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POSITIVE-DEFININTE FUNCTIONS ON FREE SEMIGROUPS

GELU POPESCU

ABSTRACT. An extension of the Naimark dilation theorem [N], [SzF2] to positive-definite functions on free semigroups is given. This is used to extend the operatorial trigonometric moment problem [A] to a non-commutative setting and to characterize the classes $\mathcal{C}_\rho$ ($\rho > 0$) of all n-tuples of operators that have a ρ-isometric dilation (see [SzF2] for the case $n = 1$). It is also shown that $\mathcal{C}_\rho \subset \mathcal{C}_{\rho'}$ and $\mathcal{C}_\rho \neq \mathcal{C}_{\rho'}$ for $0 < \rho < \rho' < \infty$.

The von Neumann inequality [vN], [Po2] is extended to the classes $\mathcal{C}_\rho$. This is used to prove that any element in $\mathcal{C}_\rho$ is simultaneously similar to an element in $\mathcal{C}_1$.

1. **Introduction and preliminaries.** Let us consider the full Fock space [E]

$$F^2(H_n) = \mathbf{C}1 \oplus \bigoplus_{m\geq 1} H_n^{\otimes m}$$

where H_n is an n-dimensional complex Hilbert space with orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ if n is finite, or $\{e_1, e_2, \ldots\}$ if $n = \infty$. For each $i = 1, 2, \ldots$, $S_i \in B\big(F^2(H_n)\big)$ is the left creation operator with e_i, *i.e.*, $S_i\xi = e_i \otimes \xi$, $\xi \in F^2(H_n)$. We shall denote by $\mathcal{P}_n$ the set of all $p \in F^2(H_n)$ of the form

$$p = a_0 + \sum_{\substack{1\leq i_1,\ldots,i_k\leq n\\ 1\leq k\leq m}} a_{i_1\cdots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k}, \quad m \in \mathbb{N}, \tag{1.1}$$

where $a_0, a_{i_1\cdots i_k} \in \mathbf{C}$ and the sum contains only a finite number of summands. The set $\mathcal{P}_n$ may be viewed as the algebra of polynomials in n non-commuting indeterminates, with $p \otimes q$, $p, q \in \mathcal{P}_n$, as multiplication. Define F_n^∞ as the set of all $g \in F^2(H_n)$ such that

$$\|g\|_\infty := \sup\{\|g \otimes p\|_2 : p \in \mathcal{P}_n, \|p\|_2 \leq 1\} < \infty$$

where $\|\cdot\| = \|\cdot\|_{F^2(H_n)}$. $(F_n^\infty, \|\cdot\|_\infty)$ is a non-commutative Banach algebra [Po2]. We denote by $\mathcal{A}_n$ the closure of $\mathcal{P}_n$ in $(F_n^\infty, \|\cdot\|_\infty)$. The Banach algebra F_n^∞ (resp. $\mathcal{A}_n$) can be viewed as a non-commutative analogue of the Hardy space H^∞ (resp. disc algebra); when $n = 1$ they coincide.

Let $\big(B(\mathcal{H})^n\big)_1$ denote the unit ball of $\big(B(\mathcal{H})^n\big)_1$, *i.e.*,

$$\big(B(\mathcal{H})^n\big)_1 = \{(T_1, \ldots, T_n) \in B(\mathcal{H})^n : \sum_{i=1}^n T_iT_i^* \leq I_{\mathcal{H}}\}.$$

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For any sequence $T_1, T_2, \dots, T_n \in B(\mathcal{H})$ and $p \in \mathcal{P}_n$ given by (1.1) we denote by $p(T_1, \dots, T_n)$ the operator acting on $\mathcal{H}$, defined by

$$p(T_1, \dots, T_n) = a_0 I_{\mathcal{H}} + \sum a_{i_1 \cdots i_k} T_{i_1} \cdots T_{i_k}.$$

The von Neumann inequality [vN], [SzF2] for $\big(B(\mathcal{H})^n\big)_1$ (see [Po2]) asserts that if $(T_1, \dots, T_n) \in \big(B(\mathcal{H})^n\big)_1$ and $p \in \mathcal{P}_n$, then

$$\|p(T_1, \dots, T_n)\| \leq \|p(S_1, \dots, S_n)\| = \|p\|_\infty. \tag{1.2}$$

According to [Po2] the mapping

$$\Psi: \mathcal{A}_n \longrightarrow B(\mathcal{H}); \quad \Psi(f) = f(T_1, \dots, T_n)$$

is a contractive homomorphism.

2. **Positive-definite kernels.** A positive-definite kernel on a set Σ is a map

$$K: \Sigma \times \Sigma \longrightarrow B(\mathcal{H})$$

with the property that $K(\sigma, \omega) = K(\omega, \sigma)^*$, $(\sigma, \omega \in \Sigma)$ and for each $k \in \mathbf{N}$, for each choice of vectors $h_1, \dots, h_k$ in $\mathcal{H}$, and $\sigma_1, \dots, \sigma_k$ in Σ the inequality

$$\sum_{i,j=1}^{k} \langle K(\sigma_i, \sigma_j) h_j, h_i \rangle \geq 0$$

holds.

A Kolmogorov decomposition for K is a map $V: \Sigma \longrightarrow B(\mathcal{H}, \mathcal{K})$, where $\mathcal{K}$ is a Hilbert space, such that $K(\sigma, \omega) = V(\sigma)^* V(\omega)$, for any $\sigma, \omega \in \Sigma$. If $\mathcal{K} = \bigvee_{\sigma \in \Sigma} V(\sigma)\mathcal{H}$ the decomposition is said to be minimal and it is a standard fact that two minimal decompositions are equivalent in an appropriate sense [PS].

Let $\Sigma = \mathbf{F}_n^+$ be the unital free semigroup on n generators: $s_1, \dots, s_n$. A kernel K on $\mathbf{F}_n^+$ is called *Toeplitz* if it has the following properties: $K(e, e) = I_{\mathcal{H}}$, ($e$ is the neutral element in $\mathbf{F}_n^+$) and

$$K(\sigma, \omega) = \begin{cases} K(w, e); & \text{if } \sigma = \omega w \text{ for some } w \in \mathbf{F}_n^+ \\ K(e, w); & \text{if } \omega = \sigma w \text{ for some } w \in \mathbf{F}_n^+ \\ 0; & \text{otherwise} \end{cases}$$

Let K be a positive-definite Toeplitz kernel on $\mathbf{F}_n^+$. We say that K has a Naimark dilation if there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a sequence $V_{s_1}, \dots, V_{s_n}$ of isometries on $\mathcal{K}$, with orthogonal ranges, such that

$$K(e, \sigma) = \mathcal{P}_{\mathcal{H}} V_\sigma|_{\mathcal{H}}, \qquad \text{for any } \sigma \in \mathbf{F}_n^+,$$

where for any $\sigma = s_{i_1} \cdots s_{i_k} \in \mathbb{F}_n^+$, $V_\sigma = V_{s_{i_1}} \cdots V_{s_{i_k}}$, and if $\sigma = e$ then $V_\sigma = I_{\mathcal{K}}$. The Naimark dilation is called minimal if $\mathcal{K} = \bigvee_{\sigma \in \mathbf{F}_n^+} V_\sigma \mathcal{H}$. The sequence $\{V_{s_1}, \dots, V_{s_n}\}$ is called the minimal isometric dilation of K.

In what follows we present an extension of the Naimark dilation theorem [Theorem 7.1, pg. 25, SzF2] to free semigroups. The proof of this result uses the ideas of the classical result.

THEOREM 2.1. *A Toeplitz kernel on $\mathbb{F}_n^+$ is positive-definite if and only if it admits a minimal Naimark dilation. In this case its minimal Naimark dilation is unique up to an isomorphism.*

PROOF. Assume $K: \mathbb{F}_n^+ \times \mathbb{F}_n^+ \longrightarrow B(\mathcal{H})$ is a positive-definite Toeplitz kernel. Let $\mathcal{K}_0$ be the set of all finitely supported sequences $\{h_\sigma\}_{\sigma \in \mathbb{F}_n^+}$ in $\mathcal{H}$. Define the bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{K}_0$ by

$$\langle \{h_\omega\}_{\omega \in \mathbb{F}_n^+}, \{k_\sigma\}_{\sigma \in \mathbb{F}_n^+} \rangle := \sum_{\omega, \sigma \in \mathbb{F}_n^+} \langle K(\sigma, \omega) h_\omega, k_\sigma \rangle_{\mathcal{H}}.$$

Since K is positive-definite $\langle \cdot, \cdot \rangle$ is positive semi-definite. Consider $\mathcal{N} = \{k \in \mathcal{K}_0 : \langle k, k \rangle = 0\}$ and $\mathcal{K}_0|_{\mathcal{N}}$. Let $\mathcal{K}$ be the Hilbert space obtained by completing $\mathcal{K}_0|_{\mathcal{N}}$ with the induced inner product. Let us define the operators $V_{s_i}(i = 1, 2, \ldots, n)$ on $\mathcal{K}_0$ by

$$V_{s_i}(\{h_\sigma\}_{\sigma \in \mathbb{F}_n^+}) = \{\delta_{s_i\sigma}(t) h_\sigma\}_{t \in \mathbb{F}_n^+},$$

where $\delta_{s_i\sigma}(t) = 1$ if $t = s_i\sigma$ and $\delta_{s_i\sigma}(t) = 0$ otherwise. One can prove that $\{V_{s_1}, \ldots, V_{s_n}\}$ extend by continuity to isometries on $\mathcal{K}$ with orthogonal ranges. Indeed, since

$$\begin{aligned}\langle V_{s_i}(\{h_\omega\}), V_{s_i}(\{h'_\sigma\}) \rangle &= \sum_{s,t \in \mathbb{F}_n^+} \langle K(s,t) \delta_{s_i\omega}(t) h_\omega, \delta_{s_i\sigma}(s) h'_\sigma \rangle \\ &= \sum_{\sigma, \omega \in \mathbb{F}_n^+} \langle K(s_i\sigma, s_i\omega) h_\omega, h'_\sigma \rangle = \sum_{\sigma, \omega \in \mathbb{F}_n^+} \langle K(\sigma, \omega) h_\omega, h'_\sigma \rangle \\ &= \langle \{h_\omega\}, \{h'_\sigma\} \rangle,\end{aligned}$$

the operators $\{V_{s_i}\}_{i=1}^n$ extend by continuity to isometries on $\mathcal{K}$. Moreover, since $K(s_i\sigma, s_j\omega) = 0$ for any $i \neq j$, $\sigma, \omega \in \mathbb{F}_n^+$, it follows that they have orthogonal ranges.

Embed $\mathcal{H}$ in $\mathcal{K}$ by setting $h = \{\delta_e(t) h)\}_{t \in \mathbb{F}_n^+}$ where

$$\delta_e(t) = \begin{cases} 1; & \text{if } t = e \\ 0; & \text{if } t \neq e. \end{cases}$$

This identification is allowed since it preserves the linear and metric structure of $\mathcal{H}$. Indeed, we have

$$\begin{aligned}\langle \delta_e h, \delta_e h' \rangle_{\mathcal{K}} &= \sum_{t,s \in \mathbb{F}_n^+} \langle K(t,s) \delta_e(s) h, \delta_e(t) h' \rangle_{\mathcal{H}} \\ &= \langle K(e,e) h, h' \rangle_{\mathcal{H}} = \langle h, h' \rangle_{\mathcal{H}}.\end{aligned}$$

For any $h, h' \in \mathcal{H}$ and $\sigma \in \mathbb{F}_n^+$ we have

$$\langle V_\sigma h, h' \rangle_{\mathcal{K}} = \langle \delta_\sigma(t) h, \delta_e(t) h' \rangle_{\mathcal{K}} = \langle K(e, \sigma) h, h' \rangle_{\mathcal{H}}$$

which implies $\langle P_{\mathcal{H}} V_\sigma h, h' \rangle_{\mathcal{H}} = \langle K(e,\sigma) h, h' \rangle_{\mathcal{H}}$. Therefore $K(e, \sigma) = P_{\mathcal{H}} V_\sigma|_{\mathcal{H}}$ for any $\sigma \in \mathbb{F}_n^+$. Let us observe that every element in $\mathcal{K}_0$ can be considered as a finite sum of terms of type $\{\delta_\sigma(t)\}_{t \in \mathbb{F}_n^+}$ and hence every element $k \in \mathcal{K}_0$ can decomposed into a finite sum of terms of the type $V_\sigma h$, $\sigma \in \mathbb{F}_n^+$, $h \in \mathcal{H}$. This implies $\mathcal{K} = \bigvee_{\sigma \in \mathbb{F}_n^+} V_\sigma \mathcal{H}$, *i.e.*, the Naimark dilation is minimal.

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To prove the uniqueness let $\{V'_{s_1},\dots,V'_{s_n}\}$ be another minimal dilation of K on a Hilbert space $\mathcal{K}'\supset\mathcal{H}$. One can prove that there is a unitary operator $W:\mathcal{K}\longrightarrow\mathcal{K}'$ such that $WV_{s_i}=V'_{s_i}W$, for any $i=1,2,\dots,n$, and $W|_{\mathcal{H}}=I_{\mathcal{H}}$. To see this, it is sufficient to define

$$W\Big(\sum_{\substack{\sigma\in\mathbb{F}_n^+\\ |\sigma|\le m}} V_\sigma h_\sigma\Big)=\sum_{\substack{\sigma\in\mathbb{F}_n^+\\ |\sigma|\le m}} V'_\sigma h_\sigma \quad (h_\sigma\in\mathcal{H}) \tag{2.1}$$

for any $m=0,1,2,\dots$. Here $|\sigma|$ stands for the length of σ, *i.e.*, $|\sigma|=k$ if $\sigma=s_{i_1}\cdots s_{i_k}$. Since

$$\begin{aligned}\langle V_\omega h, V_\sigma h'\rangle_{\mathcal{K}}&=\langle\delta_\omega(t)h,\delta_\sigma(t)h'\rangle_{\mathcal{K}}\\&=\sum_{t,s\in\mathbb{F}_n^+}\langle K(s,t)\delta_\omega(t)h,\delta_\sigma(s)h'\rangle_{\mathcal{H}}=\langle K(\sigma,\omega)h,h'\rangle_{\mathcal{H}}\end{aligned}$$

the operator W defined by (2.1) is correctly defined, isometric, and in view of minimality, it extends to a unitary operator between $\mathcal{K}$ and $\mathcal{K}'$.

Conversely, let $V_{s_1},\dots,V_{s_n}$ be a sequence of isometries on $\mathcal{K}\supset\mathcal{H}$ with orthogonal ranges. Assume that $K\colon\mathbb{F}_n^+\times\mathbb{F}_n^+\longrightarrow B(\mathcal{H})$ is the kernel defined by

$$K(e,\sigma)=P_{\mathcal{H}}V_\sigma|_{\mathcal{H}},\quad \text{for any } \sigma\in\mathbb{F}_n^+.$$

Since for any finitely supported sequence $\{h_\omega\}_{\omega\in\mathbb{F}_n^+}\subset\mathcal{H}$

$$\begin{aligned}\sum_{\sigma,\omega\in\mathbb{F}_n^+}\langle K(\sigma,\omega)h_\omega,h_\sigma\rangle&=\sum_{\sigma,\omega\in\mathbb{F}_n^+}\langle V_\sigma^*V_\omega h_\omega,h_\sigma\rangle\\&=\Big\|\sum_{\sigma\in\mathbb{F}_n^+}V_\sigma h_\sigma\Big\|^2\ge 0\end{aligned}$$

we infer that K is a positive-definite Toeplitz kernel. The proof is complete. ■

COROLLARY 2.2. *Let $T_1,\dots,T_n\in B(\mathcal{H})$. Then the operator matrix $[T_1,\dots,T_n]$ is a contraction if and only if the Toeplitz kernel*

$$K_c\colon\mathbb{F}_n^+\times\mathbb{F}_n^+\longrightarrow B(\mathcal{H})$$

defined by $K_c(e,\omega)=T_\omega$, $K_c(\omega,e)=T_\omega^*$, where for any $w=s_{i_1}\cdots s_{i_k}$, $T_w:=T_{i_1}\cdots T_{i_k}$, is positive-definite.

PROOF. Assume that $[T_1,\dots,T_n]$ is a contraction. According to [Po1] there is a Hilbert space $\mathcal{K}\supset\mathcal{H}$ and a sequence $V_1,\dots,V_n$ of isometries on $\mathcal{K}$ such that

$$\sum_{i=1}^n V_iV_i^*\le I_{\mathcal{H}}\quad\text{and}\quad V_i^*|_{\mathcal{H}}=T_i^*,\quad\text{for any } i=1,2,\dots,n.$$

Thus, for any $\omega\in\mathbb{F}_n^+$ we have

$$K_c(e,\omega)=T_\omega=P_{\mathcal{H}}V_\omega|_{\mathcal{H}}\quad\text{and}\quad K_c(\omega,e)=T_\omega^*=V_\omega^*.$$

Therefore, for any finitely supported sequence $\{h_\omega\}_{\omega\in\mathbb{F}_n^+}\subset B(\mathcal{H})$ we have

$$\sum_{\sigma,\omega\in\mathbb{F}_n^+}\langle K_c(\sigma,\omega)h_\omega,h_\sigma\rangle=\sum_{\sigma,\omega\in\mathbb{F}_n^+}\langle V_\sigma^*V_\omega h_\omega,h_\sigma\rangle$$
$$=\Big\|\sum_{\omega\in\mathbb{F}_n^+}V_\omega h_\omega\Big\|^2\geq 0,$$

which proves that the Toeplitz kernel K_c is positive-definite.

Conversely, assume K_c is positive-definite. According to Theorem 2.1 there exists a Hilbert space $\mathcal{K}\supset\mathcal{H}$ and a sequence $V_{s_1},\dots,V_{s_n}$ of isometries on $\mathcal{K}$ with orthogonal ranges such that

$$T_{i_1}\cdots T_{i_k}=K_c(e,s_{i_1}\cdots s_{i_k})=P_{\mathcal{H}}V_{s_{i_1}}\cdots V_{s_{i_k}}|_{\mathcal{H}}$$

for any $1\leq i_1,\dots,i_k\leq n$.

Therefore,

$$\sum_{i=1}^n\|T_i^*h\|^2\leq\sum_{i=1}^n\|V_i^*h\|^2\leq\|h\|^2$$

which shows that $[T_1,\dots,T_n]$ is a contraction. ∎

3. **Generalized trigonometric moment problem.** An $B(\mathcal{H})$-valued semispectral measure on $\mathbf{T}=\{z\in\mathbf{C}:|z|=1\}$ is a linear positive map from $C(\mathbf{T})$, the set of continous functions on the unit circle, into $B(\mathcal{H})$. Since $C(\mathbf{T})$ is commutative μ is completely positive. The operatorial trigonometric moment problem says that, given the operators $A_k\in B(\mathcal{H})$, $k=0,1,\dots,m(A_0=I)$ there exists a semispectral measure on $\mathbf{T}$ such that $A_k=\mu(e^{ikt})$, $k=0,1,\dots,m$ if and only if the block matrix

$$\begin{bmatrix} I & A_1 & \cdots & A_m\\ A_1^* & I & \cdots & A_{m-1}\\ \vdots & \vdots & & \vdots\\ A_m^* & A_{m-1}^* & \cdots & I\end{bmatrix}$$

built up on the given operators$\{A_k\}_{k=1}^m$ is positive.

In what follows we will find a non-commutative analogue of this problem. The place of the multiplication by e^{it} is taken by the left creation operators $S_1,\dots,S_n$ on the full Fock space $F^2(H_n)$, and the place of $C(\mathbf{T})$ is taken by $C^*(S_1,\dots,S_n)$, the extension through compact operators of the Cuntz algebra O_n [Cu]. Let $\{A_{(\sigma)}\}_{\sigma\in\mathbb{F}_n^+,|\sigma|\leq m}$ be a sequence of operators in $B(\mathcal{H})$ such that $A_{(e)}=I_{\mathcal{H}}$. Define the operator matrix

$$M_m=\big(K(\omega,\sigma)\big)_{|\omega|\leq m,|\sigma|\leq m},\tag{3.1}$$

where K is the Toeplitz kernel on $\{\sigma\in\mathbb{F}_n^+,|\sigma|\leq m\}$ defined by $K(e,\sigma)=K(\sigma,e)^*=A_{(\sigma)}$. Notice that M_m is an operator on $\oplus_{k=1}^N\mathcal{H}$, where $N=1+n+n^2+\cdots+n^m$.

THEOREM 3.1. *Let $\{A_{(\sigma)}\}_{\sigma\in\mathbb{F}_n^+,|\sigma|\leq m}$ be a sequence of operators in $B(\mathcal{H})$ with $A_{(e)} = I_{\mathcal{H}}$. Then, there is a completely positive linear map*

$$\mu\colon C^*(S_1,\ldots,S_n)\longrightarrow B(\mathcal{H})$$

such that $\mu(S_\sigma) = A_{(\sigma)}$, $\sigma\in\mathbb{F}_n^+$, $|\sigma|\leq m$ if and only if the operator matrix M_m defined by (3.1) *is positive.*

PROOF. Assume that $\mu\colon C^*(S_1,\ldots,S_n)\longrightarrow B(\mathcal{H})$ is a completely positive linear map such that $\mu(S_\sigma) = A_{(\sigma)}$ for any $\sigma\in\mathbb{F}_n^+$, $|\sigma|\leq m$. According to the Stinespring theorem [S] there is a Hilbert space $\mathcal{K}\supset\mathcal{H}$ and a $*$-representation $\pi\colon C^*(S_1,\ldots,S_n)\longrightarrow B(\mathcal{K})$ such that

$$\mu(f) = P_{\mathcal{H}}\pi(f)|_{\mathcal{H}} \quad \text{for any } f\in C^*(S_1,\ldots,S_n).$$

Let $K\colon\mathbb{F}_n^+\times\mathbb{F}_n^+\longrightarrow B(\mathcal{H})$ be the Toeplitz kernel defined by $K(e,\sigma) = \mu(S_\sigma)$, $\sigma\in\mathbb{F}_n^+$. Since $\pi(S_1),\ldots,\pi(S_n)$ are isometries with orthogonal ranges, by Theorem 2.1,we infer that K is positive definite. In particular, the matrix

$$M_m = \big(K(\omega,\sigma)\big)_{|\omega|\leq m,|\sigma|\leq m}$$

is positive.

Conversely, assume that the matrix M_m is positive. Let $\mathcal{K}_m$ be the Hilbert space of all sequences of the form $\{h_\sigma\}_{|\sigma|\leq m}$ ($h_\sigma\in\mathcal{H}$) with the inner product

$$\langle\{h_\sigma\}_{|\sigma|\leq m},\{h'_\omega\}_{|\omega|\leq m}\rangle = \sum_{\omega,\sigma\in\mathbb{F}_n^+,|\omega|,|\sigma|\leq m}\langle K(\omega,\sigma)h_\sigma,h'_\omega\rangle$$

As in the proof of Theorem 2.1 we identify the zero element in $\mathcal{K}_m$ with all elements $k\in\mathcal{K}_m$ with $\|k\| = 0$. Let $\mathcal{X}$ be the subspace of $\mathcal{K}_m$ defined by $\mathcal{X} = \big\{\{h_\sigma\}_{|\sigma|\leq m-1}, h_\sigma\in\mathcal{H}\big\}$. For each $i = 1,2,\ldots,n$ let $T_{s_i}\colon\mathcal{X}\longrightarrow\mathcal{X}$ be defined by

$$T_{s_i}(\{h_\sigma\}_{|\sigma|\leq m-1}) = P_{\mathcal{X}}\{\delta_{s_i\sigma}(t)h_\sigma\}_{|t|\leq m}.$$

Embed $\mathcal{H}$ in $\mathcal{K}_m$ by setting $h = \{\delta_e(t)h\}_{|t|\leq m}$. As in the proof of Theorem 1.1 one can prove that $[T_{s_1},\ldots,T_{s_n}]$ is a contraction and

$$K(e,\sigma) = P_{\mathcal{H}}T_\sigma|_{\mathcal{H}} \quad \text{for } |\sigma|\leq m. \tag{3.2}$$

Let $V_{s_1},\ldots,V_{s_n}$ be an isometric dilation of $T_{s_1},\ldots,T_{s_n}$ on a Hilbert space $\mathcal{K}\supset\mathcal{X}\supset\mathcal{H}$ ([Po1]). This implies $T_{s_i} = P_{\mathcal{X}}V_{s_i}|_{\mathcal{X}}$ and by (3.2)

$$K(e,\sigma) = P_{\mathcal{H}}V_\sigma|_{\mathcal{H}} \quad \text{for any } |\sigma|\leq m. \tag{3.3}$$

Define $\mu\colon C^*(S_1,\ldots,S_n)\longrightarrow B(\mathcal{H})$ by

$$\mu(f) = P_{\mathcal{H}}f(V_1,\ldots,V_n)|_{\mathcal{H}}. \tag{3.4}$$

According to [Po3],$f\longmapsto f(V_1,\ldots,V_n)$ is a $*$-representation of $C^*(S_1,\ldots,S_n)$. Thus, μ is completely positive.

In particular, the relation (3.4) implies

$$\mu(S_\sigma) = P_{\mathcal{H}}V_\sigma|_{\mathcal{H}} \quad \text{for any } \sigma\in\mathbb{F}_n^+.$$

Hence and using the relation (3.3) we infer that

$$\mu(S_\sigma) = K(e,\sigma) = A_{(\sigma)} \quad \text{for any } \sigma\in\mathbb{F}_n^+,\quad |\sigma|\leq m.$$

The proof is complete. ∎

COROLLARY 3.2. *Let $\{A_{(\sigma)}\}_{\sigma\in\mathbb{F}_n^+}$ be a sequence of operators in $B(\mathcal{H})$. Then, there is a completely positive linear map $\mu: C^*(S_1,\dots,S_n)\longrightarrow B(\mathcal{H})$ such that $\mu(S_\sigma)=A_{(\sigma)}$, $\sigma\in\mathbb{F}_n^+$, if and only if the Toeplitz kernel $K:\mathbb{F}_n^+\times\mathbb{F}_n^+\longrightarrow B(\mathcal{H})$ defined by*

$$K(\sigma,e)^*=K(e,\sigma)=A_{(\sigma)}\quad \textit{for any } \sigma\in\mathbb{F}_n^+,$$

is positive-definite.

4. **ρ-contractions and similarity.**

Let C_ρ ($\rho>0$) be the set of all n-tuples of operators $T_1,\dots,T_n$ on a Hilbert $\mathcal{H}$ for which there exists an n-tuple of isometry $V_1,\dots,V_n$ on a Hilbert space $\mathcal{K}\supset\mathcal{H}$ such that

$$\sum_{i=1}^n V_iV_i^*\le I_{\mathcal{K}}\quad\text{and}\quad T_{i_1}\cdots T_{i_k}=\rho P_{\mathcal{H}}V_{i_1}\cdots V_{i_k}|_{\mathcal{H}},\tag{4.1}$$

for any $1\le i_1,\dots,i_k\le n$. In this case the sequence $T_1,\dots,T_n$ is called a ρ-contraction and $V_1,\dots,V_n$ is a dilation of it.

According to [Po1] we have $C_1=\big(B(\mathcal{H})^n\big)_1$. A dilation theory for this class was developed in [Po1].

THEOREM 4.1. *Let $A_1,\dots,A_n$ be in $B(\mathcal{H})$. Then $(A_1,\dots,A_n)\in C_\rho$ ($\rho>0$) if and only if*

$$\sum_{\sigma\in\mathbb{F}_n^+}\|h_\sigma\|^2+\frac{1}{\rho}\sum_{\substack{\sigma\in\mathbb{F}_n^+,\sigma\ne e\\ \beta\in\mathbb{F}_n^+}}\operatorname{Re}\langle A_\sigma h_\beta,h_{\beta\sigma}\rangle\ge 0\tag{4.2}$$

for any finitely supported sequence $\{h_\beta\}_{\beta\in\mathbb{F}_n^+}\subset B(\mathcal{H})$.

PROOF. Let $K_\rho:\mathbb{F}_n^+\times\mathbb{F}_n^+\longrightarrow B(\mathcal{H})$ be the Toeplitz kernel defined by

$$K_\rho(e,e)=I_{\mathcal{H}},K_\rho(e,\omega)=\frac{1}{\rho}A_\omega,\quad\text{and}\quad K_\rho(\omega,e)=\frac{1}{\rho}A_\omega^*.\tag{4.3}$$

According to the definition it is easy to see that $(A_1,\dots,A_n)\in C_\rho$ if and only if there is a Hilbert space $\mathcal{K}\supset\mathcal{H}$ and a sequence of isometries $V_{s_1},\dots,V_{s_n}$ on $\mathcal{K}$ with orthogonal ranges such that

$$\frac{1}{\rho}A_\omega=P_{\mathcal{H}}V_\omega|_{\mathcal{H}},\quad\text{for any }\omega\in\mathbb{F}_n^+,\quad \omega\ne e.$$

Applying Theorem 2.1, we deduce that $(A_1,\dots,A_n)\in C_\rho$ if and only if the Toeplitz kernel K_ρ is positive-definite. Since

$$\sum_{\sigma,\omega\in\mathbb{F}_n^+}\langle K_\rho(\sigma,\omega)h_\omega,h_\sigma\rangle=\sum_{\sigma\in\mathbb{F}_n^+}\|h\|^2+\frac{1}{\rho}\sum_{\substack{\sigma\in\mathbb{F}_n^+,\sigma\ne e\\ \beta\in\mathbb{F}_n^+}}\operatorname{Re}\langle A_\sigma h_\beta,h_{\beta\sigma}\rangle$$

for any finitely supported sequence $\{h_\beta\}_{\beta\in\mathbb{F}_n^+}$ in $\mathcal{H}$, the result follows. ∎

We will show that the class C_ρ is strictly increasing as a function of $\rho(0<\rho<\infty)$. Let us remark that if $(T_1,\dots,T_n)\in C_\rho$ ($\rho>0$) then $\|[T_1,\dots,T_n]\|\le\rho$.

PROPOSITION 4.2. *If* $\dim \mathcal{H} \geq n+1$, *the class* $\mathcal{C}_\rho$ $(0 < \rho < \infty)$ *increases with* ρ, *i.e.*, $\mathcal{C}_\rho \subset \mathcal{C}_{\rho'}$ *and* $\mathcal{C}_\rho \neq \mathcal{C}_{\rho'}$ *for* $0 < \rho < \rho' < \infty$.

PROOF. According to Theorem 2.3 we have that $\mathcal{C}_\rho \subset \mathcal{C}_{\rho'}$ for $0 < \rho < \rho' < \infty$. Now we construct for every $0 < \rho < \infty$ an operator $(T_1, \ldots, T_n) \in \big(B(\mathcal{H})\big)^n$ such that $(T_1, \ldots, T_n) \in \mathcal{C}_\rho$ and $\|[T_1, \ldots, T_n]\| = \rho$. This will prove the second part of the theorem.

Let $\{e_0, e_1, \ldots, e_n, \ell_\lambda(\lambda \in \Lambda)\}$ be an orthonormal basis of $\mathcal{H}$. For each $i = 1, 2, \ldots, n$ let $T_i \in B(\mathcal{H})$ be defined by

$$T_i e_0 = \rho e_i, \quad T_i e_j = 0 \quad (j = 1, 2, \ldots, n) \quad \text{and} \quad T_i \ell_\lambda = 0 (\lambda \in \Lambda).$$

Notice that $T_i T_j = 0$ for any $i, j = 1, 2, \ldots, n$. Let $\mathcal{K}$ be the Hilbert space defined by

$$\mathcal{K} = \mathbb{C}e_0 \oplus \bigoplus_{m \geq 1} H^{\oplus m}$$

where $H = \bigvee\{e_1, \ldots, e_n, \ell_\lambda(\lambda \in \Lambda)\}$. We identify $\mathcal{H}$ with $\mathbb{C}e_0 \oplus H \subset \mathcal{K}$. Define $V_i \colon \mathcal{K} \longrightarrow \mathcal{K}$ to be the left creation operators on $\mathcal{K}$ with e_i $(i = 1, 2, \ldots, n)$ by setting $V_i k = e_i \otimes k$ (with the convention that $e_i \otimes e_0 = e_i$). These are isometries with orthogonal ranges. Let $P_{\mathcal{H}}$ be the orthogonal projection from $\mathcal{K}$ into $\mathcal{H}$. For each $i = 1, 2, \ldots, n$ we have

$$\rho P_{\mathcal{H}} V_i e_0 = \rho P_{\mathcal{H}} e_i = \rho e_i = T_i e_0,$$
$$\rho P_{\mathcal{H}} V_i e_j = \rho P_{\mathcal{H}}(e_i \otimes e_j) = 0, \quad \text{for } j = 1, 2, \ldots, n,$$

and

$$\rho P_{\mathcal{H}} V_i \ell_\lambda = \rho P_{\mathcal{H}}(e_i \otimes \ell_\lambda) = 0, \quad \text{for } \lambda \in \Lambda.$$

Now, it is clear that

$$T_{i_1} \cdots T_{i_k} = \rho P_{\mathcal{H}} V_{i_1} \cdots V_{i_k}|_{\mathcal{H}},$$

for any $1 \leq i_1, \ldots, i_k \leq n$ that is, $(T_1, \ldots, T_n) \in \mathcal{C}_\rho$. The fact that $\|[T_1, \ldots, T_n]\| = \rho$ is obvious. The proof is complete. ■

Let $\mathcal{H}$ be a Hilbert space and $B(\mathcal{H})$ the set of bounded linear operators on $\mathcal{H}$. We identify $M_m\big(B(\mathcal{H})\big)$, the set of $m \times m$ matrices with entries from $B(\mathcal{H})$, with $B(\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{\text{m-times}})$. Thus we have a natural C^*-norm on $M_m\big(B(\mathcal{H})\big)$. If X is an operator space, *i.e.*, a closed subspace of $B(\mathcal{H})$, we consider $M_m(X)$ as a subspace of $M_m\big(B(\mathcal{H})\big)$ with the induced norm. Let X, Y be operator spaces and $u \colon X \longrightarrow Y$ be a linear map. Define $u_m \colon M_m(X) \longrightarrow M_m(Y)$ by

$$u_m\left[(x_{ij})\right] = \left[\big(u(x_{ij})\big)\right].$$

We say that u is completely bounded (*cb* in short) if

$$\|u\|_{cb} = \sup_{m \geq 1} \|u_m\| < \infty.$$

The von Neumann inequality (1.2) can be extended, in an appropiate form, to the class $\mathcal{C}_\rho$.

PROPOSITION 4.3. *If* $(T_1, \dots, T_n) \in \mathcal{C}_\rho$ $(\rho > 0)$ *then for any polynomial* $p \in \mathcal{P}_n$,

$$\|p(T_1, \dots, T_n)\| \le \|(1-\rho)p(0, \dots, 0) + \rho p\|_\infty.$$

PROOF. Since $(T_1, \dots, T_n) \in \mathcal{C}_\rho$, there is a Hilbert space $\mathcal{K} \supset \mathcal{H}$ and a sequence of isometries $V_1, \dots, V_n$ on $\mathcal{K}$ with orthogonal ranges such that

$$T_{i_1} \cdots T_{i_k} = \rho P_{\mathcal{H}} V_{i_1} \cdots V_{i_k}|_{\mathcal{H}} \quad \text{for any } 1 \le i_1, \dots, i_k \le n.$$

Hence, for any $p \in \mathcal{P}_n$ we have

$$p(T_1, \dots, T_n) = P_{\mathcal{H}}[(1-\rho)p(0, \dots, 0)I_{\mathcal{K}} + \rho p(V_1, \dots, V_n)]|_{\mathcal{H}} \tag{4.4}$$

According to the von Neumann inequality (1.2) we infer that

$$\begin{aligned} \|p(T_1, \dots, T_n)\| &\le \|(1-\rho)p(0, \dots, 0)I_{\mathcal{K}} + \rho p(V_1, \cdots, V_n)\| \\ &\le \|(1-\rho)p(0, \dots, 0) + \rho p\|_\infty. \end{aligned}$$

■

COROLLARY 4.4. *Let* $q \in \mathcal{P}_n$ *such that* $q(0, \dots, 0) = 0$ *and* $\|q\|_\infty \le 1$. *If* $(T_1, \dots, T_n) \in \mathcal{C}_\rho$ $(\rho > 0)$ *then* $q(T_1, \dots, T_n) \in \mathcal{C}_\rho$.

PROOF. Let $V_1, \dots, V_n \in B(\mathcal{H})$ be an isometric ρ-dilation of $T_1, \dots, T_n$. We have

$$q(T_1, \dots, T_n)^k = \rho P_{\mathcal{H}} q(V_1, \dots, V_n)^k|_{\mathcal{H}} \tag{4.5}$$

for any $k = 1, 2, \dots$. Since $\|q\|_\infty \le 1$ it follows by the von Neumann inequality (1.2) that $\|q(V_1, \dots, V_n)\| \le 1$. Thus, there is a unitary operator U on a larger space $\mathcal{U} \supset \mathcal{K}$ such that $q(V_1, \dots, V_n)^k = P_{\mathcal{K}} U^k|_{\mathcal{K}}$ for any $k = 1, 2, \dots$. Therefore $q(T_1, \dots, T_n)^k = \rho P_{\mathcal{H}} U^k|_{\mathcal{H}}$, $k = 1, 2, \dots$, *i.e.*, $q(T_1, \dots, T_n) \in \mathcal{C}_\rho$. ■

A sequence of operators $A_1, \dots, A_n$ is called simultaneously similar to a sequence $T_1, \dots, T_n$ if there is an invertible operator S such that $A_i = ST_iS^{-1}$, for any $i = 1, 2, \dots, n$. In what follows we extend [SzF1] to our setting.

THEOREM 4.5. *Any sequence* $(A_1, \dots, A_n) \in \mathcal{C}_\rho$ $(\rho > 0)$ *is simultaneously similar to a sequence* $(T_1, \dots, T_n) \in \mathcal{C}_1$.

PROOF. Let $V_1, \dots, V_n$ be a ρ-dilation of $(A_1, \dots, A_n)$ on a Hilbert space $\mathcal{K} \supset \mathcal{H}$. According to (4.4), for any polynomial $p_{ij} \in \mathcal{P}_n$ $1 \le i, j \le k$ we have

$$p_{ij}(A_1, \dots, A_n) = P_{\mathcal{H}}[(1-\rho)p_{ij}(0, \dots, 0)I_{\mathcal{K}} + \rho p_{ij}(V_1, \dots, V_n)]|_{\mathcal{H}}$$

Denoting by V the isometry $\mathcal{H} \subset \mathcal{K}$, we obtain:

$$\begin{aligned} \big[p_{ij}(A_1, \dots, A_n)\big] = \begin{bmatrix} V & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & V \end{bmatrix}^* &\left(\begin{bmatrix} (1-\rho)I_{\mathcal{K}} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & (1-\rho)I_{\mathcal{K}} \end{bmatrix} \big[p_{ij}(0, \dots, 0)I_{\mathcal{K}}\big] \right. \\ &\left. + \begin{bmatrix} \rho I_{\mathcal{K}} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \rho I_{\mathcal{K}} \end{bmatrix} \big[p_{ij}(V_1, \dots, V_n)\big] \right) \begin{bmatrix} V & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & V \end{bmatrix}. \end{aligned}$$

Therefore,

$$\|[p_{ij}(A_1,\ldots,A_n)]\| \le |1-\rho|\,\|[p_{ij}(0,\ldots,0)I_{\mathcal{K}}]\| + \rho\|[p_{ij}(V_1,\ldots,V_n)]\|. \tag{4.6}$$

According to (1.2) we have that $\|[p_{ij}(0,\ldots,0)]I_{\mathcal{K}}\| \le \|[p_{ij}(S_1,\ldots,S_n)]\|$ and $\|[p_{ij}(V_1,\ldots,V_n)]\| \le \|[p_{ij}(S_1,\ldots,S_n)]\|$. These relations together with (4.6) imply $\|[p_{ij}(A_1,\ldots,A_n)]\| \le (|1-\rho|+\rho)\|[p_{ij}(S_1,\ldots,S_n)]\|$. Hence the map $\Phi\colon \mathcal{P}_n \longrightarrow B(\mathcal{H})$ defined by

$$\Phi(p) = p(A_1,\ldots,A_n), \quad p \in \mathcal{P}_n$$

can be extended to a completely bounded representation of the disc algebra $\mathcal{A}_n$. Using [Po4, Theorem 2.4] (see also [P]) we infer that there is a contraction $[T_1,\ldots,T_n]$ and a invertible operator S such that $A_i = S^{-1}T_iS$, for any $i = 1,2,\ldots,n$. Thus, $(A_1,\ldots,A_n)$ is simultaneously similar $(T_1,\ldots,T_n) \in C_1$. The proof is complete. ∎

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Gelu Popescu
Division of Mathematics and Statistics,
The University of Texas at San Antonio
San Antonio, TX 78249
U.S.A.
e-mail: gpopescu@ringer.cs.utsa.edu