ON THE NUMBER OF DISJOINT EDGES IN A GRAPH

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Introduction. In what follows we prove that a finite graph of n nodes in which each node has degree $\geqslant 1$ but $\leqslant \bar{d}$ possesses a set of at least $n/(1 + \bar{d})$ pairwise disjoint edges. Our principal theorem states an analogue of this result for the case when each node has degree $\geqslant 2$: we show that in this case the graph possesses a set of at least $2n/(2 + \max(4, \bar{d}))$ mutually disjoint edges.

Both these results are established in §3, where it is also shown that they are "best possible."

§1 furnishes the needed definitions and §2 constitutes an introductory discussion. §4 contains an application of our principal result to the colour problem for graphs. Our main conclusion in §4 is that for each integer $j \geqslant 0$ there are (after isomorphism) only *finitely* many node critical graphs which are not cones (that is, do not contain a node which adjoins all other nodes) and for which the difference between order and chromatic number is $\leqslant j$.

1. Definitions and notation. We adopt the following conventions: if X is any set, its cardinal number is $|X|$; $\emptyset$ is the null set; "iff" stands for "if and only if."

A *graph* G (without loops or multiple edges or directed edges) is taken here to be an ordered pair (X, F), where F is a set of pair sets $\{x, y\}$ of distinct members of the set X. The set X constitutes the *nodes* NG of G; the set F constitutes the *edges* EG of G. Two nodes x, y of G *adjoin* in G (we write "xGy") iff $\{x, y\}$ is an edge of G.

The *order* nG of a graph G is $|NG|$, the *edge number* eG is $|EG|$. Throughout this paper we shall assume that all graphs considered have *finite* order.

The *complement* $'G$ of a graph G is the graph which has the same nodes as G and such that $x\,'Gy$ iff $x \neq y$ and not xGy. A graph H is a *subgraph* of G ($H \subseteq G$) iff $NH \subseteq NG$ and $EH \subseteq EG$. Given any $Y \subseteq NG$, there is a largest subgraph H of G such that $Y = NH$ (viz., for $x, y \in Y$, xHy iff xGy): we call this H the *restriction* G/Y *of* G *to* Y. It is convenient also to define $G - Y$ (for any $Y \subseteq NG$) to be $G/(NG - Y)$.

A graph G is *complete* iff every pair of distinct nodes adjoin. G is *discrete* iff no pairs of nodes adjoin. G is a *cone* with *apex* x iff x is a node of G which adjoins every other node of G. G is a *cone over* a graph H iff there is an apex x of G with $H = G - \{x\}$.

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If G is a graph with node x, the *degree* dxG *of* x *in* G is the number of nodes which x adjoins in G. The *subdegree* $\underline{d}G$ and *super-degree* $\bar{d}G$ of a graph G are, respectively, min $\{dxG: x \in NG\}$ and max $\{dxG: x \in NG\}$.

Suppose $x_0, x_1, \ldots, x_q$ $(q \geqslant 0)$ is a sequence with range NG such that x_iGx_j iff $|i - j| = 1$. If this sequence is one-one, G is a *path* with *end nodes* x_0 and x_q. If the sequence $x_1, \ldots, x_q$ is one-one and $q \geqslant 2$ and $x_0 = x_q$, G is a *circuit*.

A graph is *connected* iff each two nodes are the end nodes of some path $\subseteq$ the graph. A *component* of a graph is a maximal connected subgraph. A *tree* is a connected graph which includes no circuit.

2. Disjoint edges in a graph. In a given graph G there are various *disjoint sets of edges* (that is, sets of pairwise disjoint edges). We define:

$$mG = \max \{|M|: M \textit{ is a disjoint set of edges of } G\}.$$

Our problem here is *to determine a non-trivial lower bound for* mG in terms of other data concerning the structure of G.

Clearly we have always $m \leqslant \frac{1}{2}n$ (for any graph); a classic problem has been to obtain conditions for the existence of a disjoint set of edges which contain all the nodes of the graph—that is, conditions that $m \geqslant \frac{1}{2}n$. For regular graphs this amounts to the problem of finding a factor of degree 1 (cf. **1**).

In general, a non-trivial lower bound for m will depend on $\underline{d}$ and $\bar{d}$: if $\bar{d}$ is small then m is small—indeed if $\bar{d} = 0$, then $m = 0$ no matter how large n might be; and, given fixed n and $\bar{d}$, m clearly tends to increase as $\underline{d}$ is increased.

Our results in the next section establish lower bounds for m in terms of n, $\underline{d}$, and $\bar{d}$ for certain important special cases.

3. Results.

3.1. THEOREM. *For graphs with* $\underline{d} \geqslant 0$,

$$0 \cdot n \leqslant (0 + \bar{d}) \cdot m.$$

3.2. THEOREM. *For graphs with* $\underline{d} \geqslant 1$,

$$1 \cdot n \leqslant (1 + \bar{d}) \cdot m.$$

Proof. We use course-of-values induction on e, the edge number. Suppose that $\underline{d}G \geqslant 1$ and that whenever H is a graph with $\underline{d}H \geqslant 1$ and $eH < eG$ we have $nH \leqslant (1 + \bar{d}H)mH$: we shall show that $nG \leqslant (1 + \bar{d}G)mG$.

We first note that:

(1) If $H \subseteq G$, $NH = NG$, and H satisfies the conclusion of the theorem, then so does G.

(2) If $NG = N_1 \cup N_2$, where N_1 and N_2 are disjoint sets of nodes, and G/N_1 and G/N_2 both satisfy the conclusion of the theorem, then so does G.

By (1), (2), and the induction hypothesis, the desired conclusion is immediate unless G has just one component (that is, is connected) and removal of any

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edge of G causes the resulting graph to violate the hypothesis $\underline{d} \geqslant 1$ (that is, each edge of G contains at least one node of degree 1 in G, and thus no two nodes of degree >1 adjoin in G).

Assuming then that these conditions all hold, let z be a node of G with maximum possible degree. The only nodes z adjoins have degree 1, and hence adjoin only z. G being connected, NG then consists just of z and those nodes which adjoin z (that is, G is a "star" with centre z). Then $nG = 1 + dzG = 1 + \bar{d}G$, and $mG = 1$. Hence $nG \leqslant (1 + \bar{d}G)mG$, q.e.d.

3.3. *Remarks.* For each integer $j \geqslant 1$ there is a graph G with $\bar{d}G = j$, $\underline{d}G = 1$, and $nG = (1 + \bar{d}G)mG$. Indeed, reminiscent of the proof of 3.2, let G be a "star" of order $j + 1$—that is, a cone over a discrete graph of order j. Then $nG = 1 + \bar{d}G$ and $mG = 1$. Hence the inequality of 3.2 is a "best possible" result.

Our principal result (3.5) is an analogue of 3.1 and 3.2 for the condition $\underline{d} \geqslant 2$. It is not true, however, that for $\underline{d} \geqslant 2$ we always have $2n \leqslant (2 + \bar{d})m$, as strict analogy requires: the simplest counterexample is a "triangle" (a circuit of order 3). However, a relatively minor amendment in the conclusion suffices to ensure a theorem, viz., $2 \cdot n \leqslant (2 + \max(4, \bar{d})) \cdot m$. In proving this result we shall use the following lemma:

3.4. Lemma. (1) *Let G be a tree, z a node of G. Then there is a one-one function f from $NG - \{z\}$ onto EG such that always $x \in fx$.* (2) *Let G be a connected graph not a tree. Then there is a one-one function f from NG into EG such that always $x \in fx$.*

Proof. (1) For each node x other than z there is a unique path from x to z and a unique edge of this path which contains x: let fx be this edge.

(2) Let H be a subtree of G with $NH = NG$, and let $F \in EG - EH$. Choose $z \in F$. By (1) there is a one-one function f from $NG - \{z\}$ onto EH such that always $x \in fx$. Extend f to all of NG by putting $fz = F$.

3.5. Theorem. *For graphs with $\underline{d} \geqslant 2$,*

$$2 \cdot n \leqslant (2 + \max\,(4, \bar{d})) \cdot m.$$

Proof. Suppose that $\underline{d}G \geqslant 2$: we shall show that $2 \cdot nG \leqslant (2 + \max\,(4, \bar{d}G)) \cdot mG$. As in the proof of 3.2 we use course-of-values induction on eG. Remarks (1) and (2) from Theorem 3.2 hold verbatim, and—analogous with the proof of 3.2—we may assume that G is connected and that each edge of G contains at least one node of degree 2 (and hence no two nodes of degree > 2 adjoin in G).

Let T be the nodes of degree 2, S the nodes of degree > 2, $= NG - T$. We know that G/S is discrete: we shall show that we can assume also that G/T is discrete.

Suppose that $nK > 1$ for some component K of G/T. Then either K is a circuit or K is a path whose distinct end nodes x, y adjoin, respectively, unique nodes u, v of S (possibly $u = v$). If K is a circuit, then K is a component of G; since G is connected, $G = K$ and G certainly satisfies the conclusion of the theorem. If K is a path with $u \neq v$ or with $u = v$ and $duG > 3$, let $N_1 = NK$, $N_2 = NG - NK$. Then G/N_1 satisfies the conclusion of the theorem, and since $\underline{d}(G/N_2) \geqslant 2$ the induction hypothesis ensures that G/N_2 also satisfies the conclusion. By (2) of 3.2, G satisfies the conclusion.

Therefore we may assume that each component of G/T either has just one node or is a path whose distinct end nodes x, y adjoin in common a node u of S of degree 3 (and whose other nodes adjoin no nodes of S). Suppose then that K is a component of the second type. Let z be the unique node $\notin \{x, y\}$ such that zGu: since $u \in S$ and G/S is discrete, $z \in T$; let L be the component of G/T containing z. If $nL > 1$ then L has a node t other than z such that tGu. Further, $t \notin \{x, y\}$ (else $L = K$ and then $z \in \{x, y\}$), so x, y, z, t are distinct nodes adjoining u: but this is impossible since $duG = 3$. Hence L comprises just z. Let $N_1 = NK \cup \{u, z\}$, $N_2 = NG - N_1$. Then G/N_1 satisfies the conclusion of the theorem, and since $\underline{d}(G/N_2) \geqslant 2$ the induction hypothesis ensures that G/N_2 also satisfies the conclusion. By (2) of 3.2, G satisfies the conclusion.

Therefore we may assume that each component K of G/T has just one node—that is, G/T is discrete. (Since $NG = S \cup T$ and G/S is also discrete, G is thus a "Paare graph" in the sense of **(1)**.) Since $\underline{d}G \geqslant 2$, G is not a tree; by 3.4 let f be a one-one function from NG into EG such that fx is always an edge which contains x. Now fS is a disjoint set of edges of G. For if not then there are distinct nodes x, y of S and a node z of T such that $fx = \{x, z\}$, $fy = \{y, z\}$; but $dzG = 2$, so fz must be either fx or fy: this contradicts the one-one property of f.

Since each edge contains exactly one node of S and one node of T,

$$\sum \{dxG: x \in S\} = eG = \sum \{dxG: x \in T\}.$$

Hence $\bar{d}G \cdot |S| \geqslant 2 \cdot |T|$. Now $|T| = nG - |S|$, and since f is one-one $|fS| = |S|$. But fS is a disjoint set of edges of G, so $|fS| \leqslant mG$. Therefore, $\bar{d}G \cdot mG \geqslant 2(nG - mG)$, that is,

$$2 \cdot nG \leqslant (2 + \bar{d}G) \cdot mG,$$

and G satisfies the conclusion of the theorem, q.e.d.

3.6. *Remarks.* For each integer $j \geqslant 4$ there is a graph G with $\bar{d}G = j$, $\underline{d}G = 2$, and $2 \cdot nG = (2 + \max(4, \bar{d}G)) \cdot mG$ $(= (2 + \bar{d}G) \cdot mG)$. Indeed let G be a complete graph of order $j + 1$ modified by splicing each edge by a single node. Hence the inequality of 3.5 is a "best possible" result for $j \geqslant 4$. However, for *all* cases when $j = 2$ or 3, *except* those in which each component of the graph is a triangle, 3.5 appears to be *not* best possible.

One might conjecture that for each integer $i \geqslant 3$ there exists an analogue to 3.1, 3.2, 3.5 for the condition $\underline{d} \geqslant i$. However, thus far I have been unable either to obtain such an analogue for $i = 3$ or to show that none such can exist.

4. Applications to the colour problem for graphs.

4.1. DEFINITIONS AND REMARKS. A *colouring* of a graph G is a function f (whose values are the "colours") defined on NG such that if xGy then $fx \neq fy$: adjoining nodes receive different colours. The *chromaticity* (chromatic number) of G is the least number of colours needed in a colouring of G:

$$kG = \min \{|\textit{range}\, f| : f \textit{ is a colouring of } G\}.$$

The colour problem for graphs is the problem of relating kG to other structural information about G: one important quantity which enters in frequently is the difference $nG - kG$ between the order and the chromaticity of G. This has no standard name, but we shall call it the *plexity* pG of G. Clearly always $pG \geqslant 0$, with equality iff G is complete.

A graph G is *node critical* iff for every $x \in NG$, $k(G - \{x\}) < kG$. It is easily verified that a node critical graph G has no node of degree $nG - 2$ and that $\underline{d}G \geqslant kG - 1$. In many respects the colour problem for arbitrary graphs reduces to the colour problem for *non-conical node critical* graphs: this may be seen from the following facts ((1) and (2) are easily verified and (3) and (4) are immediate from (2)):

(1) Every graph G includes a node critical graph H with $kH = kG$.

(2) If H is any graph and G is a cone over H, then $kG = kH + 1$, $pG = pH$, and G is node critical iff H is node critical.

(3) If j is any integer $\geqslant 0$, then there is a node critical graph of plexity j iff there are infinitely many node critical graphs of plexity j.

(4) If j is any integer $\geqslant 0$, then every node critical graph of plexity j is a repeated cone over some *non-conical* node critical graph of plexity j.

4.2. LEMMA. *For all graphs G,*

$$m\,'G \leqslant pG.$$

Proof. Let M be a disjoint set of $m\,'G$ edges of $'G$. For $x \in NG$ let $fx = F$ if $x \in F \in M$; if x belongs to no edge of M let $fx = x$. Then f is a colouring of G: the $2m\,'G$ nodes contained in edges of M receive $m\,'G$ colours and the remaining nodes of G receive $nG - 2\,m\,'G$ colours. Then $kG \leqslant |\text{range}\, f| = nG - m\,'G$, and thus $m\,'G \leqslant nG - kG = pG$, q.e.d.

4.3. COROLLARY. *For non-conical graphs with no nodes of degree $n - 2$,*

$$n \leqslant \tfrac{1}{2}p(2 + \max(4, n - 1 - \underline{d})).$$

Proof. We have $\underline{d}' \geqslant 2$, so that by 3.5 $2n = 2n' \leqslant m'(2 + \max(4, \bar{d}'))$. Now $\bar{d}' = n - 1 - \underline{d}$, and by 4.2 $m' \leqslant p$.

4.4. THEOREM. *For non-conical node critical graphs,*

$$n \leqslant \tfrac{1}{2}p \max(p + 2, 6).$$

Proof. Node critical graphs have no nodes of degree $n - 2$, so the conclusion

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follows from 4.3, noting that $\underline{d} \geqslant k - 1$ for node critical graphs, and thus $n - 1 - \underline{d} \leqslant p$ for node critical graphs.

4.5. *Remark.* Thus for each integer $j \geqslant 0$ there are (after isomorphism) only *finitely* many non-conical node critical graphs of plexity $\leqslant j$: indeed their order is bounded by $\frac{1}{2}j \max(j + 2, 6)$. In view of (3) of 4.1 this fact is curious and noteworthy. Our main result in 3.5 is not needed to establish this fact: for such purpose 3.1 suffices. However, the bound we have obtained using 3.5 definitely improves that $(n \leqslant p(p + 1))$ which may be deduced in like manner from 3.1.

It is easily seen that the only node critical graphs of plexity 0 are the complete graphs and that there are no node critical graphs of plexity 1. The node critical graphs of plexity 2 are given by (**2**, Proposition IV); they may also be obtained by the following easy application of 4.4.

4.6. Corollary. *If G is a non-conical node critical graph of plexity* 2, *then G is a circuit of order* 5.

Proof. We have $\underline{d}G \geqslant kG - 1 = nG - 3 \geqslant \bar{d}G$. Thus, for each node x of G, $dxG = nG - 3$, and hence $dx\,'G = 2$. Therefore the components of $'G$ are circuits. By 4.4 $nG \leqslant 6$. The reader may easily construct the five graphs of order $\leqslant 6$ the components of whose complements are circuits and may directly verify that a circuit of order 5 is the only node critical graph among them.

Added in proof. Quite recently I have discovered results confirming the conjecture at 3.6. I shall endeavour to discuss these in a future paper.

References

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