$, and define $B_r^0(E)$ similarly using only the seminorm $\|\cdot\|_r$. (These spaces may not be complete because E need not be complete for the left and right seminorms separately.) The map $T \mapsto T \oplus \widehat{T}^*$ isometrically embeds $B_l(E)$ as a C^*-subalgebra of $B_l^0(E) \oplus B_r^0(E)$. Since $B_r(E)$ is the opposite algebra of $B_l(E)$ it is a C^*-algebra too.

Lemma 2 shows that $L_a \in B_r(E)$ and $R_a \in B_l(E)$ for all $a \in A$. Linearity and multiplicativity of the representations are clear. Preservation of adjoints follows from the calculation

$$\langle ax, y\rangle_r = \langle x^*a^*, y\rangle = \langle x^*, a^*y\rangle = \langle x, a^*y\rangle_r.$$

(Hence $L_{a^*} = L_a^*$, and $R_{a^*} = R_a^*$ is proven analogously.) ∎

2 Simple Examples

We now list several simple examples of Hilbert $*$-bimodules.

Example 6 (The case** $A = \mathbf{C}$**) For X any measure space, $L^2(X)$ has a canonical bilinear $\mathbf{C}$-valued map $(f,g) \mapsto \int fg$ and involution $f \mapsto \bar{f}$.

If H is any real Hilbert space then the complex Hilbert space $H+iH$ has the bilinear form

$$(v_1 + iw_1, v_2 + iw_2) \mapsto \langle v_1, v_2\rangle + i\langle w_1, v_2\rangle + i\langle v_1, w_2\rangle - \langle w_1, w_2\rangle$$

and carries the natural involution $v + iw \mapsto v - iw$.

Let H be a complex Hilbert space and let H^* be its dual space; for $v \in H$ write δ_v for the linear functional $w \mapsto \langle w, v\rangle$. Then $H \oplus H^*$ with bilinear form

$$(v_1 \oplus \delta_1, v_2 \oplus \delta_2) = \delta_2(v_1)$$

and involution $v \oplus \delta_w \mapsto w \oplus \delta_v$ provides a simple example where the seminorms $\|\cdot\|_l$ and $\|\cdot\|_r$ do not agree. Indeed $\|v \oplus 0\|_l = \|v\|$ and $\|v \oplus 0\|_r = 0$ for any $v \in H$.

If A is any C^*-algebra and ϕ is a state on A, then the bilinear map $(x,y) \mapsto \phi(xy)$ together with the original involution on A makes A a pre-Hilbert $*$-bimodule over $\mathbf{C}$. This is just the GNS construction and here the left and right seminorms coincide only if ϕ is a trace. Similarly, if M is a von Neumann algebra and ϕ is a weight on M then the same prescription makes a pre-Hilbert $*$-bimodule over $\mathbf{C}$ of the set $\{x \in M : \phi(xx^*), \phi(x^*x) < \infty\}$. ∎

https://doi.org/10.4153/CMB-2001-036-8 Published online by Cambridge University Press

Example 7 (Examples derived from tensor products) Let A be a C^*-algebra, I a closed ideal of A, and H a real Hilbert space. Then the algebraic tensor product $I \otimes_{\mathbf{R}} H$ is a pre-Hilbert $*$-bimodule over A. The bilinear map into A is defined by $(a \otimes v, b \otimes w) \mapsto \langle v, w\rangle ab$ and the involution by $(a \otimes v)^* = a^* \otimes v$. Taking $H = l^2(\mathbf{N})$ and $I = A$ in this construction yields the uncompleted "standard" Hilbert module $l^2(A)$. Completing recovers precisely the sequences $(a_n) \subset A$ such that $\sum a_n^* a_n$ and $\sum a_n a_n^*$ both converge in norm. ■

In Example 7, if $I = A$ and $H = \mathbf{R}$ then the left and right seminorms coincide since $\|xx^*\| = \|x^*x\|$. However, in general they disagree, even for $H = \mathbf{R}^2$. In fact this happens whenever A is noncommutative, by the following theorem of Akemann [1].

Theorem 8 **(Akemann)** *Let A be a C^*-algebra that is not commutative. Then there exist $x, y \in A$ such that $\|xx^* + yy^*\| \neq \|x^*x + y^*y\|$.*

Proof Since A is not commutative there exists a pure state f which gives rise to an irreducible representation on a Hilbert space of dimension at least two. Let (a_α) be a decreasing net of positive norm one elements which satisfies $f(a_\alpha) = 1$ for all α and excises f ([2], Proposition 2.2). This means that

$$\|a_\alpha b a_\alpha - f(b)a_\alpha^2\| \to 0$$

for every $b \in A$.

By Kadison's transitivity theorem ([13], Theorem 2.7.5) we can find a unitary $u \in A$ such that $f(u^* a_{\alpha_0} u) < 1$ for some fixed α_0. Define $b_\alpha = u^* a_\alpha u$. Then $\|f(b_{\alpha_0})a_\alpha^2\| < 1$ and so the excision condition implies that $\|a_\beta b_{\alpha_0} a_\beta\| < 1$ for some $\beta > \alpha_0$. Since the net is decreasing, it follows that $\|a_\beta b_\beta a_\beta\| < 1$. Thus $\|a_\beta b_\beta^{1/2}\|, \|b_\beta^{1/2} a_\beta\| < 1$, hence $\|a_\beta b_\beta\|, \|b_\beta a_\beta\| < 1$.

Set $x = u^* a_\beta$ and $y = a_\beta$. Then

$$\|x^*x + y^*y\| = \|a_\beta^2 + a_\beta^2\| = 2$$

but

$$\begin{aligned}\|xx^* + yy^*\| &= \|b_\beta^2 + a_\beta^2\| \leq \|b_\beta + a_\beta\| \\ &= \|b_\beta^2 + b_\beta a_\beta + a_\beta b_\beta + a_\beta^2\|^{1/2} \\ &< (1 + 1 + 1 + 1)^{1/2} = 2.\end{aligned}$$

Thus $\|x^*x + y^*y\| \neq \|xx^* + yy^*\|$. ■

Example 9 (Examples arising from maps into subalgebras) If $B \subset A$ are C^*-algebras and $\phi \colon A \to B$ is a conditional expectation, then A is a B-B-bimodule and the maps $(x, y) \mapsto \phi(xy)$ and $x \mapsto x^*$ make it a pre-Hilbert $*$-bimodule.

https://doi.org/10.4153/CMB-2001-036-8 Published online by Cambridge University Press

Similarly, if $N \subset M$ are von Neumann algebras and $\phi \colon M_+ \to \widehat{N_+}$ is an operator-valued weight [7], then $E = \{x \in M : \|\phi(x^*x)\|, \|\phi(xx^*)\| < \infty\}$ is a pre-Hilbert $*$-bimodule over N with bilinear form $(x, y) \mapsto \hat{\phi}(xy)$, where $\hat{\phi}$ is the unique linear extension of ϕ. ■

Example 10 (The local commutative case) Let X be a locally compact space and let B be a Fell bundle of real Hilbert spaces over X [8]. Let $S_0(B)$ be the space of continuous sections of B which vanish at infinity, and let $S_0^{\mathbf{C}}(B) = S_0(B) + iS_0(B)$ be its complexification. Then $S_0^{\mathbf{C}}(B)$ is a bimodule over $C_0(X)$ with coincident left and right actions given by fiberwise multiplication. The involution on $S_0^{\mathbf{C}}(B)$ is defined by $(f + ig)^* = f - ig$ and the bilinear form by bilinear extension of the fiberwise inner product on $S_0(B)$. ■

An important special case of Example 10 arises when X is a Riemannian manifold and B is its tangent bundle, as mentioned in the introduction. This has a simple noncommutative generalization when we have an action of a real Lie group G on a C^*-algebra A [5]. Fixing an inner product on the Lie algebra g of G, the Hilbert $*$-bimodule $E = A \otimes_{\mathbf{R}} g$ formed as in Example 7 plays the role of the tangent bimodule. Sauvageot's construction generalizes this class of examples (see Section 4).

Example 10 has the following converse.

Theorem 11 *Let $A = C_0(X)$ be a commutative C^*-algebra and let E be a Hilbert $*$-bimodule over A. Suppose that any inner product of self-adjoint elements of E is a self-adjoint element of A. Then there is a Fell bundle B of real Hilbert spaces over X such that $E \cong S_0^{\mathbf{C}}(B)$.*

Proof Let Y be the one-point compactification of X and let $A^\sim = C(Y)$. Then E is also a Hilbert $*$-bimodule over $A^\sim$. We want to show that the left and right actions of $A^\sim$ on E coincide. Fix $p \in Y$. For any $a \in A^\sim$ such that $a \geq 0$ and $a(p) = 0$, and any $x \in E$ such that $x = x^*$, set $b = a^{1/2}$; then

$$\begin{aligned}\langle bx + xb, bx + xb\rangle(p) &= b(p)\langle x, bx\rangle(p) + b(p)\langle x, xb\rangle(p)\\ &\quad + \langle xb, bx\rangle(p) + \langle xb, x\rangle(p)b(p)\\ &= \langle xb, bx\rangle(p),\end{aligned}$$

so that $\langle x, ax\rangle(p) \geq 0$. But also

$$\begin{aligned}\langle bx + xb, ibx - ixb\rangle(p) &= ib(p)\langle x, bx\rangle(p) - ib(p)\langle x, xb\rangle(p) + i\langle xb, bx\rangle(p)\\ &\quad - i\langle xb, x\rangle(p)b(p)\\ &= i\langle xb, bx\rangle(p).\end{aligned}$$

Since both $bx + xb$ and $ibx - ixb$ are self-adjoint, so is their inner product, so this computation shows that $\langle x, ax\rangle(p)$ must be purely imaginary. But we already showed that it is real, so it follows that $\langle x, ax\rangle(p) = 0$. By linearity, we have $\langle x, ax\rangle(p) = 0$ for any $a \in A^\sim$ such that $a(p) = 0$.

https://doi.org/10.4153/CMB-2001-036-8 Published online by Cambridge University Press

Now let $a \in A^{\sim}$ and $x \in E$ and suppose a is real and $x = x^*$. Since the quantity

$$\langle ax + xa, iax - ixa \rangle = i\langle x, a^2x \rangle - ia^2\langle x, x \rangle$$

is real and $a^2\langle x, x\rangle$ is also real, it follows that $\mathrm{Re}\langle x, a^2x\rangle = a^2\langle x, x\rangle$. Therefore

$$\begin{aligned}\langle ax - xa, ax - xa \rangle &= \mathrm{Re}\langle ax - xa, ax - xa \rangle \\ &= -a^2\langle x, x\rangle + 2\,\mathrm{Re}\, a\langle x, ax\rangle - \mathrm{Re}\langle x, a^2x\rangle \\ &= 2\,\mathrm{Re}(a^2\langle x, x\rangle - a\langle x, ax\rangle),\end{aligned}$$

and evaluating this expression at $p \in Y$ yields

$$2\,\mathrm{Re}\Big(a(p)\,\big\langle x, \big(a(p) - a\big)x\big\rangle\,(p)\Big),$$

which is zero by the last paragraph. Since this is true for all $p \in Y$, we have $ax - xa = 0$ as desired. Taking linear combinations, we conclude that this is true for any $a \in A^{\sim}$ and $x \in E$.

Define $E_{\mathrm{sa}} = \{x \in E : x = x^*\}$, so that $E = E_{\mathrm{sa}} + iE_{\mathrm{sa}}$. Then E_{sa} is a real Hilbert module over $C(Y; \mathbf{R})$, hence $E_{\mathrm{sa}} \cong S(B)$ for some real Hilbert Fell bundle over Y [22]. Since the inner product on E takes values in $C_0(X)$, this bundle must have zero fiber over the point at infinity, so actually $E_{\mathrm{sa}} \cong S_0(B)$ for some Hilbert Fell bundle B over X. Thus $E \cong S_0^{\mathbf{C}}(B)$. ■

The left and right actions do not coincide in general in the commutative case; for instance consider a module constructed as in Example 9 using a conditional expectation from a C^*-algebra onto a commutative but not central subalgebra. Also, the case of $H \oplus H^*$ in Example 6 shows that the reality condition on inner products can fail even if the left and right actions agree. However, the construction in Example 10 always satisfies the reality condition, so Theorem 11 is a true converse of it.

3 Operator Representation

Let A be a C^*-algebra and E a Hilbert $*$-bimodule over A. The following is the appropriate version of the linking algebra construction in this setting. Let $N = \{x \in E : \|x\|_r = 0\}$ and let E_r be the completion of E/N for $\|\cdot\|_r$, so that E_r is a right Hilbert module over A. The A-valued inner product on E_r extends $\langle\cdot\,,\cdot\rangle_r$, and we use the same notation for the extension. Then define $F = A \oplus E$ to be the direct sum of right Hilbert A-modules.

Let $B(F)$ be the space of bounded adjointable right A-linear maps from F to itself. This is a C^*-algebra. Then define $\phi\colon A \to B(F)$ by

$$\phi(a)(b \oplus y) = ab \oplus ay$$

and $\psi\colon E \to B(F)$ by

$$\psi(x)(b \oplus y) = \langle x^*, y\rangle_r \oplus xb.$$

https://doi.org/10.4153/CMB-2001-036-8 Published online by Cambridge University Press

A number of simple facts need to be verified. First, for any $a \in A$ the map $\phi(a)$ is bounded by Lemma 2, and a short computation shows that $\phi(a)^* = \phi(a^*)$, so ϕ is an injective $*$-homomorphism from A into $B(F)$. For any $x \in E$ the map $\psi(x)$ is clearly linear, and it is bounded because

$$\begin{aligned}\langle \psi(x)(b \oplus y), \psi(x)(b \oplus y)\rangle_r &= \langle y, x^*\rangle_r \langle x^*, y\rangle_r + \langle xb, xb\rangle_r \\ &\leq \|x^*\|_r^2 \langle y, y\rangle_r + \|x\|_r^2 b^* b \\ &\leq \|x\|_m^2 (\langle y, y\rangle_r + b^* b).\end{aligned}$$

This actually shows that $\|\psi(x)\| \leq \|x\|_m$, and the converse inequality follows from the computations

$$\langle \psi(x)(\langle x, x\rangle_r \oplus 0), \psi(x)(\langle x, x\rangle_r \oplus 0)\rangle_r = \langle x, x\rangle_r^3$$

(hence $\|\psi(x)\| \geq \|x\|_r$) and

$$\langle \psi(x)(0 \oplus x^*), \psi(x)(0 \oplus x^*)\rangle_r = \langle x^*, x^*\rangle_r^2$$

(hence $\|\psi(x)\| \geq \|x^*\|_r = \|x\|_l$). Also, $\psi(x)$ is adjointable and in fact $\psi(x)^* = \psi(x^*)$ by the computations

$$\begin{aligned}\langle \psi(x)(b \oplus y), (c \oplus z)\rangle_r &= \langle \langle x^*, y\rangle_r \oplus xb, c \oplus z\rangle_r \\ &= \langle y, x^*\rangle_r c + \langle xb, z\rangle_r\end{aligned}$$

and

$$\begin{aligned}\langle (b \oplus y), \psi(x^*)(c \oplus z)\rangle_r &= \langle b \oplus y, \langle x, z\rangle_r \oplus x^* c\rangle_r \\ &= b^* \langle x, z\rangle_r + \langle y, x^* c\rangle_r.\end{aligned}$$

Finally, we note that $\phi(a)\psi(x) = \psi(ax)$ and $\psi(x)\phi(a) = \psi(xa)$; these are trivially verified. We list the preceding facts in the following theorem.

Theorem 12 *Let A be a C^*-algebra and E a Hilbert $*$-bimodule over A. Then $\phi\colon A \to B(F)$ is an isometric $*$-homomorphism, $\psi\colon E \to B(F)$ is an isometric linear embedding, and for every $a \in A$ and $x, y \in E$ we have $\phi(a)\psi(x) = \psi(ax)$, $\psi(x)\phi(a) = \psi(xa)$, and $\psi(x)^* = \psi(x^*)$.* ■

Although in general $\phi(\langle x, y\rangle) \neq \psi(x)\psi(y)$, if A has a unit then we do have

$$\phi(\langle x, y\rangle)(1 \oplus 0) = \langle x, y\rangle \oplus 0 = \psi(x)\psi(y)(1 \oplus 0)$$

for all $x, y \in E$.

Of course, there is an analogous left module version of this section's construction.

4 Duality

If $A = M$ is a von Neumann algebra, it is natural to focus attention on Hilbert $*$-bimodules which are dual spaces. But given any Hilbert $*$-bimodule E over a von Neumann algebra, the linking algebra construction can be modified by using the dual module E_r' in place of E_r; this has the consequence that $B(F)$ is a von Neumann algebra [12], so that if E is not a dual space we can replace it with the weak* closure of $\psi(E)$ in $B(F)$. This is the idea behind the main result of this section. We need some terminology first.

Definition 13 Let E be a Hilbert $*$-bimodule over a von Neumann algebra M. We define the $*$-*weak topology* on E to be the weakest topology such that the maps $x \mapsto \langle x, y\rangle$ and $x \mapsto \langle y, x\rangle$ are continuous from E into M for all $y \in E$.

If the unit ball of E is $*$-weakly compact, we say that E is a *dual* bimodule. If for any bounded, ultraweakly convergent net $a_i \to a$ in M and any $x \in E$ we have $a_i x \to ax$ $*$-weakly, then we say that E is *normal*. Finally, if E is both normal and dual we call it a *W^* Hilbert $*$-bimodule.* ■

It is standard that if E is dual in the above sense then it is actually a dual Banach space. Thus, given Proposition 5, the following proposition is routinely verified.

Proposition 14 *If E is a dual Hilbert $*$-bimodule over a von Neumann algebra then $B_l(E)$ and $B_r(E)$ are von Neumann algebras. In either case a bounded net of operators (T_i) converges ultraweakly to T if and only if $T_i x \to Tx$ $*$-weakly for all $x \in E$.* ■

The normality condition is symmetric, because $a_i x \to ax$ $*$-weakly if and only if $x^* a_i^* \to x^* a^*$ $*$-weakly. Also, note that part of normality is automatic: if $a_i \to a$ then

$$\langle a_i x, y\rangle = a_i\langle x, y\rangle \to a\langle x, y\rangle = \langle ax, y\rangle$$

for all $x, y \in E$. But $\langle xa_i, y\rangle \to \langle xa, y\rangle$ need not always hold, for instance in the case of a module constructed as in Example 9 using a non-normal conditional expectation of von Neumann algebras.

If we modify the linking algebra construction by replacing E_r with E_r' as suggested above, normality of E is crucial because the map $\phi\colon M \to B(F)$ is then ultraweakly continuous if and only if E is a normal module.

We can now formalize the dualization procedure indicated at the start of this section.

Lemma 15 *Let M and N be von Neumann algebras and let A be an ultraweakly dense $*$-subalgebra of M. Suppose $\phi\colon A \to N$ is a bounded $*$-homomorphism, and $a_i \to 0$ boundedly and ultraweakly in $A \subset M$ implies $\phi(a_i) \to 0$ ultraweakly in N. Then ϕ extends to an ultraweakly continuous $*$-homomorphism from M to N.*

Proof Let M' be the ultraweak closure of $A' = \{a \oplus \phi(a) : a \in A\}$ in $M \oplus N$. Then the natural projection $\pi_M\colon M' \to M$ has zero kernel and hence is an isomorphism, and the map $\pi_N \circ \pi_M^{-1}\colon M \to N$ is ultraweakly continuous. It is clear that the restriction of this map to A agrees with ϕ. ■

Theorem 16 *Let M be a von Neumann algebra, let A be an ultraweakly dense $*$-subalgebra of M, and let E be a pre-Hilbert $*$-bimodule over A. Suppose that for any bounded net (a_i) in A and any $x, y \in E$, $a_i \to 0$ ultraweakly implies $\langle xa_i, y\rangle \to 0$ ultraweakly. Then E modulo its null space densely embeds in a unique W^* Hilbert $*$-bimodule over M.*

In particular, if E is a normal Hilbert $*$-bimodule over M then it densely embeds in a unique W^* Hilbert $*$-bimodule over M.

Proof First let $N = \{x \in E : \|x\|_m = 0\}$ and replace E with E/N. Now define E_r' to be the set of right A-linear maps from E into M which are bounded for the seminorm $\|\cdot\|_r$. It follows from [12] that E_r' is a self-dual right Hilbert module over M whose inner product extends $\langle \cdot\,,\,\cdot\rangle_r$ on E_r when $x \in E_r$ is identified with the map $y \mapsto \langle x, y\rangle_r$.

Let $F = M \oplus E_r'$ be the direct sum of right Hilbert modules and define maps $\phi\colon M \to B(F)$ and $\psi\colon E \to B(F)$ as in Section 3. For a bounded net (T_i) in $B(F)$, ultraweak convergence is equivalent to ultraweak convergence of $\langle T_i(x), y\rangle_r$ in M for all $x, y \in F$. Thus Lemma 15 implies that there is an ultraweakly continuous extension of $\phi|_A$ to M, and restriction of operators to $M \oplus 0$ shows that this extension must be ϕ. So ϕ is ultraweakly continuous.

Define E' to be the ultraweak closure of $\psi(E)$ in $B(F)$. This is a bimodule over $M \cong \phi(M)$ via operator multiplication, and normality and duality are trivial. It is also straightforward to check that the bimodule structure of E' extends that of $\psi(E) \cong E$. The inner product and adjoint can either be extended from E by continuity or defined directly by $\langle x, y\rangle = x\big(y(1 \oplus 0)\big)$ and operator adjoints.

For uniqueness, let E'' be any other bimodule with the same properties, and define a map $T\colon E' \to E''$ by $T(\lim_{E'} x_i) = \lim_{E''} x_i$ for any bounded universal net (x_i) in E. This map is well-defined and unitary since

$$\langle \lim_{E''} x_i, y\rangle = \lim_{M} \langle x_i, y\rangle = \langle \lim_{E'} x_i, y\rangle$$

for any $y \in E$, which is enough. ■

5 Locality

In this section we formulate a $*$-bimodule version of a condition on Hilbert modules which was independently introduced in [18] and [20]. The purposes to which it was put in these two papers were very different, and even the definitions are not obviously equivalent. (Their equivalence follows from Proposition 5.4.2 of [18].) Our interest in the condition is that it has strong consequences for the structure of the bimodule which are analogous to facts about self-dual Hilbert modules [12].

In [18] and [20] the property of interest was a C^* version of the centered condition given next. The appropriate $*$-bimodule property incorporates a self-adjointness requirement, which we identified in the commutative case in Theorem 11.

Definition 17 Let E be a W^* Hilbert $*$-bimodule over a von Neumann algebra M. The *center* of E is the set

$$Z(E) = \{x \in E : ax = xa \text{ for all } a \in M\}.$$

We say that E is *centered* if $MZ(E)$ is $*$-weakly dense in E. We say that E is *local* if it is centered and the inner product of any two self-adjoint elements of $Z(E)$ is self-adjoint in M. ■

In the next result we say that two subspaces $F, F' \subset E$ are *orthogonal* if $\langle x, y\rangle = \langle y, x\rangle = 0$ for all $x \in F$ and $y \in F'$. We also use the notation $Z(M)_{\text{sa}}$ or $Z(E)_{\text{sa}}$ for the set of self-adjoint elements in $Z(M)$ or $Z(E)$.

Theorem 18 *Let E be a local W^* Hilbert $*$-bimodule over a von Neumann algebra M and let F be a centered, $*$-weakly closed, self-adjoint sub-bimodule of E. Then there is another centered, $*$-weakly closed, self-adjoint sub-bimodule F' of E which is orthogonal to F and such that $E = F + F'$.*

Proof Let S be a subspace of $Z(F)_{\text{sa}}$ which is finitely generated as a module over $Z(M)_{\text{sa}}$. We claim that we can find a finite set $\{x_1, \ldots, x_n\}$ which spans S over $Z(M)_{\text{sa}}$ such that $\langle x_i, x_j\rangle = 0$ whenever $i \neq j$. To see this let $\{x_1, \ldots, x_n\}$ be any finite set which spans S and assume inductively that $\langle x_i, x_j\rangle = 0$ for $i, j \leq n-1$, $i \neq j$. Since

$$a\langle x, y\rangle = \langle ax, y\rangle = \langle xa, y\rangle = \langle x, ay\rangle = \langle x, ya\rangle = \langle x, y\rangle a$$

for any $x, y \in Z(F)$ and $a \in M$, it follows that the inner product of any two elements of $Z(F)$ is in $Z(M)$. Thus $Z(F)$ satisfies the hypothesis of Theorem 11 as a Hilbert $*$-bimodule over $Z(M)$. Using the conclusion of Theorem 11 it is easy to verify that

$$y = x_n - \sum_{i=1}^{n-1} \frac{\langle x_n, x_i\rangle}{\langle x_i, x_i\rangle} x_i,$$

is well-defined and orthogonal to x_i ($i \leq n-1$) and $\{x_1, \ldots, x_{n-1}, y\}$ spans S. This proves the claim.

Now fix $y \in Z(E)_{\text{sa}}$. For any $S \subset Z(F)_{\text{sa}}$ as above, let $\{x_1, \ldots, x_n\}$ verify the claim and define

$$y_S = \sum_{i=1}^{n} \frac{\langle y, x_i\rangle}{\langle x_i, x_i\rangle} x_i \in S.$$

This expression is sensible and $\|y_S\| \leq \|y\|$ by appeal to Theorem 11. Direct the subspaces S by inclusion and let $T(y)$ be a cluster point of the net (y_S); then $T(y) \in Z(F)_{\text{sa}}$ and $\langle y - T(y), x\rangle = 0$ for all $x \in Z(F)_{\text{sa}}$, and as there is at most one element of $Z(F)_{\text{sa}}$ which can have this property T is a well-defined (orthogonal) projection from $Z(E)_{\text{sa}}$ onto $Z(F)_{\text{sa}}$.

Let F' be the $*$-weak closure of the set $M \cdot \ker(T)$. It is clear that F' is orthogonal to F. Also $Z(F)_{\text{sa}} + Z(F')_{\text{sa}} = Z(E)_{\text{sa}}$, so $F + F'$ is $*$-weakly dense in E. So for any $x \in E$

https://doi.org/10.4153/CMB-2001-036-8 Published online by Cambridge University Press

we can find a bounded net $(y_i + y_i')$ such that $y_i \in F$ and $y_i' \in F'$ and $y_i + y_i' \to x$ $*$-weakly. By orthogonality, the nets (y_i) and (y_i') are also bounded and so they have cluster points $y \in F$ and $y' \in F'$, and $y + y' = x$ by continuity. So $E = F + F'$. ■

As we mentioned earlier, there is a strong structure theorem for local W^* Hilbert $*$-bimodules. Let $\{p_i\}$ be a family of central projections in a von Neumann algebra M. Then it is easy to check that the algebraic direct sum of the family $\{p_iM\}$ is a pre-Hilbert $*$-bimodule with respect to the bilinear form

$$\langle \oplus a_i, \oplus b_i \rangle = \sum a_i b_i$$

and involution

$$(\oplus a_i)^* = \oplus a_i^*$$

and that it satisfies the normality condition of Theorem 16. Thus it has a W^* Hilbert $*$-bimodule completion, which we denote $\bigoplus p_iM$; this completion consists of those elements $\oplus a_i$ with the property that both of the sums $\sum a_i a_i^*$ and $\sum a_i^* a_i$ converge ultraweakly. The center of $\bigoplus p_iM$ is $\bigoplus Z(p_iM)$ and it is therefore centered and local. The next theorem gives a converse to this fact.

Theorem 19 *Let E be a local W^* Hilbert $*$-bimodule over a von Neumann algebra M. Then there is a family $\{p_i\}$ of central projections of M such that $E \cong \bigoplus p_iM$.*

Proof As in the proof of Theorem 18, regard $Z(E)$ as a local W^* Hilbert $*$-module over $Z(M)$. Observe that for any $x \in Z(E)_{\mathrm{sa}}$ the sequence

$$(\langle x, x\rangle^{1/2} + n^{-1})^{-1}x$$

is bounded and if y is a $*$-weak cluster point of this sequence then $\langle y, y\rangle$ is a projection in $Z(M)$. Now let $\{x_i\}$ be a maximal family of orthogonal elements of $Z(E)_{\mathrm{sa}}$ with the property that $\langle x_i, x_i\rangle$ is a projection in $Z(M)$. It follows from Theorem 18 and the preceding observation that $M \cdot \mathrm{span}\,\{x_i\}$ is $*$-weakly dense in E. Also the sub-bimodules Mx_i are pairwise orthogonal. So it suffices to show that each Mx_i is isomorphic to p_iM where $p_i = \langle x_i, x_i\rangle$. But the kernel of the map $a \mapsto ax_i$ is an ultraweakly closed ideal of M, hence is of the form q_iM for some central projection q_i, and clearly $q_i = 1 - p_i$. Also, for any $a, b \in p_iM$ we have $ab = \langle ax_i, bx_i\rangle$ and $(ax_i)^* = a^*x_i$. So indeed $Mx_i \cong p_iM$ as Hilbert $*$-bimodules. ■

6 Sauvageot's Construction

Our most sophisticated class of examples of Hilbert $*$-bimodules, which were the original motivation for this investigation, arise from a construction in noncommutative geometry given in [18]. A related construction also appears in [3]. We now present a simplified version of this construction which exhibits its symmetry and also shows its resemblance to Kähler differentials (see *e.g.* [9]). Several instances of the construction are detailed in [18].

The ingredients of the construction are a von Neumann algebra M and a C_0^*-semigroup (see [4]) of completely positive maps $\phi_t \colon M \to M$ ($t \geq 0$) such that

https://doi.org/10.4153/CMB-2001-036-8 Published online by Cambridge University Press

(1) $\phi_0 = \mathrm{id}_M$,
(2) $\phi_t(1) = 1$ for all t, and
(3) the set

$$A_\infty = \{a \in M : a \in D(\Delta^n) \text{ for all } n\}$$

is an algebra, where Δ is the generator of (ϕ_t).
We also require the existence of a faithful, normal, semifinite trace τ on M such that
(4) $\tau(a^*a) < \infty$ for all $a \in A_\infty$ and
(5) $\tau\big(a\phi_t(b)\big) = \tau\big(\phi_t(a)b\big)$ for all $a, b \in A_\infty$.

Condition (5) is a noncommutative version of symmetry for Markov processes. In [18] τ is only a weight, but in [19] it is also required to be a trace.

The construction proceeds as follows. For a, b, c, $d \in A_\infty$ define

$$(a \otimes b)^* = b^* \otimes a^*$$

and

$$\langle a \otimes b, c \otimes d\rangle = a\Delta(bc)d,$$

and extend both by linearity to $A_\infty \otimes A_\infty$. This bilinear form does not satisfy $\langle x, x^*\rangle \geq 0$ on $A_\infty \otimes A_\infty$, but it does hold on the sub-A_∞-A_∞-bimodule

$$E_0 = \mathrm{span}\,\{a \otimes bc - ab \otimes c : a, b, c \in A_\infty\}$$

([6], Theorem 14.7). Thus E_0 is a pre-Hilbert $*$-bimodule over A_∞.

The hypothesis of Theorem 16 is verified by observing that if $(a_i) \subset A_\infty$ is bounded and $a_i \to 0$ ultraweakly in M then

$$\begin{aligned}
&\tau\big(\langle(a \otimes bc - ab \otimes c)a_i, (a' \otimes b'c' - a'b' \otimes c')\rangle d\big)\\
&\quad = \tau\big(a\Delta(bca_ia')b'c'd - a\Delta(bca_ia'b')c'd - ab\Delta(ca_ia')b'c'd + ab\Delta(ca_ia'b')c'd\big)\\
&\quad = \tau\big(bca_ia'\Delta(b'c'da) - bca_ia'b'\Delta(c'da) - ca_ia'\Delta(b'c'dab) + ca_ia'b'\Delta(c'dab)\big)
\end{aligned}$$

for all a, b, c, a', b', c', $d \in A_\infty$; the last expression converges to zero since τ is normal, and therefore so does

$$\langle(a \otimes bc - ab \otimes c)a_i, (a' \otimes b'c' - a'b' \otimes c')\rangle$$

since τ is faithful and semifinite (see *e.g.* [10]). Thus E_0 densely embeds in a unique W^* Hilbert $*$-bimodule E over M.

E plays the role of the module of bounded measurable 1-forms, and we have an exterior derivative $d_0 \colon A_\infty \to E$ defined by

$$d_0(a) = i(1 \otimes a - a \otimes 1).$$

It is easy to check that d_0 is a self-adjoint derivation. In fact, it is ultraweakly to $*$-weakly closable because (a_i) and $\big(d_0(a_i)\big)$ bounded and $a_i \to 0$ ultraweakly imply that

$$\begin{aligned}\tau\big(\langle i(1 \otimes a_i - a_i \otimes 1), (a \otimes bc - ab \otimes c)\rangle d\big)\\ &= i\tau\big(\Delta(a_i a)bcd - \Delta(a_i ab)cd - a_i\Delta(a)bcd + a_i\Delta(ab)cd\big)\\ &= i\tau\big(a_i a\Delta(bcd) - a_i ab\Delta(cd) - a_i\Delta(a)bcd + a_i\Delta(ab)cd\big)\end{aligned}$$

converges to zero, and similarly for the inner product in reverse order, which implies that $i(1 \otimes a_i - a_i \otimes 1) \to 0$ $*$-weakly by the same reasoning as in the last paragraph. Thus, in the terminology of [23] the closure d of d_0 is a W^*-derivation and its domain is a noncommutative Lipschitz algebra.

References

[1] C. A. Akemann, personal communication.
[2] C. A. Akemann, J. Anderson and G. K. Pedersen, *Excising states of C^*-algebras*. Canad. J. Math. **38**(1986), 1239–1260.
[3] W. Arveson, *Dynamical invariants for noncommutative flows*. Operator algebras and quantum field theory (Rome, 1996), Internat. Press, 1998, 476–514.
[4] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I*. (second edition), Springer-Verlag, 1987.
[5] A. Connes, *C^* algèbres et géométrie différentielle*. C. R. Acad. Sci. Paris Sér. A **290**(1980), A599–A604.
[6] D. E. Evans and J. T. Lewis, *Dilations of irreversible evolutions in algebraic quantum theory*. Comm. Dublin Inst. Adv. Stud. Ser. A **24**(1977).
[7] T. Falcone and M. Takesaki, *Operator valued weights without structure theory*. Trans. Amer. Math. Soc. **351**(1999), 323–341.
[8] J. M. G. Fell, *An extension of Mackey's method to Banach $*$-algebraic bundles*. Mem. Amer. Math. Soc. **90**(1969).
[9] R. Hartshorne, *Algebraic Geometry*. Springer-Verlag, Graduate Texts in Math. **52**, 1977.
[10] R. V. Kadison and J. R. Ringrose, *Fundamentals of the Theory of Operator Algebras II*. Academic Press, 1986.
[11] C. Lance, *Hilbert C^*-modules*, LMS Lecture Note Series **210**, Cambridge University Press, 1995.
[12] W. L. Paschke, *Inner product modules over B^*-algebras*. Trans. Amer. Math. Soc. **182**(1973), 443–468.
[13] G. K. Pedersen, *C^*-algebras and their Automorphism Groups*. Academic Press, 1979.
[14] N. C. Phillips and N. Weaver, *Modules with norms which take values in a C^*-algebra*. Pacific J. Math. **185**(1998), 163–181.
[15] M. A. Rieffel, *Induced representations of C^*-algebras*. Adv. Math. **13**(1974), 176–257.
[16] ———, *Morita equivalence for C^*-algebras and W^*-algebras*. J. Pure Appl. Algebra **5**(1974), 51–96.
[17] ———, *Morita equivalence for operator algebras*. Proc. Symp. Pure Math. **38**(1982), 285–298.
[18] J.-L. Sauvageot, *Tangent bimodule and locality for dissipative operators on C^*-algebras*. Quantum Probability and App. IV, Springer, Lecture Notes in Math. **1396**, 1989, 322–338.
[19] ———, *Quantum Dirichlet forms, differential calculus and semigroups*. Quantum Probability and App. V, Springer, Lecture Notes in Math. **1442**, 1990, 334–346.
[20] M. Skeide, *Hilbert modules in quantum electro dynamics and quantum probability*. Comm. Math. Phys. **192**(1998), 569–604.
[21] R. G. Swan, *Vector bundles and projective modules*. Trans. Amer. Math. Soc. **105**(1962), 264–277.
[22] A. Takehashi, *Fields of Hilbert Modules*. Dissertation, Tulane University, 1971.

[23] N. Weaver, *Lipschitz algebras and derivations of von Neumann algebras*. J. Funct. Anal. **139**(1996), 261–300.

[24] S. L. Woronowicz, *Differential calculus on compact matrix pseudogroups (quantum groups)*. Comm. Math. Phys. **122**(1989), 125–170.

Department of Mathematics
Washington University
St. Louis, Missouri 63130
U.S.A.
e-mail: nweaver@math.wustl.edu

https://doi.org/10.4153/CMB-2001-036-8 Published online by Cambridge University Press