Nagoya Math. J., **235** (2019), 42–57
DOI 10.1017/nmj.2017.43

GROUP ACTION ON THE DEFORMATIONS OF A FORMAL GROUP OVER THE RING OF WITT VECTORS

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Abstract. A recent result by the authors gives an explicit construction for a universal deformation of a formal group Φ of finite height over a finite field k. This provides in particular a parametrization of the set of deformations of Φ over the ring $\mathcal{O}$ of Witt vectors over k. Another parametrization of the same set can be obtained through the Dieudonné theory. We find an explicit relation between these parameterizations. As a consequence, we obtain an explicit expression for the action of $\mathrm{Aut}_k(\Phi)$ on the set of $\mathcal{O}$-deformations of Φ in the coordinate system defined by the universal deformation. This generalizes a formula of Gross and Hopkins and the authors' result for one-dimensional formal groups.

§1. Introduction

Let Φ be a d-dimensional formal group of height h over a perfect field k of characteristic $p>0$, and $\mathcal{O}$ denote the ring of Witt vectors over k. The deformation functor $\mathcal{D}ef_\Phi$ assigns to any Artinian local $\mathcal{O}$-algebra R the set of $\star$-isomorphism classes of deformations of Φ over R, where two deformations of Φ are $\star$-isomorphic if there exists an isomorphism between them with identity reduction. Grothendieck [11] proved that $\mathcal{D}ef_\Phi$ is representable by the formal spectrum of the ring of formal power series in $d(h-d)$ variables with coefficients in $\mathcal{O}$. In other words, there exists a formal group Γ over $\mathcal{O}[[t_1,\dots,t_{d(h-d)}]]$ called a universal deformation of Φ such that for any deformation F of Φ over R, there is a unique homomorphism $\mu:\mathcal{O}[[t_1,\dots,t_{d(h-d)}]]\to R$ such that $\mu_*\Gamma$ is $\star$-isomorphic to F. Notice that Grothendieck's proof does not allow one to construct

Received April 28, 2017. Revised November 6, 2017. Accepted November 6, 2017.
2010 Mathematics subject classification. 14L05, 11S31.
The first author has been supported by the Russian Science Foundation grant no. 6.53.924.2016.

Γ explicitly. In the case where Φ is one-dimensional, an explicit universal deformation was constructed by Hazewinkel [9]. For arbitrary Φ it was done by the authors in [5].

For the $\star$-isomorphism classes of the deformations of Φ over $\mathcal{O}$, there is another explicit description due to the Dieudonné theory. The Dieudonné module $\mathcal{D}(\Phi)$ of Φ is a certain profinite module over the Dieudonné ring E. Fontaine [7] assigned a submodule of $\mathcal{D}(\Phi)$ to any deformation of Φ over $\mathcal{O}$ so that $\star$-isomorphic deformations correspond to the same submodule. Honda [10] described $\mathcal{D}(\Phi)$ as a factor module of E^d. Thus one can show that $\mathcal{D}ef_\Phi(\mathcal{O})$ is naturally identified with the set of certain double cosets in $\mathrm{GL}_d(E)$. The first purpose of the present paper is to establish an explicit relation between two parameterizations of $\mathcal{D}ef_\Phi(\mathcal{O})$: through the explicit universal deformation and through the Dieudonné theory. For a formal group of dimension one such relation was established in the authors' work [4] where a period map was constructed and an explicit formula for it was obtained. In order to treat the general case we exercise a similar approach. In addition to Hazewinkel's universal p-typical formal group [9] and Honda theory [10] that were involved in [4], we make use of covariant Honda theory developed in [3].

It is necessary to mention that in his fundamental work [13], Zink introduced the theory of $3n$-displays which describes formal groups of finite height over an excellent ring R. Moreover, he developed a corresponding deformation theory which generalizes the description of $\mathcal{D}ef_\Phi(\mathcal{O})$ based on the Dieudonné module $\mathcal{D}(\Phi)$. It would be highly interesting to relate Zink's description of $\mathcal{D}ef_\Phi(R)$ and the parametrization obtained through the explicit universal deformation. Being more involved, this task is left to a future investigation.

The group of the automorphisms of Φ over k acts naturally on the right on the functor $\mathcal{D}ef_\Phi$. For $\phi \in \mathrm{Aut}_k(\Phi)$ and $F \in \mathcal{D}ef_\Phi(\mathcal{O})$, the action is defined by $[F]\phi = [g^{-1} \circ F(g, g)]$, where g is a lift of ϕ over $\mathcal{O}$. In the one-dimensional case, this action was introduced by Lubin and Tate [12]. Its deep importance for the representation theory was revealed by Drinfeld [6] and in a more specific way by Carayol [1]. Later on, it was investigated by several researchers. Gross and Hopkins [8] defined the p-adic period map from the generic fiber of the deformation functor to the projective space and proved that it is equivariant with respect to the action of the automorphism group. It provides a linearization of the action for the deformations over $\mathcal{O}$. Chai [2] studied this action on the closed fiber of the deformation space and described it by certain recurrence relation. Another explicit description for the action which also involved the p-adic period map was established by the authors [4]. Our second purpose in the present work is to study this action in the higher dimensional case. We proceed similarly to what was done in [4]. Namely, applying Honda's description of $\mathrm{Aut}_k(\Phi)$ and the explicit formula for the period map we obtain an explicit expression for the action in terms of the parametrization introduced in [5].

The outline of the paper is as follows. Section 2 recalls the basic definitions related to formal groups and describing the main results of Honda theory [10] and covariant Honda theory [3] which are our main tools. In Section 3, Hazewinkel's universal d-dimensional p-typical formal group F_V over the ring $\mathbb{Z}[V]$ of polynomials in an infinite set V of independent variables is introduced. Any p-typical formal group Φ over k can be obtained from F_V by a specialization of the variables of V in k. We recall the main result of [5] which states that it is possible to choose a finite set $\{v_\psi\}_{\psi\in\Psi^+}$ of variables from V and to define a homomorphism $\gamma:\mathbb{Z}[V]\to\mathcal{O}[[t_\psi]]_{\psi\in\Psi^+}$ so that $\Gamma=\gamma_*F_V$ is a universal deformation of Φ. In particular, it gives a one-to-one correspondence between $\mathcal{D}ef_\Phi(\mathcal{O})$ and $p\mathcal{O}^{|\Psi^+|}$.

In Section 4, a period map from the set of $\star$-isomorphism classes of $\mathcal{O}$-deformations of Φ to $\mathrm{GL}_d(\mathcal{O})\backslash\mathrm{GL}_d(E)/I+\mathrm{M}_d(E)\widehat{u}$ is constructed, where $E=\mathcal{O}[[\blacktriangle]]$ is an $\mathcal{O}$-algebra with multiplication rule $\blacktriangle a=\Delta(a)\blacktriangle$, $a\in\mathcal{O}$, Δ denotes the Frobenius automorphism, and $\widehat{u}\in\mathrm{M}_d(E)$ is Honda type of a deformation of Φ. It gives another parametrization of $\mathcal{D}ef_\Phi(\mathcal{O})$. Then an explicit relation between these two parameterizations is established. For a natural example, we deduce an explicit formula which expresses the parameters obtained from the cosets in $\mathrm{GL}_d(E)$ via the parameters obtained from the universal deformation with the aid of certain rigid analytic functions. It turns out that in the one-dimensional case, these functions coincide up to a constant factor with the functions used by Gross and Hopkins [8] for the definition of their p-adic period map.

Section 5 describes the action of the automorphism group of Φ on $\mathcal{D}ef_\Phi(\mathcal{O})$ in the coordinate system introduced in Section 3. Honda theory implies that $\mathrm{Aut}_k(\Phi)$ acts on $\mathrm{GL}_d(\mathcal{O})\backslash\mathrm{GL}_d(E)/I+\mathrm{M}_d(E)\widehat{u}$ by the right multiplication. We show that this action agrees with the natural action on $\mathcal{D}ef_\Phi(\mathcal{O})$ (Theorem 5.1). Thus our period map linearizes the action of the automorphism group on the moduli space which provides its explicit description (Theorem 5.2). In the case of a formal group of dimension

1, this result implies the equivariance of the Gross–Hopkins period map [8, Proposition 23.5].

Throughout the paper, the following notations are used. $\mathrm{M}_{d',d}(A)$ denotes the set of $d' \times d$-matrices with entries in the ring A, $\mathrm{M}_d(A) = \mathrm{M}_{d,d}(A)$, $\mathrm{GL}_d(A)$ stands for the subset of $\mathrm{M}_d(A)$ consisting of the invertible matrices. The identity matrix is denoted by I. An entry of a matrix $M \in \mathrm{M}_d(A)$ in the ith row and the jth column is denoted by $M(i, j)$. The matrix obtained from M by raising all its entries to the qth power is denoted by $M^{(q)}$, i.e., $M^{(q)}(i, j) = M(i, j)^q$, $1 \leqslant i, j \leqslant d$. If $\mu : A \to A'$ is a ring homomorphism and $f \in A[[X]]$, then $\mu_* f \in A'[[X]]$ denotes a power series obtained by applying μ to the coefficients of f.

§2. Formal groups and Honda theory

Let A be a ring. We denote by X the d-tuple of independent variables $(x_1, \ldots, x_d)$ and write $X^q, q \in \mathbb{N}$, for the d-tuple $(x_1^q, \ldots, x_d^q)$. We also consider the ring $A[[X]]_0$ of formal power series over A in variables $x_1, \ldots, x_d$ without constant term.

A d-dimensional *formal group* over A is a d-tuple of formal power series $F \in A[[X, Y]]^d$ satisfying the following properties

(i) $F(X, 0) = X$;
(ii) $F(X, F(Y, Z)) = F(F(X, Y), Z)$;
(iii) $F(X, Y) = F(Y, X)$.

Let F and F' be d- and d'-dimensional formal groups over A. A d'-tuple of formal power series $g \in A[[X]]_0^{d'}$ is called a *homomorphism* from F to F', if $g(F(X, Y)) = F'(g(X), g(Y))$. The matrix $D \in \mathrm{M}_{d',d}(A)$ satisfying $g(X) \equiv DX \mod \deg 2$ is called the *linear coefficient* of g. It is easy to see that a homomorphism g is an isomorphism if and only if its linear coefficient is an invertible matrix. A homomorphism with identity linear coefficient is called a *strict isomorphism*.

If A is a $\mathbb{Q}$-algebra, then for any d-dimensional formal group F over A, there exists a unique strict isomorphism f from F to $F_a^d(X, Y) = X + Y$ (see for instance [10, Theorem 1]). This f is called the *logarithm* of F. It determines the formal group F uniquely, since $F(X, Y) = f^{-1}(f(X) + f(Y))$.

If A is a ring of characteristic 0 and F is a formal group over A, the logarithm of F is by definition the logarithm of $F_{A\otimes\mathbb{Q}}$. If, in addition, F' is another formal group over A and g is a homomorphism from F to F' with linear coefficient D, then $g = f'^{-1} \circ (Df)$, where f, f' are the logarithms of F, F', respectively (see [10, Proposition 1.6]).

Let k be a perfect field of characteristic $p \neq 0$, $\mathcal{O}$ be the ring of Witt vectors over k, $\mathcal{K}$ be the quotient field of $\mathcal{O}$, and $\Delta : \mathcal{K} \to \mathcal{K}$ be the Frobenius automorphism. Let $E = \mathcal{O}[[\blacktriangle]]$ denote the $\mathcal{O}$-algebra of noncommutative formal power series in the variable $\blacktriangle$ over $\mathcal{O}$ with the multiplication rule $\blacktriangle a = a^{\Delta}\blacktriangle$ for any $a \in \mathcal{O}$. Extend the $\mathcal{O}$-module structure on $\mathcal{K}[[X]]_0$ to a left E-module structure by the formula $\blacktriangle f(X) = f^{\Delta}(X^p)$. It determines a bilinear map $\mathrm{M}_{d',d}(E) \times \mathcal{K}[[X]]_0^d \to \mathcal{K}[[X]]_0^{d'}$. In particular, it gives a left $\mathrm{M}_d(E)$-module structure on $\mathcal{K}[[X]]_0^d$.

Denote $\mathfrak{P} = p\mathcal{O}[[X]]_0^d$. Let $u \in \mathrm{M}_d(E)$ be such that $u \equiv pI \mod \blacktriangle$. We say that u is a *type* of $f \in \mathcal{K}[[X]]_0^d$ if $f(X) \equiv X \mod \deg 2$ and $uf \in \mathfrak{P}$.

The next technical result will be employed in what follows.

LEMMA 2.1. [10, Lemmas 4.2 and 2.3]

(1) *Let $u \in \mathrm{M}_d(E)$, $u \equiv pI \mod \blacktriangle$, $f \in \mathcal{K}[[X]]_0^d$ be of type u, $\psi_1 \in \mathcal{K}[[X]]_0^d$ and $\psi_2 \in \mathcal{O}[[X]]_0^d$. Then $f \circ \psi_1 \equiv f \circ \psi_2 \mod \mathfrak{P}$ iff $\psi_1 \equiv \psi_2 \mod \mathfrak{P}$.*
(2) *Let $u, v \in \mathrm{M}_d(E)$, $u \equiv pI \mod \blacktriangle$, $f \in \mathcal{K}[[X]]_0^d$ be of type u and $g \in \mathcal{O}[[X]]_0^d$. Then $v(f \circ g) \equiv (vf) \circ g \mod \mathfrak{P}$.*

Honda theory [10] describes the logarithms of formal groups over $\mathcal{O}$ as formal power series of certain Honda type.

THEOREM 2.2. [10, Theorem 2 and Proposition 3.3]

(1) *Let $u \in \mathrm{M}_d(E)$, $u \equiv pI \mod \blacktriangle$, $f \in \mathcal{K}[[X]]_0^d$ be of type u. Then f is the logarithm of a formal group over $\mathcal{O}$.*
(2) *Let F be a formal group over $\mathcal{O}$ with the logarithm $f \in \mathcal{K}[[X]]_0^d$. Then there exists $u \in \mathrm{M}_d(E)$, $u \equiv pI \mod \blacktriangle$, such that f is of type u.*

With the aid of Honda theory, the category of formal groups over k can be also described and the Dieudonné module can be constructed. Under *reduction* of a formal power series with coefficients in $\mathcal{O}$, we always mean the reduction modulo p.

PROPOSITION 2.3. *Let F_1, F_2 be formal groups over $\mathcal{O}$ with the logarithms $f_1, f_2 \in \mathcal{K}[[X]]_0^d$, respectively. Then the reductions of F_1 and F_2 are equal iff there exists $v \in I + \blacktriangle\mathrm{M}_d(E)$ such that $f_1 \equiv v f_2 \mod \mathfrak{P}$.*

Proof. The "if" part follows immediately from [5, Proposition 1]. To prove the "only if" part suppose that the reductions of F_1 and F_2 are equal.

According to [5, Proposition 1], there exists $q \in \mathrm{GL}_d(E)$ such that $f_1 \equiv q f_2 \mod \mathfrak{P}$. Let $c \in \mathrm{M}_d(\mathcal{O})$ be such that $q \equiv c \mod \blacktriangle$. Since $X \equiv f_1(X) \equiv q f_2(X) \equiv cX \bmod (\mathfrak{P}, \deg 2)$, we conclude $I \equiv c \mod p$, i.e., there exists $c' \in \mathrm{M}_d(\mathcal{O})$ such that $c - I = pc'$. Let $u_2 \in \mathrm{M}_d(E)$ be such that $u_2 \equiv pI \mod \blacktriangle$ and f_2 is of type u_2. Denote $v = q - c'u_2$. Then $v \equiv I \mod \blacktriangle$ and $f_1 \equiv q f_2 = v f_2 + c' u_2 f_2 \equiv v f_2 \mod \mathfrak{P}$. □

PROPOSITION 2.4. [10, Proposition 2.6] *Let $v \in \mathrm{M}_{d',d}(E)$, $u \in \mathrm{M}_d(E)$, $u \equiv pI \mod \blacktriangle$ and $f \in \mathcal{K}[[X]]_0^d$ be of type u. If $vf \in p\mathcal{O}[[X]]_0^{d'}$, then there exists $s \in \mathrm{M}_{d',d}(E)$ such that $v = su$.*

Let F be a d-dimensional formal group over $\mathcal{O}$ with logarithm $f \in \mathcal{K}[[X]]_0^d$ of type $u \in \mathrm{M}_d(E)$, and $\bar{F}$ denote the reduction of F. The E-module $\mathcal{D}(\bar{F}) = \mathrm{M}_{1,d}(E)f / \mathrm{M}_{1,d}(E)f \cap p\mathcal{O}[[X]]_0$ is called the *Dieudonné module* of $\bar{F}$. Proposition 2.3 implies that $\mathcal{D}(\bar{F})$ depends only on $\bar{F}$. Since $uf \in \mathfrak{P}$, the E-linear map $\mathrm{M}_{1,d}(E) \to \mathcal{D}(\bar{F})$ defined by $s \mapsto sf + p\mathcal{O}[[X]]_0$ induces a homomorphism $\mathrm{M}_{1,d}(E)/\mathrm{M}_{1,d}(E)u \to \mathcal{D}(\bar{F})$ which is an isomorphism by Proposition 2.4. A formal group $\bar{F}$ is said to be of finite height if $\mathcal{D}(\bar{F})$ is a free $\mathcal{O}$-module of finite rank (see [7, Proposition III.6.1]). In this case the rank of $\mathcal{D}(\bar{F})$ is called the *height* of $\bar{F}$.

The following theorem describes $\mathrm{Aut}_k(\bar{F})$ in terms of Honda types.

THEOREM 2.5. [10, Theorems 5 and 6] *Let F be a formal group over $\mathcal{O}$ with the logarithm $f \in \mathcal{K}[[X]]_0^d$ of type $u \in \mathrm{M}_d(E)$, and $w \in \mathrm{M}_d(E)$. Then*

(1) *$f^{-1} \circ (wf) \in \mathcal{O}[[X]]_0^d$ iff there exists $z \in \mathrm{M}_d(E)$ such that $uw = zu$.*
(2) *If $f^{-1} \circ (wf) \in \mathcal{O}[[X]]_0^d$, then its reduction is an automorphism of the reduction of F.*
(3) *If ϕ is an automorphism of the reduction of F, then there exists $w \in \mathrm{M}_d(E)$ such that $f^{-1} \circ (wf) \in \mathcal{O}[[X]]_0^d$ and ϕ is the reduction of $f^{-1} \circ (wf)$.*

Let $\mathcal{K}[[t]]_p = \{\sum_{i=0}^{\infty} b_i t^{p^i} | b_i \in \mathcal{K}\}$ and $\mathcal{K}[[X]]_p = \{\sum_{j=1}^{d} f_j(x_j) | f_j \in \mathcal{K}[[t]]_p\}$. For any $\sum_{i=0}^{\infty} a_i \blacktriangle^i \in E$, define $t^{p^j} \sum_{i=0}^{\infty} a_i \blacktriangle^i = \sum_{i=0}^{\infty} a_i^{\Delta^j} t^{p^{i+j}}$. This provides to $\mathcal{K}[[t]]_p$ a right E-module structure which in general looks as follows:

$$\left(\sum_{j=0}^{\infty} b_j t^{p^j}\right)\left(\sum_{i=0}^{\infty} a_i \blacktriangle^i\right) = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} b_j a_{i-j}^{\Delta^j}\right) t^{p^i}.$$

https://doi.org/10.1017/nmj.2017.43 Published online by Cambridge University Press

It determines a right $\mathrm{M}_d(E)$-module structure on $\mathcal{K}[[X]]_p^d$. In fact, associating $\sum_{i=0}^{\infty} b_i t^{p^i}$ with $\sum_{i=0}^{\infty} b_i \blacktriangle^i$ identifies $\mathcal{K}[[t]]_p$ with $\mathcal{K}[[\blacktriangle]]$ provided with screw multiplication $\blacktriangle a = a^{\Delta}\blacktriangle$. Then the right action of E on $\mathcal{K}[[t]]_p$ corresponds to the right multiplication in $\mathcal{K}[[\blacktriangle]]$. Similarly, $\mathcal{K}[[X]]_p^d$ and $\mathrm{M}_d(\mathcal{K}[[\blacktriangle]])$ can be identified, and the right action of $\mathrm{M}_d(E)$ on $\mathcal{K}[[X]]_p^d$ corresponds to the right multiplication in $\mathrm{M}_d(\mathcal{K}[[\blacktriangle]])$.

Covariant Honda theory [3] gives an alternative description of the category of p-typical formal groups over $\mathcal{O}$ and can be considered as dual to Honda theory. Only one result from this theory will be used hereafter.

PROPOSITION 2.6. [3, Proposition 1(i) and (iii)]

(1) *The left* $\mathrm{M}_d(E)$*-module structure on* $\mathcal{K}[[X]]_p^d$ *induced from* $\mathcal{K}[[X]]_0^d$ *commutes with the right* $\mathrm{M}_d(E)$*-module structure.*
(2) *Let* $f \in \mathcal{K}[[X]]_p^d$ *and* $u \in \mathrm{M}_d(E)$, $u \equiv pI \mod \blacktriangle$. *Then* $uf = pX$ *iff* $fu = pX$.

§3. Universal p-typical formal group and universal deformation

We consider polynomial rings with integer coefficients in an infinite number of variables which are grouped together in matrices for convenience. Let V_n be $d \times d$-matrices of independent variables $V_n(i, j)$. The set of these independent variables is denoted by $V = (V_1, V_2, \ldots)$.

According to Hazewinkel's functional equation lemma [9, Section 10.2], the d-tuple of power series $f_V \in \mathbb{Q}[V][[X]]_0^d$ defined by the recursion formula

$$f_V(X) = X + p^{-1} \sum_{n=1}^{\infty} V_n \sigma_*^n f_V(X^{p^n}),$$

where $\sigma(V_n(i, j)) = V_n(i, j)^p$, is the logarithm of a d-dimensional formal group F_V defined over $\mathbb{Z}[V]$.

Let $a_n(V) \in \mathbb{Q}[V]$ denote the coefficients of f_V, that means $f_V(X) = \sum_{n=0}^{\infty} a_n(V) X^{p^n}$. It is clear that $a_0(V) = I$. The following recurrence relation will be employed hereafter.

LEMMA 3.1. [9, Formula 10.4.5] $p\, a_n(V) = \sum_{k=1}^{n} a_{n-k}(V) V_k^{(p^{n-k})}$.

COROLLARY. $p^n\, a_n(V) \in \mathbb{Z}[V]$.

Proof. It follows immediately by induction on n. ∎

https://doi.org/10.1017/nmj.2017.43 Published online by Cambridge University Press

For a ring A, denote by A^∞ the A-module consisting of the infinite sequences of elements of A. Then A^∞_m is the submodule of A^∞ consisting of the sequences starting with m zeros.

A formal group F over a ring A is called *p-typical*, if there exists $\Xi : \mathbb{Z}[V] \to A$ such that $\Xi_* F_V = F$. A morphism $\Xi : \mathbb{Z}[V] \to A$ is uniquely defined by the sequence $(\Xi(V_1), \Xi(V_2), \ldots) \in \mathrm{M}_d(A)^\infty$ where $\Xi(V_n) \in \mathrm{M}_d(A)$ is defined by $(\Xi(V_n))(i, j) = \Xi(V_n(i, j))$. Thus we can write by abuse of the notation $\Xi = (\Xi(V_1), \Xi(V_2), \ldots)$. We will also write $F_{V(\Xi)}$ instead of $\Xi_* F_V$.

Let k, $\mathcal{O}$, E be as in Section 2. Let Φ be a d-dimensional formal group over k of height h. According to [9, Corollary 25.4.29], Φ is isomorphic to a p-typical formal group over k. Thus without loss of generality we can suppose that Φ is p-typical, i.e., there exists $\Xi = (\Xi_1, \Xi_2, \ldots) \in \mathrm{M}_d(k)^\infty$ such that $\Phi = F_{V(\Xi)}$. Let $\widehat{\Xi} = (\widehat{\Xi}_1, \widehat{\Xi}_2, \ldots) \in \mathrm{M}_d(\mathcal{O})^\infty$ be the sequence of matrices which are composed of the multiplicative representatives in $\mathcal{O}$ of the entries of the matrices $\Xi_1, \Xi_2, \ldots$ Then $\widehat{F} = F_{V(\widehat{\Xi})}$ is a formal group over $\mathcal{O}$, and its reduction is equal to Φ.

Given a sequence $T = (T_1, T_2, \ldots) \in \mathrm{M}_d(k)^\infty$, we introduce

$$Y_n(\Xi, T) = T_1 \Xi_{n-1}^{(p)} + \cdots + T_{n-1} \Xi_1^{(p^{n-1})} \in \mathrm{M}_d(k)$$

for $n \geqslant 2$ and $Y_1(\Xi, T) = 0$. Then $Y_\Xi(T) = (Y_1(\Xi, T), Y_2(\Xi, T), \ldots)$ defines a k-linear operator on $\mathrm{M}_d(k)^\infty$.

For $n \geqslant 1$, $1 \leqslant i, j \leqslant d$, denote by $B_{(n,i,j)} \in \mathrm{M}_d(k)^\infty$ the sequence of matrices with the only nonzero entry which is equal to 1 and appears in the nth matrix at the (i, j)th position. According to [5, Proposition 8], there exists a set

$$\Psi \subset \{(n, i, j) | n \geqslant 1, 1 \leqslant i, j \leqslant d, B_{(n,i,j)} \notin \mathrm{Im} Y_\Xi\}$$

such that $\{B_\psi + \mathrm{Im} Y_\Xi | \psi \in \Psi\}$ is a basis of $\mathrm{Coker} Y_\Xi$. Moreover, $|\Psi| = dh$.

Since $Y_1(V, T) = 0$, one has $\mathrm{Im} Y_\Xi \subset \mathrm{M}_d(k)^\infty_1$, and hence $\{(1, i, j) | 1 \leqslant i, j \leqslant d\} \subset \Psi$. Define $\Psi^+ = \{(n, i, j) | (n + 1, i, j) \in \Psi\}$. Clearly, $|\Psi^+| = d(h - d)$.

Let R be a complete Noetherian local ring with residue field k, and Φ be a formal group over k. A formal group F over R with reduction Φ is called a *deformation* of Φ over R. Let F, F' be deformations of Φ over R. An isomorphism from F to F' with identity reduction is called a $\star$-*isomorphism*.

Similarly to $B_{(n,i,j)}$, denote by $\widehat{B}_{(n,i,j)} \in \mathrm{M}_d(R)^\infty$ the sequence of matrices with the only nonzero entry being equal to 1 and appearing in the nth matrix at the (i, j)th position.

Now we can formulate the main result of [5].

THEOREM 3.2. [5, Theorem 2] *Let* $\Phi = F_{V(\Xi)}$. *Then for any complete Noetherian local* $\mathcal{O}$*-algebra* R *with maximal ideal* $\mathfrak{M}$ *containing* p *and residue field* k*, and for any deformation* F *of* Φ *over* R*:*

(1) *there exists a unique dh-tuple* $(\tau_\psi)_{\psi\in\Psi}$, $\tau_\psi \in \mathfrak{M}$, *such that* F *is strictly* $\star$*-isomorphic to the formal group* $F_{V(Z)}$, *where* $Z = \widehat{\Xi} + \sum_{\psi\in\Psi} \tau_\psi \widehat{B}_\psi \in \mathrm{M}_d(R)^\infty$;
(2) *there exists a unique* $d(h-d)$*-tuple* $(\tau_\psi)_{\psi\in\Psi^+}$, $\tau_\psi \in \mathfrak{M}$, *such that* F *is* $\star$*-isomorphic to the formal group* $F_{V(Z)}$, *where* $Z = \widehat{\widehat{\Xi}} + \sum_{\psi\in\Psi^+} \tau_\psi \widehat{B}_\psi \in \mathrm{M}_d(R)^\infty$.

COROLLARY. *Let* $(t_\psi)_{\psi\in\Psi^+}$ *be a set of independent variables, and put* $\widehat{\widehat{\Xi}}^* = \widehat{\widehat{\Xi}} + \sum_{\psi\in\Psi^+} t_\psi \widehat{B}_\psi$. *Then* $\Gamma = F_{V(\widehat{\widehat{\Xi}}^*)}$ *is a universal deformation of* Φ*, i.e., for any complete Noetherian local* $\mathcal{O}$*-algebra* R *with residue field* k *and for any deformation* F *of* Φ *over* R *there exists a unique* $\mathcal{O}$*-homomorphism* $\mu : \mathcal{O}[[t_\psi]]_{\psi\in\Psi^+} \to R$ *such that* $\mu_*\Gamma$ *is* $\star$*-isomorphic to* F.

§4. Period map for deformations over $\mathcal{O}$

We keep the notation of the previous section. First, we prove an auxiliary result needed for the definition of the period map.

PROPOSITION 4.1. *Let* F_1, F_2 *be formal groups over* $\mathcal{O}$ *with the logarithms* $f_1, f_2 \in \mathcal{K}[[X]]_0^d$, *respectively, such that their reductions coincide. Then*

(1) F_1 *and* F_2 *are strictly* $\star$*-isomorphic iff* $f_1 \equiv f_2 \mod \mathfrak{P}$.
(2) F_1 *and* F_2 *are* $\star$*-isomorphic iff there exists* $c \in \mathrm{GL}_d(\mathcal{O})$ *such that* $f_1 \equiv cf_2 \mod \mathfrak{P}$.

Proof. By definition formal groups F_1, F_2 are strictly $\star$-isomorphic (resp. $\star$-isomorphic) iff $f_1^{-1} \circ f_2 \equiv X \mod \mathfrak{P}$ (resp. $f_1^{-1} \circ (cf_2) \equiv X \mod \mathfrak{P}$ for some $c \in \mathrm{GL}_d(\mathcal{O})$). According to Theorem 2.2(2), there exists $u \in \mathrm{M}_d(E)$ with $u \equiv pI \mod \blacktriangle$ such that f_1 is of type u. Then Lemma 2.1(1) implies the required statement. ∎

Denote $\widehat{f} = f_{V(\widehat{\Xi})}$. Clearly, $\widehat{f}$ is the logarithm of $\widehat{F}$. Further, denote $\widehat{u} = pI - \sum_{n=1}^\infty \widehat{\Xi}_n \blacktriangle^n \in \mathrm{M}_d(E)$.

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LEMMA 4.2. [5, Lemma 3] $\widehat{u}\widehat{f} = pX$.

Let F be a deformation of Φ over $\mathcal{O}$ with logarithm f. We denote the strict $\star$-isomorphism classes of F by $[F]$, and the $\star$-isomorphism classes of F by $[F]'$. By Proposition 2.3, there is $v \in I + \blacktriangle \mathrm{M}_d(E)$ such that $f \equiv v\widehat{f} \mod \mathfrak{P}$. Then one can define the map χ from the set of the strict $\star$-isomorphism classes of deformations of Φ over $\mathcal{O}$ to

$$I + \mathrm{M}_d(p\mathcal{O} + \blacktriangle E)/I + \mathrm{M}_d(E)\widehat{u}$$

and the map χ' from the set of the $\star$-isomorphism classes of deformations of Φ over $\mathcal{O}$ to

$$I + \mathrm{M}_d(p\mathcal{O})\backslash I + \mathrm{M}_d(p\mathcal{O} + \blacktriangle E)/I + \mathrm{M}_d(E)\widehat{u}$$

as follows: $\chi([F]) = v(I + \mathrm{M}_d(E)\widehat{u})$, $\chi'([F]') = (I + \mathrm{M}_d(p\mathcal{O}))v(I + \mathrm{M}_d(E)\widehat{u})$.

PROPOSITION 4.3. *The maps χ and χ' are well defined and bijective.*

Proof. By Proposition 4.1(1), deformations of Φ with logarithms f_1 and f_2 are strictly $\star$-isomorphic if and only if $f_1 \equiv f_2 \mod \mathfrak{P}$, i.e., $(v_2 - v_1)\widehat{f} \in \mathfrak{P}$. According to Lemma 4.2 and Proposition 2.4, it is equivalent to the existence of $r \in \mathrm{M}_d(E)$ such that $v_2 = v_1(I + v_1^{-1}r\widehat{u})$, i.e., $v_2 \in v_1(I + \mathrm{M}_d(E)\widehat{u})$. Thus χ is well defined and injective. For any $v \in I + \mathrm{M}_d(p\mathcal{O} + \blacktriangle E)$, one can find $b \in \mathrm{M}_d(\mathcal{O})$ such that $\tilde{v} = v(I + b\widehat{u}) \in I + \mathrm{M}_d(\blacktriangle E)$. Then $\widehat{u}\tilde{v}^{-1} \equiv pI \mod \blacktriangle$ and $\widehat{u}\tilde{v}^{-1}(\tilde{v}\widehat{f}) = pX$. Hence, Theorem 2.2(1) implies that $\tilde{v}\widehat{f}$ is the logarithm of a formal group over $\mathcal{O}$. By Proposition 2.3, this formal group must be a deformation of Φ, and thus χ is surjective.

Further, according to Proposition 4.1(2), formal groups with logarithms f_1 and f_2 are $\star$-isomorphic iff $f_1 \equiv cf_2 \mod \mathfrak{P}$ for some $c \in \mathrm{GL}_d(\mathcal{O})$, i.e., $(v_2 - cv_1)\widehat{f} \in \mathfrak{P}$. Proposition 2.4 implies that this holds iff there exists $r \in \mathrm{M}_d(E)$ such that $v_2 = cv_1(I + v_1^{-1}r\widehat{u})$, i.e., $v_2 \in \mathrm{GL}_d(\mathcal{O})v_1(I + \mathrm{M}_d(E)\widehat{u})$. Thus χ' is well defined and injective. Since χ is surjective, χ' is also surjective. ∎

One can see that χ and χ' describe deformations of Φ over $\mathcal{O}$ up to strict $\star$-isomorphism and up to $\star$-isomorphism, respectively. Theorem 3.2 provides an alternative description of the same deformations. Our aim is to establish an explicit connection between these two descriptions.

For $Z = (Z_1, Z_2, \ldots) \in \mathrm{M}_d(\mathcal{O})^\infty$ such that $Z \equiv \widehat{\Xi} \mod p$, we define

$$v_Z = I + p^{-1} \sum_{n=1}^{\infty} \sum_{k=1}^{n} a_{n-k}(Z) \left(Z_k^{(p^{n-k})} - \widehat{\Xi}_k^{(p^{n-k})} \right) \blacktriangle^n.$$

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Since $Z \equiv \widehat{\Xi} \mod p$, we conclude that $Z^{(p^m)} \equiv \widehat{\Xi}^{(p^m)} \mod p^{m+1}$ for any $m \geqslant 0$. Moreover, Corollary of Lemma 3.1 implies $p^m a_m(Z) \in \mathrm{M}_d(\mathcal{O})^\infty$ for any $m \geqslant 0$. Therefore $a_{n-k}(Z)\left(Z_k^{(p^{n-k})} - \widehat{\Xi}_k^{(p^{n-k})}\right) \in \mathrm{M}_d(p\mathcal{O})$ for any $n \geqslant k$, and hence $v_Z \in \mathrm{GL}_d(E)$.

PROPOSITION 4.4.

(1) *If* $Z = \widehat{\Xi} + \sum_{\psi\in\Psi} \tau_\psi \widehat{B}_\psi$, *for some* $\tau_\psi \in p\mathcal{O}$, $\psi \in \Psi$, *then* $\chi([F_{V(Z)}]) = v_Z(I + \mathrm{M}_d(E)\widehat{u})$.

(2) *If* $Z = \widehat{\Xi} + \sum_{\psi\in\Psi^+} \tau_\psi \widehat{B}_\psi$, *for some* $\tau_\psi \in p\mathcal{O}$, $\psi \in \Psi^+$, *then* $\chi'([F_{V(Z)}]') = (I + \mathrm{M}_d(p\mathcal{O}))v_Z(I + \mathrm{M}_d(E)\widehat{u})$.

Proof. Since $\widehat{\Xi}_n^{(p)} = \widehat{\Xi}_n^{\Delta}$ for any $n \geqslant 1$, the recurrence formula of Lemma 3.1 can be rewritten as follows

$$pa_n(Z) - \sum_{k=1}^{n} a_{n-k}(Z)\widehat{\Xi}_k^{\Delta^{n-k}} = \sum_{k=1}^{n} a_{n-k}(Z)\left(Z_k^{(p^{n-k})} - \widehat{\Xi}_k^{(p^{n-k})}\right).$$

Then according to Proposition 2.6 and Lemma 4.2, one gets

$$f_{V(Z)}\widehat{u} = \left(\sum_{n=0}^{\infty} a_n(Z)X^{p^n}\right)\left(pI - \sum_{k=1}^{\infty} \widehat{\Xi}_k \blacktriangle^k\right) = v_Z(pX) = v_Z(\widehat{f}\widehat{u}) = (v_Z\widehat{f})\widehat{u}.$$

Since $\widehat{u} \equiv pI \mod \blacktriangle$ and the right multiplication by $\widehat{u}$ in $\mathcal{K}[[X]]_p^d$ corresponds to the right multiplication by $\widehat{u}$ in $\mathrm{M}_d(\mathcal{K}[[\blacktriangle]])$, we deduce that it is injective, and hence $f_{V(Z)} = v_Z\widehat{f}$. This implies (1). Clearly, (2) follows from (1). ∎

We have a natural bijection from the set $I + \mathrm{M}_d(p\mathcal{O} + \blacktriangle E)/I + \mathrm{M}_d(E)\widehat{u}$ to the set $\mathrm{M}_d(p\mathcal{O} + \blacktriangle E)/\mathrm{M}_d(E)\widehat{u}$ which maps $v(I + \mathrm{M}_d(E)\widehat{u})$ to $v - I + \mathrm{M}_d(E)\widehat{u}$. The correspondence $pcI + \blacktriangle q + \mathrm{M}_d(E)\widehat{u} \mapsto (pcI - cu)\blacktriangle^{-1} + q + \mathrm{M}_d(E)\widehat{u}$ is an $\mathcal{O}$-module isomorphism from $\mathrm{M}_d(p\mathcal{O} + \blacktriangle E)/\mathrm{M}_d(E)\widehat{u}$ to $\mathrm{M}_d(E)/\mathrm{M}_d(E)\widehat{u}$. The latter module is isomorphic as an $\mathcal{O}$-module to the direct sum of d copies of $\mathcal{D}(\Phi)$. Therefore $\mathrm{M}_d(p\mathcal{O} + \blacktriangle E)/\mathrm{M}_d(E)\widehat{u}$ is a free $\mathcal{O}$-module of rank dh. Fix an $\mathcal{O}$-basis of $\mathrm{M}_d(p\mathcal{O} + \blacktriangle E)/\mathrm{M}_d(E)\widehat{u}$. Taking into account that χ is bijective (Proposition 4.3), one obtains a parametrization of the set of strict $\star$-isomorphism classes of deformations of Φ over $\mathcal{O}$. Another parametrization of the same set is provided by Theorem 3.2(1) applied for $R = \mathcal{O}$. Using Proposition 4.4(1) and the arithmetic properties of the ring $\mathrm{M}_d(E)$, one can give an explicit formula which expresses the

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first parametrization through the second one. The same applies to two parameterizations of the set of $\star$-isomorphism classes of deformations, one constructed with the aid of χ' and the other coming from Theorem 3.2(2).

We illustrate the last observation for the set of $\star$-isomorphism classes of deformations in a particular example where $\Xi_m = I$, for some $m \geqslant 1$, $\Xi_n = 0$ for $n \neq m$. This example is a direct generalization of the setting considered by Gross and Hopkins [8] for one-dimensional formal groups. In this case, one has $\widehat{u} = pI - I\blacktriangle^m$ and the height of Φ is equal to dm. Moreover, $\Psi = \{(n, i, j) | 1 \leqslant i, j \leqslant d; 1 \leqslant n \leqslant m\}$ (see [5, Example 1]), and hence, $\Psi^+ = \{(n, i, j) | 1 \leqslant i, j \leqslant d; 1 \leqslant n \leqslant m-1\}$.

For any $(\tau_\psi)_{\psi\in\Psi^+}$, $\tau_\psi \in p\mathcal{O}$, denote by C_n the matrix in $\mathrm{M}_d(p\mathcal{O})$ such that $C_n(i, j) = \tau_{(n,i,j)}$, $(n, i, j) \in \Psi^+$. If $Z = \widehat{\Xi} + \sum_{\psi\in\Psi^+} \tau_\psi \widehat{B}_\psi$, then $Z_n = C_n$ for $1 \leqslant n \leqslant m-1$, $Z_m = I$, $Z_n = 0$ for $n \geqslant m+1$, and

$$v_Z = I + p^{-1} \sum_{n=1}^{\infty} \sum_{k=1}^{\min(n,m-1)} a_{n-k}(Z) C_k^{(p^{n-k})} \blacktriangle^n.$$

We can write v_Z in the form $v_Z = \sum_{n=0}^{\infty} \nu_n \blacktriangle^n$ with $\nu_n \in \mathrm{M}_d(\mathcal{O})$, $\nu_0 = I$. Since $\widehat{u} = pI - I\blacktriangle^m$, one gets $I\blacktriangle^{km} \equiv p^k I \bmod \mathrm{M}_d(E)\widehat{u}$ and therefore $v_Z \equiv \sum_{i=0}^{m-1} \alpha_i \blacktriangle^i \bmod \mathrm{M}_d(E)\widehat{u}$ where $\alpha_i = \sum_{j=0}^{\infty} p^j \nu_{jm+i} \in \mathrm{M}_d(\mathcal{O})$. This means that there exists $r \in \mathrm{M}_d(E)$ such that $\sum_{i=0}^{m-1} \alpha_i \blacktriangle^i = v_Z + r\widehat{u} = v_Z(I + v_Z^{-1} r\widehat{u})$. Thus $\sum_{i=0}^{m-1} \alpha_i \blacktriangle^i \in v_Z(I + \mathrm{M}_d(E)\widehat{u})$, and it is obviously a unique element of this form in $v_Z(I + \mathrm{M}_d(E)\widehat{u})$. Moreover, $\alpha_0 \equiv I \mod p$ which implies that α_0 is invertible in $\mathrm{M}_d(\mathcal{O})$ and $\alpha_0^{-1} \equiv I \mod p$. For $1 \leqslant i \leqslant m-1$, denote $\beta_i = \alpha_0^{-1}\alpha_i$. Then

$$I + \sum_{i=1}^{m-1} \beta_i \blacktriangle^i = \alpha_0^{-1} \sum_{i=0}^{m-1} \alpha_i \blacktriangle^i \in (I + \mathrm{M}_d(p\mathcal{O})) v_Z (I + \mathrm{M}_d(E)\widehat{u}).$$

We consider β_i, $1 \leqslant i \leqslant m-1$, as functions of $(\tau_\psi)_{\psi\in\Psi^+}$ and establish explicit formulas for them.

Proposition 4.5. *For any* $(\tau_\psi)_{\psi\in\Psi^+}$, $\tau_\psi \in p\mathcal{O}$, $1 \leqslant i \leqslant m-1$

$$\alpha_i = \lim_{k\to\infty} p^k a_{km+i}(Z), \qquad \beta_i = \lim_{k\to\infty} a_{km}^{-1}(Z) a_{km+i}(Z).$$

Proof. We write as above $v_Z = \sum_{n=0}^{\infty} \nu_n \blacktriangle^n$, $\nu_n \in \mathrm{M}_d(\mathcal{O})$. Then $\nu_0 = I$ and

$$\nu_n = p^{-1} \sum_{k=1}^{\min(n,m-1)} a_{n-k}(Z) C_k^{(p^{n-k})}.$$

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According to Lemma 3.1, $\nu_n = a_n(Z)$ for $0 \leqslant n \leqslant m-1$, $\nu_n = a_n(Z) - p^{-1}a_{n-m}(Z)$ for $n \geqslant m$. These formulas imply

$$\begin{aligned}\alpha_i &= \sum_{j=0}^{\infty} p^j \nu_{jm+i} = \lim_{k\to\infty} \sum_{j=0}^{k} p^j \nu_{jm+i} \\ &= \lim_{k\to\infty} \left(a_i(Z) + \sum_{j=1}^{k} p^j (a_{jm+i}(Z) - p^{-1} a_{(j-1)m+i}(Z)) \right) \\ &= \lim_{k\to\infty} p^k a_{km+i}(Z),\end{aligned}$$

for any $0 \leqslant i \leqslant m-1$. The required formula for β_i follows immediately. ∎

Remark that for any $0 \leqslant i \leqslant m-1$, the infinite sequence $p^k a_{km+i}(\widehat{\Xi}^*) \in \mathrm{M}_d(\mathcal{O}[[t_\psi]]_{\psi\in\Psi^+})$ of matrices of polynomials converges in the rigid metric to a matrix of rigid analytic functions. Thus α_i can be considered as a matrix of rigid analytic functions of variables $(\tau_\psi)_{\psi\in\Psi^+}$.

When $d = 1$, the functions α_i, $0 \leqslant i \leqslant m-1$, coincide up to a constant factor with the homogeneous coordinates of the p-adic period map introduced by Gross and Hopkins; see [8, 21.6, 21.13 and 23.6].

§5. Action of the automorphism group

Let F be a deformation of Φ over $\mathcal{O}$. For $\phi \in \mathrm{Aut}_k(\Phi)$, we put $[F]'\phi = [g^{-1}(F(g,g))]'$, where $g \in \mathcal{O}[[X]]_0^d$ is such that its reduction is equal to ϕ. This defines a right action of $\mathrm{Aut}_k(\Phi)$ on the set of $\star$-isomorphism classes of the deformations of Φ over $\mathcal{O}$. In the case of dimension one, this definition is due to Lubin and Tate [12]. We give an explicit description of this action in terms of the parametrization provided by Theorem 3.2(2).

It is easy to notice that the correspondence $(I + \mathrm{M}_d(p\mathcal{O}))v(I + \mathrm{M}_d(E)\widehat{u}) \mapsto \mathrm{GL}_d(\mathcal{O})v(I + \mathrm{M}_d(E)\widehat{u})$ is a bijection from $I + \mathrm{M}_d(p\mathcal{O})\backslash I + \mathrm{M}_d(p\mathcal{O} + \blacktriangle E)/I + \mathrm{M}_d(E)\widehat{u}$ to $\mathrm{GL}_d(\mathcal{O})\backslash \mathrm{GL}_d(E)/I + \mathrm{M}_d(E)\widehat{u}$. Thus we can identify these two sets. Define a right action of $\mathrm{Aut}_k(\Phi)$ on $\mathrm{GL}_d(\mathcal{O})\backslash \mathrm{GL}_d(E)/I + \mathrm{M}_d(E)\widehat{u}$ in the following way: by Theorem 2.5(3), for any $\phi \in \mathrm{Aut}_k(\Phi)$ there exists $w \in \mathrm{GL}_d(E)$ such that ϕ is equal to the reduction of $\widehat{f}^{-1}(w\widehat{f})$ modulo p. For $v \in \mathrm{GL}_d(E)$, put $\mathrm{GL}_d(\mathcal{O})v(I + \mathrm{M}_d(E)\widehat{u})\phi = \mathrm{GL}_d(\mathcal{O})vw(I + \mathrm{M}_d(E)\widehat{u})$. Theorem 2.5(1) implies that this expression does not depend on the choice of v in the corresponding coset. If $w' \in \mathrm{GL}_d(E)$ is such that the reduction of $\widehat{f}^{-1}(w'\widehat{f})$ modulo p is equal to ϕ, then by Lemma 2.1(1), $w\widehat{f} \equiv w'\widehat{f} \mod \mathfrak{P}$ and Proposition 2.4 implies that

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there exists $s \in \mathrm{M}_d(E)$ such that $w' - w = s\widehat{u}$, i.e., $w' = w(1 + w^{-1}s\widehat{u}) \in w(I + \mathrm{M}_d(E)\widehat{u})$. Thus the definition does not depend on the choice of w.

THEOREM 5.1. *χ' is equivariant with respect to the action of $\mathrm{Aut}_k(\Phi)$.*

Proof. Let F be a deformation of Φ over $\mathcal{O}$ with logarithm f and $\phi \in \mathrm{Aut}_k(\Phi)$. By Proposition 2.3, there exists $v \in I + \mathrm{M}_d(\blacktriangle E)$ such that $v\widehat{f} \equiv f \mod \mathfrak{P}$. By Theorem 2.5(3), ϕ is equal to the reduction of $g = \widehat{f}^{-1}(w\widehat{f})$ for some $w \in \mathrm{GL}_d(E)$. The formal group with logarithm $w_0^{-1} f \circ g$ belongs to the coset $[F]'\phi$, where $w_0 \in \mathrm{GL}_d(\mathcal{O})$ is such that $w \equiv w_0 \mod \blacktriangle$. On the other hand, Lemma 2.1(2) implies $w_0^{-1} f \circ g \equiv w_0^{-1}(v\widehat{f}) \circ \widehat{f}^{-1}(w\widehat{f}) \equiv w_0^{-1} v w \widehat{f} \mod \mathfrak{P}$, and hence, $\chi'([F]'\phi) = \mathrm{GL}_d(\mathcal{O}) v w (I + \mathrm{M}_d(E)\widehat{u}) = \chi'([F]')\phi$. ∎

THEOREM 5.2. *Let $w \in \mathrm{GL}_d(E)$ and $Z^{(i)} \in \mathrm{M}_d(\mathcal{O})^\infty$ be such that the reduction of $\widehat{f}^{-1}(w\widehat{f})$ is equal to $\phi \in \mathrm{Aut}_k(\Phi)$ and $Z^{(i)} = \widehat{\Xi} + \sum_{\psi \in \Psi^+} \tau_\psi^{(i)} \widehat{B}_\psi$ for some $\tau_\psi^{(i)} \in p\mathcal{O}$, $i = 1, 2$. Then $[F_{V(Z^{(1)})}]'\phi = [F_{V(Z^{(2)})}]'$ iff there exist $c \in \mathrm{GL}_d(\mathcal{O})$ and $q \in \mathrm{M}_d(E)$ such that $v_{Z^{(2)}} = c v_{Z^{(1)}} w (I + q\widehat{u})$.*

Proof. According to Theorem 5.1, Proposition 4.4 and the definition of the action of $\phi \in \mathrm{Aut}_k(\Phi)$ on $\mathrm{GL}_d(\mathcal{O}) \backslash \mathrm{GL}_d(E) / I + \mathrm{M}_d(E)\widehat{u}$, we get

$$\begin{aligned} \chi'([F_{V(Z^{(2)})}]') &= \chi'([F_{V(Z^{(1)})}]'\phi) = \chi'([F_{V(Z^{(1)})}]')\phi \\ &= \mathrm{GL}_d(\mathcal{O}) v_{Z^{(1)}} (I + \mathrm{M}_d(E)\widehat{u})\phi = \mathrm{GL}_d(\mathcal{O}) v_{Z^{(1)}} w (I + \mathrm{M}_d(E)\widehat{u}). \end{aligned}$$

On the other hand, $\chi'([F_{V(Z^{(2)})}]') = \mathrm{GL}_d(\mathcal{O}) v_{Z^{(2)}} (I + \mathrm{M}_d(E)\widehat{u})$. This implies the required statement. ∎

COROLLARY. *In the setting of Theorem 5.2, suppose that $\Xi_m = I$, $\Xi_n = 0$ for $n \neq m$, and choose $w_i \in \mathrm{M}_d(\mathcal{O})$ such that $w \equiv \sum_{i=0}^{m-1} w_i \blacktriangle^i \mod \mathrm{M}_d(E)\widehat{u}$. Then $[F_{V(Z^{(1)})}]'\phi = [F_{V(Z^{(2)})}]'$ iff there exists $c \in \mathrm{GL}_d(\mathcal{O})$ such that*

$$\left(\alpha_0^{(2)}, \ldots, \alpha_{m-1}^{(2)}\right) = c \left(\alpha_0^{(1)}, \ldots, \alpha_{m-1}^{(1)}\right) C(w),$$

where

$$C(w) = \begin{pmatrix} w_0 & w_1 & w_2 & & w_{m-1} \\ p w_{m-1}^{\Delta} & w_0^{\Delta} & w_1^{\Delta} & \cdots & w_{m-2}^{\Delta} \\ p w_{m-2}^{\Delta^2} & p w_{m-1}^{\Delta^2} & w_0^{\Delta^2} & & w_{m-3}^{\Delta^2} \\ & \vdots & & \ddots & \vdots \\ p w_1^{\Delta^{m-1}} & p w_2^{\Delta^{m-1}} & p w_3^{\Delta^{m-1}} & \cdots & w_0^{\Delta^{m-1}} \end{pmatrix}.$$

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Proof. As we showed above $\sum_{i=0}^{m-1} \alpha_i^{(2)} \blacktriangle^i \in v_{Z^{(2)}}(I + \mathrm{M}_d(E)\widehat{u})$. Besides, Theorem 5.2 implies

$$\begin{aligned}\mathrm{GL}_d(\mathcal{O}) v_{Z^{(2)}}(I + \mathrm{M}_d(E)\widehat{u}) &= \mathrm{GL}_d(\mathcal{O}) v_{Z^{(1)}} w(I + \mathrm{M}_d(E)\widehat{u}) \\ &= \mathrm{GL}_d(\mathcal{O}) \left(\sum_{i=0}^{m-1} \alpha_i^{(1)} \blacktriangle^i \right) w(I + \mathrm{M}_d(E)\widehat{u}).\end{aligned}$$

Hence, there exist $c \in \mathrm{GL}_d(\mathcal{O})$ and $q \in \mathrm{M}_d(E)$ such that

$$\begin{aligned}\sum_{i=0}^{m-1} \alpha_i^{(2)} \blacktriangle^i &= c \left(\sum_{i=0}^{m-1} \alpha_i^{(1)} \blacktriangle^i \right) w(1 + q\widehat{u}) \\ &\equiv c \left(\sum_{i=0}^{m-1} \alpha_i^{(1)} \blacktriangle^i \right) w \bmod \mathrm{M}_d(E)\widehat{u}.\end{aligned}$$

It is easy to check directly that

$$\left(\sum_{i=0}^{m-1} \alpha_i^{(1)} \blacktriangle^i \right) w \equiv \left(\alpha_0^{(1)}, \ldots, \alpha_{m-1}^{(1)} \right) C(w)(1, \blacktriangle, \ldots, \blacktriangle^{m-1})^T \bmod \mathrm{M}_d(E)\widehat{u}.$$

Therefore

$$\sum_{i=0}^{m-1} \alpha_i^{(2)} \blacktriangle^i \equiv c \left(\alpha_0^{(1)}, \ldots, \alpha_{m-1}^{(1)} \right) C(w) \left(1, \blacktriangle, \ldots, \blacktriangle^{m-1}\right)^T \bmod \mathrm{M}_d(E)\widehat{u}.$$

This means that there exists $r \in \mathrm{M}_d(E)$ such that

$$\sum_{i=0}^{m-1} \alpha_i^{(2)} \blacktriangle^i - c \left(\alpha_0^{(1)}, \ldots, \alpha_{m-1}^{(1)} \right) C(w) \left(1, \blacktriangle, \ldots, \blacktriangle^{m-1}\right)^T = r\widehat{u}.$$

Since the highest power of $\blacktriangle$ which appears in the left side of the equality is less or equal to $m-1$, we conclude that r must be equal to 0, which implies the required statement. □

In the case $d = 1$, the last Corollary provides the result by Gross and Hopkins on equivariance of their p-adic period map [8, Proposition 23.5].

https://doi.org/10.1017/nmj.2017.43 Published online by Cambridge University Press

Acknowledgment. The authors thank the referee for a careful reading and several helpful suggestions.

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