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RATIONALITY OF MODULI SPACES OF VECTOR BUNDLES ON RATIONAL SURFACES

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Abstract. Let X be a smooth rational surface. In this paper, we prove the rationality of the moduli space $M_{X,L}(2; c_1, c_2)$ of rank two L-stable vector bundles E on X with $\det(E) = c_1 \in \mathrm{Pic}(X)$ and $c_2(E) = c_2 \gg 0$.

§1. Introduction

Let X be a smooth algebraic surface over the complex field, $\overline{M}_{X,L}(2; c_1, c_2)$ the moduli space of rank two torsion free sheaves E on X semistable with respect to a polarization L (in the sense of Gieseker-Maruyama) with $\det(E) = c_1 \in \mathrm{Pic}(X)$ and $c_2(E) = c_2 \in \mathbb{Z}$ and $M_{X,L}(2; c_1, c_2)$ the open subscheme parameterizing L-stable (in the sense of Mumford-Takemoto) locally free sheaves. (We will write $\overline{M}_L(2; c_1, c_2)$ and $M_L(2; c_1, c_2)$ when there is no confusion.) Moduli spaces for stable vector bundles on algebraic surfaces were constructed in the 1970's. Since then, many mathematicians have studied their structure, from the point of view of algebraic geometry, of topology and of differential geometry; giving very pleasant connections between these areas. For instance, it is well known that $\overline{M}_L(2; c_1, c_2)$ (resp. $M_L(2; c_1, c_2)$) is a projective (resp. quasi-projective) variety and for $c_2 \gg 0$ it is non-empty (see [Gie77] and [Mar75]), generically smooth of dimension $4c_2 - c_1^2 - 3\chi(O_X)$ (see [Don86] and [Zuo91]) and irreducible (see [GL96] and [O'G96]).

In this paper, we turn our attention to the study of the rationality of the moduli space $M_L(2; c_1, c_2)$. To be more precise, we are interested in the following question:

Question. Let X be a smooth, rational, projective surface. Fix a polarization L, $c_1 \in \mathrm{Pic}(X)$ and $0 \ll c_2 \in \mathbb{Z}$. Is $M_L(2; c_1, c_2)$ rational?

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For $X = \mathbb{P}^2$, the answer is affirmative (See [Mar85], [ES87] and [Mae90]). Here we recall that, if X is a smooth rational surface, $X \neq \mathbb{P}^2$, then X can be obtained as the blowup $\pi : X \longrightarrow X_e$ of a Hirzebruch surface $X_e \cong \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-e))$ for some $e \geq 0$. We recall also that $\operatorname{Pic}(X_e) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by classes C_0, F such that $C_0^2 = -e$, $F^2 = 0$ and $C_0 F = 1$ (See [Har77, Chapter V, Corollary 2.13]). For simplicity we write C_0, F for $\pi^* C_0$, $\pi^* F$. The goal of this paper is to prove the following result:

THEOREM A. *Let X be a Hirzebruch surface or the blowup $\pi : X \rightarrow X_e$ of a Hirzebruch surface X_e, L a polarization on X such that $L(K_X + F) < 0$, $c_1 \in \operatorname{Pic}(X)$ and $0 \ll c_2 \in \mathbb{Z}$. Then, the moduli space $M_L(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$.*

Note that, by [Wal98, Lemma 9], there always exist polarizations L on X such that $L(K_X + F) < 0$.

Next, we outline the ideas used for proving Theorem A and the structure of the paper. Section 2 is devoted to provide the reader with the general background that we need in the sequel. The aim of Section 3 is to establish criteria of rationality for moduli spaces $M_L(2; c_1, c_2)$ of rank two, L-stable vector bundles on smooth, rational surfaces. The first criterion (Criterion 3.4) holds for anticanonical, rational surfaces. We use prioritary sheaves (see Definition 3.5) in order to obtain the second criterion of rationality (Criterion 3.8) which generalizes to arbitrary rational surfaces the first one. In addition, we prove that given two ample divisors L_1 and L_2 on a smooth, rational surface X, $X \neq \mathbb{P}^2$, verifying $L_i(K_X + F) < 0$ for $i = 1, 2$, $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are birationally equivalent, whenever non-empty. This result generalizes [CMR99, Theorem 3.9] and allows us to fix the polarization L for many purposes.

In Section 4, we prove Theorem A. According to the birational classification of smooth rational surfaces; in Subsection 4.1, X is a Hirzebruch surface or $\mathbb{P}^2$, i.e., a minimal rational surface (Theorem 4.7) and, in Subsection 4.2, X is a blowup of a Hirzebruch surface (Theorem 4.13). In all cases, we analyze separately all possible values of the first Chern class and we prove the rationality using either criteria stated in Section 3 or constructing suitable families of rank 2 vector bundles on X over a big enough rational base.

§2. Background material

We start by collecting the main results on torsion free sheaves and moduli spaces needed in the sequel.

2.1. *Let X be a smooth algebraic surface with canonical line bundle K and let V be a rank two vector bundle on X. It holds*

(1) *If V is H-stable and $c_1(V)H < 0$ then $H^0V = 0$.*

(2) *If V is H-stable, $\chi(V) > 0$ and $c_1(V^* \otimes K)H < 0$ then $H^0V \neq 0$.*

LEMMA 2.2. *Let X be a smooth, rational, algebraic surface with canonical divisor K. Let U (resp. $\mathcal{F}$) be the family of rank 2 torsion free sheaves (resp. vector bundles) E on X given by a non-trivial extension*

$$0 \longrightarrow O_X(-D) \longrightarrow E \longrightarrow O_X(D+c_1) \otimes I_Z \longrightarrow 0$$

where $Z \subset X$ is a 0-cycle of length l. Assume that $\pm(2D+c_1)$ and $2D+c_1+K$ are non effective divisors. Then, U (resp. $\mathcal{F}$) is an irreducible, rational, projective (resp. quasi-projective) variety.

Proof. See [CMR99, Lemma 4.1.1]. □

Remark 2.3. Lemma 2.2 is true if instead of assuming that $\pm(2D+c_1)$ and $2D+c_1+K$ are non effective divisors, we assume that $-(2D+c_1)$ and $2D+c_1+K$ are non effective divisors and $H^0E(D) = 1$ for a generic E of U (resp. $\mathcal{F}$).

THEOREM 2.4. *Let X be a smooth algebraic surface, L an ample divisor on X and $c_1 \in \mathrm{Pic}(X)$. For all $c_2 \gg 0$, $M_L(2; c_1, c_2)$ is a smooth, irreducible, quasi-projective variety of the expected dimension $4c_2 - c_1^2 - 3\chi(O_X)$.*

Proof. See [Don86] and [Zuo91] for the smoothness of $M_L(2; c_1, c_2)$ and [GL96] and [O'G96] for the irreducibility. □

Remark 2.5. For smooth, projective, anticanonical, rational surfaces, we can omit the hypothesis $c_2 \gg 0$. The irreducibility and smoothness of $M_L(2; c_1, c_2)$ holds whenever $M_L(2; c_1, c_2)$ is non-empty (See [CMR99, Proposition 3.11]).

The following result will be generalized in section three and allows us to fix a suitable polarization L for many purposes.

Theorem 2.6. *Let X be a smooth, projective, anticanonical, rational surface, $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbb{Z}$. Assume $4c_2 - c_1^2 > 2 - 3K_X^2/2$. For any two ample divisors L_1 and L_2 on X, $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are birational whenever non-empty.*

Proof. See [CMR99, Theorem 3.9]. □

Remark 2.7. Notice that if $M_L(2; c_1, c_2)$ is non-empty, by Bogomolov's inequality, $c_1^2 - 4c_2 < 0$, the condition $4c_2 - c_1^2 > 2 - 3K_X^2/2$ is automatically satisfied whenever the underlying surface is a Hirzebruch surface or a Fano surface.

Now we gather all relevant results on smooth, irreducible, rational surfaces and cohomology of divisors on rational surfaces which we need in the sequel.

Theorem 2.8. *A smooth, minimal, rational surface is either isomorphic to $\mathbb{P}^2$ or to a Hirzebruch X_e with $e \neq 1$; and any smooth rational surface $X \neq \mathbb{P}^2$ can be obtained as the blowup $\pi : X \to X_e$ of a Hirzebruch surface for some $e \geq 0$.*

For any integer $e \geq 0$, let $X_e \cong \mathbb{P}(O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-e))$ be a non singular, Hirzebruch surface. We denote by C_0 and F the standard basis of $\mathrm{Pic}(X_e) \cong \mathbb{Z} \oplus \mathbb{Z}$ such that $C_0^2 = -e$. They correspond to sections and q-fibers respectively of the natural projection map $q : X_e \to \mathbb{P}^1$. We have $C_0^2 = -e$, $F^2 = 0$, $C_0F = 1$ and the canonical divisor $K_{X_e} = -2C_0 - (e+2)F$.

Lemma 2.9. *We consider the line bundle $O_{X_e}(aH + bF)$ on a smooth Hirzebruch surface X_e and $q : X_e := \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ the natural projection. We have*

$$\begin{aligned} &H^i(X_e, O_{X_e}(aH + bF)) \\ &= \begin{cases} 0 & \text{if } a = -1 \\ H^i(\mathbb{P}^1, S^a(\mathcal{E}) \otimes O_{\mathbb{P}^1}(b)) & \text{if } a \geq 0 \\ H^{2-i}(\mathbb{P}^1, S^{-2-a}(\mathcal{E}) \otimes O_{\mathbb{P}^1}(-e-b-2)) & \text{if } a \leq -2 \end{cases} \end{aligned}$$

being $S^a(\mathcal{E})$ the a-th symmetric power of $\mathcal{E} = O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-e)$.

Proof. It follows from the fact that $R^i q_* O_{X_e}(aH+bF) = 0$ for $i > 0$ and $a > -2$, the degeneration of the Leray Spectral sequence

$$H^i(\mathbb{P}^1, R^j q_* O_{\mathbb{P}(\mathcal{E})}(aH+bF)) \Longrightarrow H^{i+j}(\mathbb{P}(\mathcal{E}), O_{\mathbb{P}(\mathcal{E})}(aH+bF))$$

and Serre's duality. □

LEMMA 2.10. *Let X_e be a smooth, Hirzebruch surface. Then, for any ample divisor H on X_e, $(K_{X_e}+F)H < 0$.*

Proof. It follows from the fact that for any ample divisor H on a surface X and any non-zero effective divisor C on X, $CH > 0$ □

Let $X \neq \mathbb{P}^2$ be a smooth rational surface. By Theorem 2.8, X can be obtained as the blowup $\pi : X \to X_e$ of a Hirzebruch surface X_e for some $e \geq 0$. So, there exists an integer $0 \leq s \in \mathbb{Z}$ and a sequence of monomial transformations $X = Y_0 \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_s} Y_s = X_e$ such that $\pi = \pi_s \circ \cdots \circ \pi_2 \circ \pi_1$. For each $i = 1, \dots, s$, let $E_i' \subset Y_{i-1}$ be the exceptional curve of the monomial transformation $\pi_i : Y_{i-1} \to Y_i$ and we denote by $E_i := (\pi_s^* \circ \cdots \circ \pi_i^*)(E_i')$ the total transform of E_i' on X. For simplicity, we denote C_0, F for $\pi^* C_0$, $\pi^* F$. Applying [Har77, Chapter V, Proposition 3.2], $\mathrm{Pic}(X) \cong \mathrm{Pic}(X_e) \oplus \mathbb{Z}^s \cong \mathbb{Z}^{s+2}$ generated by C_0, F, $E_1, \dots, E_s$, with $C_0^2 = -e$, $C_0F = 1$, $F^2 = 0$, $C_0E_i = FE_i = 0$, for $1 \leq i \leq s$, $1 = -E_1^2 = \cdots = -E_s^2$, and $E_iE_j = 0$ if $i \neq j$. Moreover, $K_X = -2C_0 - (e+2)F + \sum_{i=1}^s E_i$.

We end this section with an easy Lemma on cohomology groups of line bundles that we will use in Section 4. We left the proof to the readers.

LEMMA 2.11. *Let $\pi : X \to X_e$ be a blowup of a Hirzebruch surface X_e, $e \geq 0$, and let $D = aC_0 + bF + \sum_{i=1}^s b_iE_i$ be a divisor on X. The following is satisfied*

(a) If $a < 0$ or $b < 0$, then $H^0 O_X(D) = 0$;

(b) If $DE_i \leq 0$, then $H^0 O_X(D + aE_i) = H^0 O_X(D)$ for all $a \geq 0$.

§3. Prioritary sheaves and criteria of rationality

The aim of this section is to supply criteria of rationality for moduli spaces $M_L(2; c_1, c_2)$ of L-stable, rank two vector bundles over a smooth, irreducible, rational surface X. In [CMR99, Theorem 3.12] we gave a criterion of rationality assuming that X is a smooth, anticanonical, rational surface. Since we will use it in the forthcoming section, in order to make the work as self-contained as possible, we will recall it here. Afterwards, we will prove a second criterion of rationality which generalizes to arbitrary rational surfaces the previous one. Let us start recalling the notions of walls and chambers introduced by Qin, needed later on.

DEFINITION 3.1. (See [Qin93, Definition I.2.1.5]) For $\xi \in \mathrm{Num}(X)$ let $W^\xi := C_X \cap \{x \in \mathrm{Num}(X) \otimes \mathbb{R} \mid x \cdot \xi = 0\}$. W^ξ is called the wall of type (c_1, c_2) defined by ξ if, and only if, there exists $G \in \mathrm{Pic}(X)$ with $G \equiv \xi$ such that $G + c_1$ is divisible by 2 in $\mathrm{Pic}(X)$ and $c_1^2 - 4c_2 \leq G^2 < 0$. W^ξ is non-empty if there is a polarization L with $L\xi = 0$. The wall W^ξ separates the ample divisors L_1 and L_2 if $\xi L_1 < 0 < \xi L_2$. In this case $H = (\xi L_2)L_1 - (\xi L_1)L_2$ belongs to the wall W^ξ. Let $W(c_1, c_2)$ be the union of walls of type (c_1, c_2). A chamber of type (c_1, c_2) is a connected component of $C_X \setminus W(c_1, c_2)$. A face of type (c_1, c_2) is the intersection between a chamber of type (c_1, c_2) and a wall of the same type.

DEFINITION 3.2. Let ξ be a numerical equivalence class defining a wall of type (c_1, c_2). We define $E_\xi(c_1, c_2)$ to be the quasi-projective variety parameterizing rank 2 vector bundles E on X given by an extension

$$0 \longrightarrow O_X(G) \longrightarrow E \longrightarrow O_X(c_1 - G) \otimes I_Z \longrightarrow 0$$

where G is a divisor with $2G - c_1 \equiv \xi$ and Z is a locally complete intersection 0-cycle of length $c_2 + (\xi^2 - c_1^2)/4$. Moreover, we require that E is not given by the trivial extension when $\xi^2 = c_1^2 - 4c_2$.

We define $D(\xi) := \dim E_\xi(c_1, c_2)$ and we put $d_\xi(c_1, c_2) := d(\xi) = D(\xi) - (4c_2 - c_1^2 - 3\chi(O_X))$; i.e., $d(\xi)$ is the difference between the dimension of $E_\xi(c_1, c_2)$ and the expected dimension of a non-empty moduli space $M_L(2; c_1, c_2)$.

Remark 3.3. If X is a smooth, anticanonical, rational surface and $4c_2 - c_1^2 > 2 - 3K_X^2/2$, then $d(\xi) \leq 0$ and $d(\xi) = 0$ if, and only if, $\xi^2 = c_1^2 - 4c_2$ and $\xi^2 + \xi K_X + 2 = 0$ (see [CMR99, Remarks 3.2 and 3.5 and Corollary 3.4]).

FIRST CRITERION OF RATIONALITY 3.4. *Let X be a smooth, projective, anticanonical, rational surface, $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbb{Z}$. Assume $4c_2 - c_1^2 > 2 - 3K_X^2/2$ and that there exists a numerical equivalence class ξ which defines a non-empty wall of type (c_1, c_2) such that $d(\xi) = 0$. Then, the following holds*

(1) *There exists an ample divisor* $\widetilde{L}$ *on* X *such that* $M_{\widetilde{L}}(2; c_1, c_2)$ *is a smooth, irreducible, rational projective variety of dimension* $4c_2 - c_1^2 - 3$ *whenever non-empty and* $\mathrm{Pic}(M_{\widetilde{L}}(2; c_1, c_2)) \cong \mathbb{Z}$.

(2) *For any ample divisor* L *on* X, $M_L(2; c_1, c_2)$ *is a smooth, irreducible, rational, quasi-projective variety of dimension* $4c_2 - c_1^2 - 3$, *whenever non-empty.*

Proof. See the proof of [CMR99, Theorem 3.12]. □

The main tool we will use for extending Criterion 3.4 to arbitrary rational surfaces will be prioritary sheaves. Prioritary sheaves were introduced on $\mathbb{P}^2$ (resp. on birationally ruled surfaces) by Hirschowitz-Laszlo in [HL93] (resp. Walter in [Wal98]) as a generalization of semistable sheaves.

DEFINITION 3.5. Let $\pi : X \to \mathbb{P}^1$ be a birationally ruled surface and we consider $F \in \mathrm{Num}(X)$ the numerical class of a fiber of π. A coherent sheaf E on X is said to be prioritary if it is torsion free and if $\mathrm{Ext}^2(E, E(-F)) = 0$.

Remark 3.6. If H is an ample divisor on X such that $H(K_X + F) < 0$, then any H-semistable, torsion free sheaf is prioritary (see the proof of [Wal98, Theorem 1]).

For a given $1 \le r \in \mathbb{Z}$, $c_1 \in \mathrm{Pic}(X)$, and $c_2 \in \mathbb{Z}$, we denote by $\mathrm{Prior}(r; c_1, c_2)$ the stack of prioritary sheaves E on X of rank r with Chern classes c_1 and c_2, and by $\mathrm{Spl}(r; c_1, c_2)$ the moduli space of simple, prioritary, torsion free sheaves E on X of rank r with Chern classes c_1 and c_2. Using Remark 3.6 we are able to generalize Theorem 2.6 to arbitrary rational surfaces.

THEOREM 3.7. *Let* $\pi : X \to \mathbb{P}^1$ *be a birationally ruled surface,* $F \in \mathrm{Num}(X)$ *the numerical class of a fiber of* π, $c_1 \in \mathrm{Pic}(X)$ *and* $c_2 \in \mathbb{Z}$ *such that* $c_2 \gg 0$. *Then, for any two ample divisors* L_1 *and* L_2 *on* X *with* $L_i(K_X + F) < 0$, $i = 1, 2$, $M_{L_1}(2; c_1, c_2)$ *and* $M_{L_2}(2; c_1, c_2)$ *are birationally equivalent.*

Proof. By Theorem 2.4 and Remark 3.6, $M_{L_1}(2; c_1, c_2)$ and $M_{L_2}(2; c_1, c_2)$ are non-empty open substacks of $\mathrm{Prior}(2; c_1, c_2)$ and the result follows from the smoothness and irreducibility of $\mathrm{Prior}(2; c_1, c_2)$ ([Wal98, Proposition 2]). □

The following criterion of rationality can be viewed as a generalization of Criterion 3.4 to arbitrary rational surfaces.

SECOND CRITERION OF RATIONALITY 3.8. *Let $\pi : X \to \mathbb{P}^1$ be a birationally ruled surface, $F \in \mathrm{Num}(X)$ the numerical class of a fiber of π, $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbb{Z}$. Assume $4c_2 - c_1^2 > 2 - 3K_X^2/2$ and that there exists a numerical equivalence class ξ which defines a non-empty wall of type (c_1, c_2) and that it satisfies*

(a) $\xi^2 = c_1^2 - 4c_2, \quad \xi^2 + \xi K_X + 2 = 0,$

(b) $H^0 O_X(\xi + 3K_X) = H^0 O_X(\xi + K_X + F) = H^0 O_X(K_X + F - \xi) = 0.$

Then, the following holds

(1) *There exists an ample divisor $\widetilde{L}$ on X such that the moduli space $M_{\widetilde{L}}(2; c_1, c_2)$ is a smooth, irreducible, rational projective variety of dimension $4c_2 - c_1^2 - 3$ whenever non-empty and $\mathrm{Pic}(M_{\widetilde{L}}(2; c_1, c_2)) \cong \mathbb{Z}$.*

(2) *For $c_2 \gg 0$ and any ample divisor L on X such that $L(K_X + F) < 0$, $M_L(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$.*

Proof. (1) Let L be any ample divisor on X, $c_2 \in \mathbb{Z}$ with $4c_2 - c_1^2 > 2 - 3K_X^2/2$ and ξ a numerical equivalence class verifying (a) and (b).

CLAIM 1. *For any $E \in M_L(2; c_1, c_2)$, we have $h^0 E(-(c_1 + \xi)/2) > 0$.*

Proof of Claim 1. Applying Riemann-Roch's Theorem, we easily see that

$$c_1\Big(E\Big(-\frac{c_1+\xi}{2}\Big)\Big) = -\xi, \; c_2\Big(E\Big(-\frac{c_1+\xi}{2}\Big)\Big) = 0 \;\text{ and }\; \chi\Big(E\Big(-\frac{c_1+\xi}{2}\Big)\Big) = 1.$$

By hypothesis, $H^0 O_X(\xi + 3K_X) = 0$ and by Serre's duality $H^2 O_X(-\xi - 2K_X) = 0$. Therefore, applying again Riemann-Roch's Theorem we get $h^0 O_X(-\xi - 2K_X) - h^1 O_X(-\xi - 2K_X) = 2(4c_2 - c_1^2 - 2) + 3K_X^2 > 0$ which gives us $h^0 O_X(-\xi - 2K_X) > 0$ or, equivalently, $-(\xi + 2K_X)$ is effective. Hence, $-(2K_X + \xi)L \geq 0$ for any ample divisor L on X or, equivalently, $c_1((E(-(c_1+\xi)/2))^* \otimes K_X)L = (2K_X + \xi)L \leq 0$. If the last inequality is strict we obtain $h^0 E(-(c_1 + \xi)/2) > 0$ (see Fact 2.1). If the equality $c_1((E(-(c_1+\xi)/2))^* \otimes K_X)L = 0$ holds, we get

$$h^0 E\Big(-\frac{c_1+\xi}{2}\Big) > 0 \;\text{ or }\; h^2 E\Big(-\frac{c_1+\xi}{2}\Big) > 0$$

and we will prove that the last inequality is not possible. Indeed, by Serre's duality, $0 < h^2E(-(c_1+\xi)/2) = h^0E^*((c_1+\xi)/2+K_X)$. A non-zero section $\sigma \in H^0E^*((c_1+\xi)/2+K_X)$ defines an injection $O_X((c_1-\xi)/2-K_X) \hookrightarrow E$ and from the L-stability of E we have $((c_1-\xi)/2-K_X)L < c_1L/2$ which contradicts the fact $(2K_X+\xi)L = 0$. Therefore, $h^0E(-(c_1+\xi)/2) > 0$, which proves Claim 1.

CLAIM 2. *If $\xi L \geq 0$ then, $M_L(2; c_1, c_2) = \emptyset$.*

Proof of Claim 2. Assume $M_L(2; c_1, c_2) \neq \emptyset$. For any $E \in M_L(2; c_1, c_2)$, we take a nonzero section $s \in H^0E(-(c_1+\xi)/2)$. It defines an injection $O_X((c_1+\xi)/2) \hookrightarrow E$. Since E is L-stable, we have $((c_1+\xi)/2)L < c_1L/2$ which contradicts the hypothesis $\xi L \geq 0$. Hence, $M_L(2; c_1, c_2) = \emptyset$ which proves Claim 2.

Let $\widetilde{L}$ be an ample divisor on X such that $\xi\widetilde{L} < 0$ and $\widetilde{L} \in \mathcal{C}$ with $W^\xi \cap \overline{\mathcal{C}} \neq \emptyset$. For such $\widetilde{L}$ and $\mathcal{C}$ we have ([Qin93, Proposition 1.3.1])

$$M_{\widetilde{L}}(2; c_1, c_2) = M_{\mathcal{F}}(2; c_1, c_2) \sqcup (\sqcup_\mu E_\mu(c_1, c_2))$$

where $\mathcal{F}$ is the face of the chamber $\mathcal{C}$ contained in W^ξ, $\mu\widetilde{L} < 0$ for some $\widetilde{L} \in \mathcal{C}$ and μ runs over all numerical equivalence classes which define the wall W^ξ. For any $L' \in \mathcal{F}$, $L'\xi = 0$. So, by Claim 2, $M_{\mathcal{F}}(2; c_1, c_2) = \emptyset$. Moreover, $W^\mu = W^\eta$ if, and only if, $\mu = \lambda\eta$, for some $\lambda \in \mathbb{R}$. Therefore, we conclude $M_{\widetilde{L}}(2; c_1, c_2) \cong E_\xi(c_1, c_2)$.

By definition, any $E \in E_\xi(c_1, c_2)$, sits in an exact sequence

$$0 \longrightarrow O_X(G) \longrightarrow E \longrightarrow O_X(c_1 - G) \otimes I_Z \longrightarrow 0$$

where G is a divisor with $2G - c_1 \equiv \xi$ and Z is a locally complete intersection 0-cycle with $l(Z) = c_2 + (\xi^2 - c_1^2)/4$. By hypothesis $\xi^2 = c_1^2 - 4c_2$ (see (a)). Therefore, $Z = \emptyset$ and

$$M_{\widetilde{L}}(2; c_1, c_2) \cong E_\xi(c_1, c_2) \cong \mathbb{P}(\mathrm{Ext}^1(O_X(c_1 - G), O_X(G))) \cong \mathbb{P}^{4c_2 - c_1^2 - 3}$$

where the last isomorphism follows from the hypothesis (b), the fact that $\xi \equiv 2G - c_1$ defines a non-empty wall of type (c_1, c_2) and Riemann-Roch's Theorem.

Therefore, $M_{\widetilde{L}}(2; c_1, c_2)$ is a smooth, irreducible, rational, projective variety of dimension $4c_2 - c_1^2 - 3$, whenever non-empty and $\mathrm{Pic}(M_{\widetilde{L}}(2; c_1, c_2)) \cong \mathbb{Z}$.

(2) Let L be an ample divisor on X with $L(K_X+F)<0$. By Theorem 2.4, we only need to prove that the moduli space $M_L(2;c_1,c_2)$ is rational. It follows from the proof of (1) that there exists an ample divisor $\widetilde{L}$ on X such that $M_{\widetilde{L}}(2;c_1,c_2)\cong E_\xi(c_1,c_2)$.

Claim 3. *Any $E\in E_\xi(c_1,c_2)$ is a prioritary sheaf.*

Proof of Claim 3. Since E is a rank two vector bundle, we only need to check that $\operatorname{Ext}^2(E,E(-F))=0$. By assumption, $\xi^2=c_1^2-4c_2$, so every $E\in E_\xi(c_1,c_2)$ is given by a non-trivial extension

$$0\longrightarrow O_X\Big(\frac{c_1+\xi}{2}\Big)\longrightarrow E\longrightarrow O_X\Big(\frac{c_1-\xi}{2}\Big)\longrightarrow 0. \tag{1}$$

By Serre's duality, $\dim\operatorname{Ext}^2(E,E(-F))=\dim\operatorname{Hom}(E,E(K_X+F))$. Applying $\operatorname{Hom}(\,\cdot\,,E(K_X+F))$ to the sequence (1) we get the exact sequence

$$\begin{aligned}0\longrightarrow \operatorname{Hom}\Big(O_X\Big(\frac{c_1-\xi}{2}\Big),E(K_X+F)\Big)&\longrightarrow \operatorname{Hom}(E,E(K_X+F))\\ \longrightarrow \operatorname{Hom}\Big(O_X\Big(\frac{c_1+\xi}{2}\Big),E(K_X+F)\Big)&\longrightarrow\cdots.\end{aligned}$$

We consider the long exact cohomology sequence

$$\begin{aligned}0\longrightarrow H^0O_X(F+K_X+\xi)&\longrightarrow H^0E\Big(\frac{\xi-c_1}{2}+K_X+F\Big)\\ \longrightarrow H^0O_X(F+K_X)&\longrightarrow\cdots\end{aligned}$$

associated to the exact sequence (1). Since $F+K_X$ is not an effective divisor and by assumption $H^0O_X(F+K_X+\xi)=0$, we get

$$\operatorname{Hom}\Big(O_X\Big(\frac{c_1-\xi}{2}\Big),E(K_X+F)\Big)=H^0E\Big(\frac{\xi-c_1}{2}+K_X+F\Big)=0.$$

Using the long exact cohomology sequence

$$\begin{aligned}0\longrightarrow H^0O_X(F+K_X)&\longrightarrow H^0E\Big(-\frac{\xi+c_1}{2}+K_X+F\Big)\\ \longrightarrow H^0O_X(F+K_X-\xi)&\longrightarrow\cdots\end{aligned}$$

associated to (1) and the hypothesis $H^0O_X(F+K_X-\xi)=0$, we obtain

$$\operatorname{Hom}\Big(O_X\Big(\frac{c_1+\xi}{2}\Big),E(K_X+F)\Big)=H^0E\Big(-\frac{\xi+c_1}{2}+K_X+F\Big)=0$$

which proves that $\operatorname{Hom}(E,E(K_X+F))=0$. Therefore, E is a prioritary sheaf and Claim 3 is proved.

It follows from Claim 3 that $M_{\widetilde{L}}(2; c_1, c_2) \cong E_\xi(c_1, c_2) \cong \mathbb{P}^{4c_2 - c_1^2 - 3} \subset \mathrm{Prior}(2; c_1, c_2)$.

Since $c_2 \gg 0$, $M_{\widetilde{L}}(2; c_1, c_2)$ and $M_L(2; c_1, c_2)$ are smooth and irreducible. It follows from Claim 3 (resp. Remark 3.6) that $M_{\widetilde{L}}(2; c_1, c_2)$ (resp. $M_L(2; c_1, c_2)$) is an open substack of $\mathrm{Prior}(2; c_1, c_2)$. By [Wal98, Proposition 2] $\mathrm{Prior}(2; c_1, c_2)$ is smooth and irreducible and we have proved that $M_{\widetilde{L}}(2; c_1, c_2)$ is rational. Therefore $M_L(2; c_1, c_2)$ is rational, which proves what we want. ∎

§4. The rationality of moduli spaces on rational surfaces

4.1. Moduli spaces on minimal rational surfaces

The goal of this subsection is to prove the rationality of the moduli space $M_L(2; c_1, c_2)$ of rank two, L-stable vector bundles E on smooth, minimal, rational surfaces X with fixed Chern classes $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbb{Z}$. According to Theorem 2.8, if X is a minimal rational surface, then X is either isomorphic to $\mathbb{P}^2$ or a Hirzebruch surface X_e with $e \neq 1$. The case $X \cong \mathbb{P}^2$ has been studied by several authors (see [ES87], [Mae90] and [Mar85]). Hence, we will study the rationality of the moduli space $M_L(2; c_1, c_2)$ of rank two, L-stable vector bundles E on a Hirzebruch surface X_e with fixed Chern classes $c_1 \in \mathrm{Pic}(X_e)$ and $c_2 \in \mathbb{Z}$.

Remark 4.1. We will prove the rationality of $M_L(2; c_1, c_2)$ distinguishing different cases, according to the value of $c_1 \in \mathrm{Pic}(X_e)$. Since a rank 2 vector bundle E on X_e is L-stable if, and only if, $E \otimes O_{X_e}(G)$ is L-stable for any divisor $G \in \mathrm{Pic}(X_e)$, we may assume, without loss of generality, that $c_1(E)$ is one of the following: 0, $C_0 + \alpha F$ with $\alpha \in \{0, 1\}$ or F.

PROPOSITION 4.2. *Let X_e be a smooth, Hirzebruch surface, $c_2 \in \mathbb{Z}$ and $\alpha \in \{0, 1\}$. Then, the following is satisfied*

(1) *There exists an ample divisor $\widetilde{L}$ on X_e such that $M_{\widetilde{L}}(2; C_0 + \alpha F, c_2)$ is a smooth, irreducible, rational, projective variety of dimension $4c_2 + e - 2\alpha - 3$, whenever non-empty and* $\mathrm{Pic}(M_{\widetilde{L}}(2; C_0 + \alpha F, c_2)) \cong \mathbb{Z}$.

(2) *For any ample divisor L on X_e, $M_L(2; C_0 + \alpha F, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of the expected dimension $4c_2 + e - 2\alpha - 3$, whenever non-empty.*

Proof. First of all, notice that by [Nak93, Theorem 1.5] we have: $c_2 \geq 1$. Now, we will apply Criterion 3.4. To this end, we take the numerical equivalence class, say $\xi = C_0 - (2c_2 - \alpha)F$.

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Claim. *ξ defines a non-empty wall of type $(C_0+\alpha F, c_2)$ and $d(\xi)=0$.*

Proof of the Claim. Notice that $\xi + c_1 = 2C_0 - 2(c_2-\alpha)F$, $\xi^2 = 2\alpha - e - 4c_2 = c_1^2 - 4c_2 < 0$ and $d(\xi) = 0$ (see Remarks 2.7 and 3.3). Hence, we only have to check that there exist ample divisors L and L' on X_e such that $\xi L \le 0 < \xi L'$. Take the ample divisors $L = C_0 + (e+1)F$ and $L' = C_0 + (e + 2c_2 + 1)F$ on X_e. We have $L\xi = \alpha - 2c_2 + 1 \le 0$ and $L'\xi = \alpha + 1 > 0$.

Since a smooth, Hirzebruch surface is an anticanonical, rational surface we can apply Criterion 3.4 (see Remark 2.7) and this leads us to prove the proposition. ∎

Before studying the case $c_1 = 0$, we need a low bound for c_2 which is given by the following result.

Lemma 4.3. *Let X_e be a smooth, Hirzebruch surface and L an ample divisor on X_e. If E is a rank two, L-stable, vector bundle on X_e with Chern classes $(0, c_2)$, then $c_2 \ge 2$.*

Proof. It follows from Bogomolov's inequality, $4c_2 - c_1^2 > 0$, that $c_2 > 0$. Assume that $c_2 = 1$. By Riemann-Roch's Theorem we have $\chi(E) = 1$. Since E is L-stable and $c_1(E) = 0$, we get $h^2E = h^0E(K_{X_e}) = 0$ and $h^0E \ge 1$. On the other hand, a non-zero section defines an injection $O_{X_e} \hookrightarrow E$ which contradicts the L-stability of E. Therefore, $c_2 \ge 2$ which proves what was stated. ∎

Proposition 4.4. *Let X_e be a smooth, Hirzebruch surface, $c_2 \in \mathbb{Z}$ and L any ample divisor on X_e. Then, $M_L(2; 0, c_2)$ is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - 3$.*

Proof. Since $c_2 \ge 2$ (Lemma 4.3), the result follows from [Mae90] and [Art90, Theorem 1.7 and Corollary 3.4]. ∎

In order to study the last case, which corresponds to $c_1 = F$, we will distinguish two cases according to the parity of c_2. Let us start with the odd case.

Proposition 4.5. *Let X_e be a smooth, Hirzebruch surface, $\alpha \in \{1,3\}$ and L an ample divisor on X_e. Then, $M_L(2; F, 4m+\alpha)$ is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension $4(4m+\alpha)-3$.*

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Proof. Let L be an ample divisor on X_e such that $M_L(2; F, 4m + \alpha)$ is non-empty. We consider X the smooth, rational surface obtained by blowing up a single point of X_e and the divisor $L_n := n\pi^*L - E_1$ on X, where $\pi : X \to X_e$ is the blow up and E_1 is the exceptional divisor. For n sufficiently large, L_n is an ample divisor on X and there is an open immersion ([Nak93, Theorem 1])

$$M_{X_e,L}(2; F, 4m + \alpha) \hookrightarrow M_{X,L_n}(2; F, 4m + \alpha).$$

Furthermore, by Remark 2.5, $M_{X_e,L}(2; F, 4m+\alpha)$ is a smooth, irreducible, quasi-projective variety of dimension $4(4m+\alpha)-3$ and $M_{X,L_n}(2; F, 4m+\alpha)$ is a smooth, irreducible, quasi-projective variety of the same dimension. Therefore, we only need to check that $M_{X,L_n}(2; F, 4m + \alpha)$ is rational. To this end, we consider the irreducible family $\mathcal{F}$ of rank two torsion free sheaves E on X given by a non-trivial extension

$$\epsilon : 0 \longrightarrow O_X(-D) \longrightarrow E \longrightarrow O_X(D + F) \otimes I_Z \longrightarrow 0 \tag{2}$$

where Z is a locally complete intersection 0-cycle of length $|Z| = 6m + 3(\alpha - 1)/2$ verifying $H^0 I_Z(2D + F) = 0$ being

$$D = C_0 + bF - cE_1 = \begin{cases} C_0 + (m + n - 1)F & \text{if } e = 2n \text{ and } \alpha = 1 \\ C_0 + (m + n)F - E_1 & \text{if } e = 2n \text{ and } \alpha = 3 \\ C_0 + (m + n)F - E_1 & \text{if } e = 2n + 1 \text{ and } \alpha = 1 \\ C_0 + (m + n)F & \text{if } e = 2n + 1 \text{ and } \alpha = 3. \end{cases}$$

Let us show:

(a) $h^0 E(D) = 1$.

(b) $\dim \mathcal{F} = 4(4m + \alpha) - 3$.

(c) Any $E \in \mathcal{F}$ is a simple prioritary sheaf with Chern classes $c_1(E) = F$ and $c_2(E) = 4m + \alpha$.

(a) Since $H^0 I_Z(2D + F) = 0$, from the exact cohomology sequence associated to the exact sequence (2) we easily get $h^0 E(D) = 1$, which proves (a).

(b) By construction we have

$$\begin{aligned} \dim \mathcal{F} &= \#\,\mathrm{moduli}(Z) + \dim \mathrm{Ext}^1(I_Z(D + F), O_X(-D)) - h^0 E(D) \\ &= 2\,\mathrm{length}(Z) + \dim \mathrm{Ext}^1(I_Z(D + F), O_X(-D)) - 1 \end{aligned} \tag{3}$$

where the last equality follows from (a). By Serre's duality we have

$$\dim \operatorname{Ext}^1(I_Z(D+F), O_X(-D)) = h^1 I_Z(2D+F+K_X).$$

Using again Serre's duality and Lemma 2.11; (a) we get

$$\begin{aligned} H^2 I_Z(2D+F+K_X) &= H^2 O_X(2D+F+K_X) \\ &= H^0 O_X(-2C_0-(2b+1)F+2cE_1)^* = 0. \end{aligned}$$

Using Lemma 2.9 and Lemma 2.11; (b) we obtain

$$\begin{aligned} h^0 O_X(2D+F+K_X) &= h^0 O_X((2b-1-e)F-(2c-1)E_1) \\ &\leq h^0 O_X((2b-1-e)F) = 2b-e. \end{aligned}$$

Hence, since $|Z| = 6m+3(\alpha-1)/2 > h^0 O_X(2D+F+K_X)$, for a generic $Z \in \operatorname{Hilb}^{|Z|}(X)$ we have $H^0 I_Z(2D+F+K_X) = 0$.

Therefore, putting this results together, we get $h^1 I_Z(2D+F+K_X) = -\chi(I_Z(2D+F+K_X)) = -\chi(O_X(2D+F+K_X)) + |Z|$ and by Riemann-Roch's Theorem we have $\chi(O_X(2D+F+K_X)) = 2b-e-2c^2+c$. Finally, we substitute in (3) and we get $\dim \mathcal{F} = 4(4m+\alpha)-3$ which proves (b).

(c) It is easy to check that any $E \in \mathcal{F}$ is a rank two torsion free sheaf with Chern classes $c_1(E) = F$ and $c_2(E) = 4m+\alpha$. Let us see that E is a prioritary sheaf. Since E is torsion free, we only need to check that $\operatorname{Ext}^2(E, E(-F)) = 0$ (see Definition 3.5). Applying $\operatorname{Hom}(\,\cdot\,, E(-F))$ to (2), we get the exact sequence

$$\begin{aligned} \cdots \longrightarrow \operatorname{Ext}^2(I_Z(D+F), E(-F)) &\longrightarrow \operatorname{Ext}^2(E, E(-F)) \\ &\longrightarrow \operatorname{Ext}^2(O_X(-D), E(-F)) \longrightarrow 0. \end{aligned}$$

Claim 1. $\operatorname{Ext}^2(O_X(-D), E(-F)) = 0$.

Proof of Claim 1. We consider the exact cohomology sequence

$$\cdots \longrightarrow H^2 O_X(-F) \longrightarrow H^2 E(D-F) \longrightarrow H^2 I_Z(2D) \longrightarrow 0$$

associated to (2). By Serre's duality and Lemma 2.11 we have $H^2 O_X(-F) = H^2 I_Z(2D) = 0$. Hence, $\operatorname{Ext}^2(O_X(-D), E(-F)) = H^2 E(D-F) = 0$.

Claim 2. $\operatorname{Ext}^2(I_Z(D+F), E(-F)) = 0$.

Proof of Claim 2. Applying $\mathrm{Hom}(I_Z(D+2F),\,\cdot\,)$ to (2), we get

$$\begin{aligned}\cdots \longrightarrow \mathrm{Ext}^2(I_Z(D+2F), O_X(-D)) &\longrightarrow \mathrm{Ext}^2(I_Z(D+2F), E)\\ \longrightarrow \mathrm{Ext}^2(I_Z(D+2F), I_Z(D+F)) &\longrightarrow 0.\end{aligned}$$

Since $|Z| > h^0 O_X(2D+2F+K_X)$, using Serre's duality, for a generic $Z \in \mathrm{Hilb}^{|Z|}(X)$ we have $\mathrm{Ext}^2(I_Z(D+2F), O_X(-D)) = H^0 I_Z(2D+2F+K_X)^* = 0$. Again by Serre's duality we get $\dim \mathrm{Ext}^2(I_Z(D+2F), I_Z(D+F)) \le h^0 O_X(F+K_X) = 0$. Therefore, $\mathrm{Ext}^2(I_Z(D+2F), E) = 0$ which proves Claim 2.

It easily follows from Claim 1 and Claim 2 that E is a prioritary sheaf. Now we will see that E is simple, i.e., $\dim \mathrm{Hom}(E,E) = 1$. Applying the functor $\mathrm{Hom}(\,\cdot\,, E)$ to the exact sequence (2), we get the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(I_Z(D+F), E) \longrightarrow \mathrm{Hom}(E,E) \longrightarrow \mathrm{Hom}(O_X(-D), E) \longrightarrow \cdots.$$

From (a) we have $\dim \mathrm{Hom}(O_X(-D), E) = h^0 E(D) = 1$. Hence, we only need to check that $\mathrm{Hom}(I_Z(D+F), E) = 0$. To this end, we apply the functor $\mathrm{Hom}(I_Z(D+F),\,\cdot\,)$ to the exact sequence (2) and we obtain the long exact sequence

$$\begin{aligned}0 &\longrightarrow \mathrm{Hom}(I_Z(D+F), O_X(-D)) \longrightarrow \mathrm{Hom}(I_Z(D+F), E)\\ &\longrightarrow \mathrm{Hom}(I_Z(D+F), I_Z(D+F)) \xrightarrow{\delta} \mathrm{Ext}^1(I_Z(D+F), O_X(-D))\\ &\longrightarrow \cdots.\end{aligned}$$

Applying Serre's duality and Lemma 2.11; (a) we get

$$\mathrm{Hom}(I_Z(D+F), O_X(-D)) = H^0 O_X(-2D-F) = 0.$$

Since the extension ϵ given in (2) is non-trivial, the map δ defined by $\delta(1) = \epsilon$, is an injection. Hence, $\mathrm{Hom}(I_Z(D+F), E) = 0$ and E is simple.

We have a morphism $\phi : \mathcal{F} \to \mathrm{Spl}(2; F, 4m+\alpha)$ from $\mathcal{F}$ to the moduli space $\mathrm{Spl}(2; F, 4m+\alpha)$ of simple prioritary sheaves, which is an injection. Indeed, assume that there are two non-trivial extensions

$$\begin{aligned}0 \longrightarrow O_X(-D) \xrightarrow{\alpha_1} E \xrightarrow{\alpha_2} O_X(D+F)\otimes I_Z \longrightarrow 0\,;\\ 0 \longrightarrow O_X(-D) \xrightarrow{\beta_1} E \xrightarrow{\beta_2} O_X(D+F)\otimes I_{Z'} \longrightarrow 0.\end{aligned}$$

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By assumption we have $\operatorname{Hom}(O_X(-D), I_Z(D+F)) = H^0 I_Z(2D+F) = 0$. Thus, $\beta_2 \circ \alpha_1 = \alpha_2 \circ \beta_1 = 0$. So, there exists $\gamma \in \operatorname{Aut}(O_X(-D)) \cong k$ such that $\beta_1 = \alpha_1 \circ \gamma$. Therefore, $Z = Z'$ and ϕ is an injection.

Now, let us see that $\operatorname{Spl}(2; F, 4m+\alpha)$ is rational. In fact, since the moduli space $\operatorname{Spl}(2; F, 4m+\alpha)$ of simple prioritary sheaves is smooth and irreducible (see [Wal98, Proposition 2]), its rationality follows from the fact that ϕ is an injection, Remark 2.3, which states that $\mathcal{F}$ is rational and the fact that $\dim \mathcal{F} = \dim \operatorname{Spl}(2; F, 4m+\alpha)$.

By Lemma 2.10, $L(K_{X_e}+F) < 0$. Thus, $L_n(K_X+F) < 0$ for $n \gg 0$ and $M_{X,L_n}(2; F, 4m+\alpha)$ is an open subscheme of the moduli space $\operatorname{Spl}(2; F, 4m+\alpha)$ of simple prioritary sheaves (Remark 3.6). Therefore, $M_{X,L_n}(2; F, 4m+\alpha)$ is also rational and, as we pointed out at the beginning of the proof, this implies that $M_{X_e,L}(2; F, 4m+\alpha)$ is rational, which proves what we want. ∎

Proposition 4.6. *Let X_e be a smooth, Hirzebruch surface and L an ample divisor on X_e. Then, the moduli space $M_L(2; F, 2n)$ is either empty or a smooth, irreducible, rational, quasi-projective variety of dimension* $4(2n)-3$.

Proof. Assume that $M_L(2; F, 2n)$ is non-empty. Then, from Bogomolov's inequality we get $4(2n) > 2 - 3K_{X_e}^2/2$. Therefore, since X_e is an anticanonical rational surface, we can apply Remark 2.5 and Theorem 2.6 and we only need to check the rationality of $M_L(2; F, 2n)$ for a suitable ample divisor L on X_e. We take $L = C_0 + (2e^2+n)F$.

We consider the irreducible family $\mathcal{F}_n$ of rank 2 vector bundles E on X_e given by a non trivial extension

$$0 \longrightarrow O_{X_e}(-D) \longrightarrow E \longrightarrow O_{X_e}(D+F) \otimes I_Z \longrightarrow 0 \tag{4}$$

where $D = (n-1)F$ and Z is a locally complete intersection 0-cycle of length $2n$ such that $H^0 I_Z(2D+F) = 0$.

Notice that since $|Z| = 2n$ and $h^0 O_X(2D+F) = 2n$ (see Lemma 2.9), the condition $H^0 I_Z(2D+F) = 0$ is satisfied for all generic $Z \in \operatorname{Hilb}^{2n}(X)$. By [MR93, Proposition 1.3], $\mathcal{F}_n$ is non-empty.

Let us show:

(a) $h^0 E(D) = 1$.

(b) $\dim \mathcal{F}_n = 4(2n) - 3$.

(c) There is an injection $\mathcal{F}_n \hookrightarrow M_L(2; F, 2n)$.

(a) It follows from the fact that $H^0 I_Z(2D+F) = 0$ and the exact cohomology sequence associated to the exact sequence (4).

(b) By definition we have

$$\begin{aligned}\dim \mathcal{F}_n &= \#\,\mathrm{moduli}(Z) + \dim \mathrm{Ext}^1(I_Z(D+F), O_{X_e}(-D)) - h^0 E(D)\\ &= 2\,\mathrm{length}(Z) + \dim \mathrm{Ext}^1(I_Z, O_{X_e}(-2D-F)) - 1\end{aligned}$$

where the last equality follows from (a). Consider the exact sequence

$$\begin{aligned}0 &\longrightarrow H^1 O_{X_e}(-2D-F) \longrightarrow \mathrm{Ext}^1(I_Z, O_{X_e}(-2D-F))\\ &\longrightarrow H^0 O_Z \longrightarrow H^2 O_{X_e}(-2D-F) \longrightarrow \mathrm{Ext}^2(I_Z, O_{X_e}(-2D-F))\\ &\longrightarrow 0.\end{aligned}$$

Since $-2D-F$ is not effective, we obtain

$$\begin{aligned}&\dim \mathrm{Ext}^1(I_Z, O_{X_e}(-2D-F))\\ &\quad = \dim \mathrm{Ext}^2(I_Z, O_{X_e}(-2D-F)) + |Z| - \chi(O_{X_e}(-2D-F)).\end{aligned}$$

By Serre's duality, $\mathrm{Ext}^2(I_Z, O_{X_e}(-2D-F)) = H^0 I_Z(-2C_0 + (2n-e-3)F)^* = 0$ and applying Riemann-Roch's Theorem, we get $\chi(O_{X_e}(-2D-F)) = 2-2n$. Therefore,

$$\begin{aligned}&\dim \mathrm{Ext}^1(I_Z, O_{X_e}(-2D-F)) = |Z| - (2-2n) = 4n-2 \quad \text{and}\\ &\dim \mathcal{F}_n = 2(2n) + (4n-2) - 1 = 4(2n) - 3.\end{aligned}$$

(c) Let $E \in \mathcal{F}_n$. It is easy to check that $c_1(E) = F$ and $c_2(E) = 2n$. Let us see that E is L-stable; i.e., for any rank 1 subbundle $O_{X_e}(G)$ of E we have

$$c_1(O_{X_e}(G))L < \frac{1}{2} = \frac{c_1(E)L}{2}.$$

Indeed, since E sits in an extension of type (4) we have

$$(i)\ O_{X_e}(G) \hookrightarrow O_{X_e}(-(n-1)F) \quad \text{or} \quad (ii)\ O_{X_e}(G) \hookrightarrow O_{X_e}(nF) \otimes I_Z.$$

In the first case, $-G-(n-1)F$ is an effective divisor. Since L is an ample divisor we have $(-G-(n-1)F)L \geq 0$ and $c_1(O_{X_e}(G))L = GL \leq -(n-1)FL = -(n-1) < 1/2 = c_1(E)L/2$.

If $O_{X_e}(G) \hookrightarrow O_{X_e}(nF) \otimes I_Z$ then $nF - G$ is an effective divisor. On the other hand, we have $H^0 O_{X_e}(G+(n-1)F) \subset H^0 I_Z((2n-1)F) = H^0 I_Z(2D+F) = 0$. So $G+(n-1)F$ is not an effective divisor and writing $G = \alpha C_0 + \beta F$, we have either $\beta + n - 1 < 0$ or $\alpha < 0$ (see Lemma 2.9).

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Assume that $\beta + n - 1 < 0$ (in particular $\beta < 0$). Since $nF - G$ is an effective divisor it must be $\alpha \leq 0$ (Lemma 2.9) and we get

$$c_1(O_{X_e}(G))L = GL = \alpha(2e^2 - e + n) + \beta < \frac{1}{2} = \frac{c_1(E)L}{2}.$$

Assume that $\alpha < 0$ and $\beta + n - 1 \geq 0$. Using again the fact that $nF - G$ is an effective divisor and hence $\beta \leq n$, we obtain

$$\begin{aligned} c_1(O_{X_e}(G))L &= -\alpha e + \alpha(2e^2 + n) + \beta \\ &\leq -\alpha e + \alpha(2e^2 + n) + n < \frac{1}{2} = \frac{c_1(E)L}{2}, \end{aligned}$$

which proves the L-stability of E. Thus, we have a morphism $\phi : \mathcal{F}_n \to M_L(2; F, 2n)$ and arguing as in Proposition 4.5 we prove that it is an injection.

Finally, since $M_L(2; F, 2n)$ is smooth and irreducible (Remark 2.5), its rationality follows from (c), Remark 2.3 and the fact that $\dim \mathcal{F}_n = \dim M_L(2; F, 2n)$. ∎

Putting this results together we get the main result of this subsection.

THEOREM 4.7. *Let X be a smooth, irreducible, projective, minimal, rational surface, $c_1 \in \mathrm{Pic}(X)$ and $c_2 \in \mathbb{Z}$. Then, for any polarization L on X, the moduli space $M_L(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$, whenever non-empty.*

Proof. It follows from Theorem 2.8, Propositions 4.2, 4.4, 4.5, 4.6 and [Mae90]. ∎

4.2. Moduli spaces of vector bundles on non-minimal rational surfaces

In this subsection we prove the rationality of the moduli space $M_L(2; c_1, c_2)$ of rank two, L-stable vector bundles E with Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$, over a smooth, non-minimal, rational surface X, i.e. the underlying surface X of the moduli spaces we deal with is the blowup $\pi : X \to X_e$ of a Hirzebruch surface X_e for some $e \geq 0$ and we keep the notation introduced in §2.

Remark 4.8. Since a rank 2 vector bundle E on X is H-stable if, and only if, $E \otimes O_X(G)$ is H-stable for any divisor $G \in \mathrm{Pic}(X)$, we may assume, without loss of generality, that $c_1(E)$ is one of the following: 0, $\sum_{j=1}^{\rho} E_{i_j}$ with $1 \leq \rho \leq s$, C_0, F, C_0+F, $C_0+\sum_{j=1}^{\rho} E_{i_j}$ with $1 \leq \rho \leq s$, $F+\sum_{j=1}^{\rho} E_{i_j}$ with $1 \leq \rho \leq s$ or $C_0 + F + \sum_{j=1}^{\rho} E_{i_j}$ with $1 \leq \rho \leq s$. For simplicity, we will write $\sum_{i=1}^{\rho} E_i$ instead of $\sum_{j=1}^{\rho} E_{i_j}$.

PROPOSITION 4.9. *Let $\pi : X \to X_e$ be the blowup of a Hirzebruch surface X_e for some $e \geq 0$, H any ample divisor on X with $H(K_X+F) < 0$ and $c_1 \in \{0, C_0, F, C_0 + F\} \subset \mathrm{Pic}(X)$. For $c_2 \gg 0$, the moduli space $M_H(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$.*

Proof. Let $X = Y_0 \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_s} Y_s = X_e$ be a sequence of monomial transformations such that $\pi = \pi_s \circ \cdots \circ \pi_2 \circ \pi_1$. Take an ample divisor L on $Y_s = X_e$. Since $c_2 \gg 0$, $M_{Y_s,L}(2; c_1, c_2)$ is non-empty. We consider on Y_{s-1} the divisor $L_{s-1}^{n_{s-1}} = n_{s-1}\pi_s^* L - E'_s$. Since $L(K_{X_e}+F) < 0$ (Lemma 2.10) we also have $L_{s-1}^{n_{s-1}}(K_{Y_{s-1}} + F) < 0$ for n_{s-1} sufficiently large. Moreover, for $n_{s-1} \gg 0$, $L_{s-1}^{n_{s-1}}$ is an ample divisor on Y_{s-1} and there is an open immersion (see [Nak93, Theorem 1]) $M_{Y_s,L}(2; c_1, c_2) \hookrightarrow M_{Y_{s-1}, L_{s-1}^{n_{s-1}}}(2; c_1, c_2)$. Furthermore, for $c_2 \gg 0$ the moduli space $M_{Y_s,L}(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$ (Theorem 4.7) and $M_{Y_{s-1}, L_{s-1}^{n_{s-1}}}(2; c_1, c_2)$ is a smooth, irreducible, quasi-projective variety of the same dimension (Theorem 2.4). Hence, $M_{Y_{s-1}, L_{s-1}^{n_{s-1}}}(2; c_1, c_2)$ is rational. Repeating the process with the monomial transformation $\pi_i : Y_{i-1} \to Y_i$, $i = 1, \ldots, s-1$, we obtain an ample divisor $L_0^{n_0}$ on $X = Y_0$ such that $L_0^{n_0}(K_{Y_0} + F) < 0$ is ample and $M_{Y_0, L_0^{n_0}}(2; c_1, c_2)$ is rational and we conclude that $M_{X,H}(2; c_1, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 - c_1^2 - 3$ (Theorem 3.7). ∎

Now using Criterion 3.8, we will deal with the cases $c_1 = C_0 + \sum_{i=1}^{\rho} E_i$ and $c_1 = C_0 + F + \sum_{i=1}^{\rho} E_i$ respectively.

PROPOSITION 4.10. *Let $\pi : X \to X_e$ be the blowup of a Hirzebruch surface X_e for some $e \geq 0$ and $0 \ll c_2 \in \mathbb{Z}$. Then, the following holds*

(1) *There exists an ample divisor $\widetilde{L}$ on X such that $M_{\widetilde{L}}(2; C_0 + \alpha F + \sum_{i=1}^{\rho} E_i, c_2)$ is a projective space of dimension $4c_2 + e - 2\alpha + \rho - 3$. In particular, $\mathrm{Pic}(M_{\widetilde{L}}(2; C_0 + \alpha F + \sum_{i=1}^{\rho} E_i, c_2)) \cong \mathbb{Z}$.*

(2) *For any ample divisor L on X such that $L(K_X+F) < 0$, the moduli space $M_L(2; C_0+\alpha F+\sum_{i=1}^{\rho} E_i, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 + e - 2\alpha + \rho - 3$.*

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Proof. We take the numerical equivalence class, say $\xi = C_0 - (2c_2 - \alpha)F - \sum_{i=1}^{\rho} E_i$.

Claim. *For all $c_2 \gg 0$, $\xi \equiv C_0 - (2c_2 - \alpha)F - \sum_{i=1}^{\rho} E_i$ defines a non-empty wall of type $(C_0 + \alpha F + \sum_{i=1}^{\rho} E_i, c_2)$ and it satisfies*

(a) $\xi^2 = c_1^2 - 4c_2, \quad \xi^2 + \xi K_X + 2 = 0,$

(b) $H^0 O_X(\xi + 3K_X) = H^0 O_X(\xi + K_X + F) = H^0 O_X(K_X + F - \xi) = 0.$

Proof of the Claim. (a) Notice that since $\xi + c_1 = 2C_0 - (2c_2 - 2\alpha)F$, $\xi^2 = 2\alpha - e - 4c_2 - \rho$ and $\xi^2 + \xi K_X + 2 = 2\alpha - e - 4c_2 - \rho + (e - 2 + 4c_2 - 2\alpha + \rho) + 2 = 0$ we only need to check (see Definition 3.1) that there exist ample divisors L_1 and L_2 on X such that $\xi L_1 < 0 < \xi L_2$. In fact, we consider the ample divisors on X_e

$$L_1 = C_0 + (e + s)F \quad \text{and} \quad L_2 = C_0 + (2c_2 + e + \rho)F$$

and we define

$$\widetilde{L}_1 := n_1 \cdots n_s L_1 - \sum_{i=1}^{s} (n_1 \cdots n_{i-1}) E_i \quad \text{and}$$

$$\widetilde{L}_2 := m_1 \cdots m_s L_2 - \sum_{i=1}^{s} (m_1 \cdots m_{i-1}) E_i$$

where $n_i, m_j \in \mathbb{Z}$. For $n_i, m_j \gg 0$, by [Har77, Chapter V, Exercise 3.3], $\widetilde{L}_1$ and $\widetilde{L}_2$ are ample divisors on X. Moreover, for $n_1, m_1 \gg 0$, we have $\xi \widetilde{L}_1 < 0$ and $\xi \widetilde{L}_2 > 0$. Hence, (a) is proved.

Applying Lemma 2.11; (a) we immediately obtain what is stated in (b). This finishes the proof of the Claim.

Since by hypothesis $c_2 \gg 0$, we also have $4c_2 - c_1^2 > 2 - 3K_X^2/2$ ($c_1^2 = 2\alpha - e - \rho$). Thus, we can apply Criterion 3.8 and we get what we want. ∎

For the remaining values of c_1 there is no numerical equivalence class ξ verifying the hypothesis of Criterion 3.8. In these cases, we will prove the rationality of the moduli space $M_L(2; c_1, c_2)$ constructing a suitable family of prioritary sheaves (see Definition 3.5 and Remark 3.6) over a big enough rational base.

PROPOSITION 4.11. *Let $\pi : X \to X_e$ be the blowup of a Hirzebruch surface X_e for some $e \geq 0$. For any ample divisor L on X such that $L(K_X + F) < 0$ and for any $\mathbb{Z} \ni c_2 \gg 0$, the moduli space $M_L(2; \sum_{i=1}^{\rho} E_i, c_2)$, $1 \leq \rho \leq s$, is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 + \rho - 3$.*

Proof. Let $X = Y_0 \xrightarrow{\pi_1} Y_1 \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_s} Y_s = X_e$ be a sequence of monomial transformations such that $\pi = \pi_s \circ \cdots \circ \pi_2 \circ \pi_1$. By Theorem 2.4 we only need to check the rationality of $M_L(2; \sum_{i=1}^{\rho} E_i, c_2)$. Moreover, applying [Nak93, Theorem 1], to the monomial transformation $\pi_i : Y_{s-1} \to Y_i$, $i = \rho + 1, \ldots, s$, we can assume $s = \rho$.

We write $c_2 = 2n + \beta$ with $\beta \in \{0, 1\}$ and we consider the irreducible family $\mathcal{F}_{n,\beta}$ of rank 2 vector bundles E on X given by a non-trivial extension

$$(5) \qquad \epsilon : 0 \longrightarrow O_X(-D) \longrightarrow E \longrightarrow O_X\Big(D + \sum_{i=1}^{\rho} E_i\Big) \otimes I_Z \longrightarrow 0$$

where $D = nF - (1-\beta)E_1$ and Z is a locally complete intersection 0-cycle of length $2n + \beta$ such that $H^0 I_Z(2D + \sum_{i=1}^{\rho} E_i) = 0$. It is easy to see that such a 0-cycle Z exists and by [MR93, Proposition 1.3], $\mathcal{F}_{n,\beta}$ is non-empty.

Let us show:

(a) $h^0 E(D) = 1$.

(b) $\dim \mathcal{F}_{n,\beta} = 4c_2 + \rho - 3$.

(c) Any $E \in \mathcal{F}_{n,\beta}$ is a simple prioritary vector bundle and it has Chern classes $(\sum_{i=1}^{\rho} E_i, c_2)$.

(a) From the exact cohomology sequence associated to the exact sequence (5) and the assumption $H^0 I_Z(2D + \sum_{i=1}^{\rho} E_i) = 0$ we get $h^0 E(D) = 1$, which proves (a).

(b) By definition we have

$$\begin{aligned}
&\dim \mathcal{F}_{n,\beta} \\
&\quad = \#\,\mathrm{moduli}(Z) + \dim \mathrm{Ext}^1\Big(I_Z\Big(D + \sum_{i=1}^{\rho} E_i\Big), O_X(-D)\Big) - h^0 E(D) \\
&\quad = 2\,\mathrm{length}(Z) + \dim \mathrm{Ext}^1\Big(I_Z, O_X\Big(-2D - \sum_{i=1}^{\rho} E_i\Big)\Big) - 1
\end{aligned}$$

where the last equality follows from (a). Consider the exact sequence

$$0 \longrightarrow H^1 O_X\Big(-2D - \sum_{i=1}^{\rho} E_i\Big) \longrightarrow \operatorname{Ext}^1\Big(I_Z, O_X\Big(-2D - \sum_{i=1}^{\rho} E_i\Big)\Big) \longrightarrow H^0 O_Z$$
$$\longrightarrow H^2 O_X\Big(-2D - \sum_{i=1}^{\rho} E_i\Big) \longrightarrow \operatorname{Ext}^2\Big(I_Z, O_X\Big(-2D - \sum_{i=1}^{\rho} E_i\Big)\Big) \longrightarrow 0.$$

Since $-2D - \sum_{i=1}^{\rho} E_i$ is not effective (see Lemma 2.11; (a)), we obtain

$$\dim \operatorname{Ext}^1\Big(I_Z, O_X\Big(-2D - \sum_{i=1}^{\rho} E_i\Big)\Big)$$
$$= \dim \operatorname{Ext}^2\Big(I_Z, O_X\Big(-2D - \sum_{i=1}^{\rho} E_i\Big)\Big) + h^0 O_Z - \chi\Big(O_X\Big(-2D - \sum_{i=1}^{\rho} E_i\Big)\Big).$$

Using Serre's duality and Lemma 2.11; (a), we get $\operatorname{Ext}^2(I_Z, O_X(-2D - \sum_{i=1}^{\rho} E_i)) = H^0 I_Z(2D + \sum_{i=1}^{\rho} E_i + K_X)^* = 0$ and by Riemann-Roch's Theorem we have $\chi(O_X(-2D - \sum_{i=1}^{\rho} E_i)) = -2n + \beta - 2\beta^2 + 2 - \rho$. Putting this results together we obtain

$$\dim \operatorname{Ext}^1\Big(I_Z, O_X\Big(-2D - \sum_{i=1}^{\rho} E_i\Big)\Big) = 4n + 2\beta^2 - 2 + \rho \quad \text{and}$$
$$\dim \mathcal{F}_{n,\beta} = 8n + 2\beta + 2\beta^2 + \rho - 3 = 4(2n + \beta) + \rho - 3$$

which proves (b).

(c) It is easy to see that for any $E \in \mathcal{F}_{n,\beta}$, $c_1(E) = \sum_{i=1}^{\rho} E_i$ and $c_2(E) = 2n + \beta = c_2$. Let us see that E is a prioritary sheaf. Since E is a torsion free sheaf, we only have to check that $\operatorname{Ext}^2(E, E(-F)) = 0$. Applying $\operatorname{Hom}(\cdot, E(-F))$ to (5) we get

$$\cdots \longrightarrow \operatorname{Ext}^2\Big(I_Z\Big(D + \sum_{i=1}^{\rho} E_i\Big), E(-F)\Big) \longrightarrow \operatorname{Ext}^2(E, E(-F))$$
$$\longrightarrow \operatorname{Ext}^2(O_X(-D), E(-F)) \longrightarrow 0.$$

Claim 1. $\operatorname{Ext}^2(O_X(-D), E(-F)) = 0$.

Proof of Claim 1. We consider the exact cohomology sequence

$$\cdots \longrightarrow H^2 O_X(-F) \longrightarrow H^2 E(D - F) \longrightarrow H^2 I_Z\Big(2D - F + \sum_{i=1}^{\rho} E_i\Big) \longrightarrow 0$$

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associated to the exact sequence (5). Since Z is a 0-dimensional subscheme, using Lemma 2.11 and Serre's duality we get

$$\begin{aligned}&H^2 O_X(-F) = 0 \quad \text{and}\\ &H^2 I_Z\Big(2D - F + \sum_{i=1}^{\rho} E_i\Big) = H^0 O_X\Big(-2D + F - \sum_{i=1}^{\rho} E_i + K_X\Big)^* = 0\end{aligned}$$

which proves that $H^2 E(D-F) = 0$ or, equivalently, $\operatorname{Ext}^2(O_X(-D), E(-F)) = 0$.

Claim 2. $\operatorname{Ext}^2(I_Z(D + \sum_{i=1}^{\rho} E_i), E(-F)) = 0$.

Proof of Claim 2. Applying $\operatorname{Hom}(I_Z(D + \sum_{i=1}^{\rho} E_i), \cdot\,)$ to (5) we get

$$\begin{aligned}\cdots \longrightarrow &\operatorname{Ext}^2\Big(I_Z, O_X\Big(-2D - F - \sum_{i=1}^{\rho} E_i\Big)\Big)\\ &\longrightarrow \operatorname{Ext}^2\Big(I_Z\Big(D + \sum_{i=1}^{\rho} E_i\Big), E(-F)\Big) \longrightarrow \operatorname{Ext}^2(I_Z, I_Z(-F)) \longrightarrow 0.\end{aligned}$$

Using once more Serre's duality and Lemma 2.11; (a), we obtain

$$\operatorname{Ext}^2\Big(I_Z, O_X\Big(-2D - F - \sum_{i=1}^{\rho} E_i\Big)\Big) = H^0 I_Z\Big(2D + F + \sum_{i=1}^{\rho} E_i + K_X\Big)^* = 0.$$

Finally, we have

$$\begin{aligned}\dim \operatorname{Ext}^2(I_Z, I_Z(-F)) &\leq \dim \operatorname{Ext}^2(O_X, I_Z(-F))\\ &= h^0 O_X(K_X + F) = 0,\end{aligned}$$

which gives us $\operatorname{Ext}^2(I_Z(D + \sum_{i=1}^{\rho} E_i), E(-F)) = 0$ and this proves Claim 2.

It easily follows from Claims 1 and 2 that E is a prioritary sheaf. Let us see that E is simple, i.e., $\dim \operatorname{Hom}(E, E) = 1$. Applying $\operatorname{Hom}(\,\cdot\,, E)$ to (5) we get

$$\begin{aligned}0 \longrightarrow &\operatorname{Hom}\Big(I_Z\Big(D + \sum_{i=1}^{\rho} E_i\Big), E\Big) \longrightarrow \operatorname{Hom}(E, E)\\ &\longrightarrow \operatorname{Hom}(O_X(-D), E) \longrightarrow \cdots .\end{aligned}$$

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By (a) we have $\dim \operatorname{Hom}(O_X(-D), E) = h^0 E(D) = 1$. Hence, we only have to check that $\operatorname{Hom}(I_Z(D + \sum_{i=1}^{\rho} E_i), E) = 0$. To this end, we consider the long exact sequence

$$\begin{aligned}
0 &\longrightarrow \operatorname{Hom}\Big(I_Z\Big(D + \sum_{i=1}^{\rho} E_i\Big), O_X(-D)\Big) \longrightarrow \operatorname{Hom}\Big(I_Z\Big(D + \sum_{i=1}^{\rho} E_i\Big), E\Big) \\
&\longrightarrow \operatorname{Hom}\Big(I_Z\Big(D + \sum_{i=1}^{\rho} E_i\Big), I_Z\Big(D + \sum_{i=1}^{\rho} E_i\Big)\Big) \\
&\xrightarrow{\delta} \operatorname{Ext}^1\Big(I_Z\Big(D + \sum_{i=1}^{\rho} E_i\Big), O_X(-D)\Big) \longrightarrow \cdots
\end{aligned}$$

By Lemma 2.11; (a) we have $\operatorname{Hom}(I_Z(D+\sum_{i=1}^{\rho} E_i), O_X(-D)) = 0$. On the other hand, since E is given by a non-trivial extension ϵ, the map δ defined by $\delta(1) = \epsilon$ is an injection. Therefore, $\operatorname{Hom}(I_Z(D + \sum_{i=1}^{\rho} E_i), E) = 0$ and E is a simple vector bundle, which proves (c).

It follows from (c) that there is a morphism

$$\phi : \mathcal{F}_{n,\beta} \longrightarrow \operatorname{Spl}\Big(2; \sum_{i=1}^{\rho} E_i, c_2\Big)$$

from the irreducible family $\mathcal{F}_{n,\beta}$ to the moduli space $\operatorname{Spl}(2; \sum_{i=1}^{\rho} E_i, c_2)$ of simple prioritary sheaves and arguing as in Proposition 4.5 we prove that it is an injection.

Now, let us see that $\operatorname{Spl}(2; \sum_{i=1}^{\rho} E_i, c_2)$ is rational. In fact, since the moduli space $\operatorname{Spl}(2; \sum_{i=1}^{\rho} E_i, c_2)$ of simple prioritary sheaves is smooth and irreducible ([Wal98, Proposition 2]), its rationality follows from the fact that ϕ is an injection, Remark 2.3 and the fact that $\dim \mathcal{F}_{n,\beta} = \dim \operatorname{Spl}(2; \sum_{i=1}^{\rho} E_i, c_2)$.

Since $L(K_X + F) < 0$, the moduli space $M_L(2; \sum_{i=1}^{\rho} E_i, c_2)$ is an open dense subset of the moduli space $\operatorname{Spl}(2; \sum_{i=1}^{\rho} E_i, c_2)$ of simple prioritary sheaves (Remark 3.6). Therefore, $M_L(2; \sum_{i=1}^{\rho} E_i, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 + \rho - 3$, which proves what we want. ∎

Proposition 4.12. *Let $\pi : X \to X_e$ be the blowup of a Hirzebruch surface X_e for some $e \geq 0$. For any ample divisor L on X such that $L(K_X + F) < 0$ and for any $\mathbb{Z} \ni c_2 \gg 0$, the moduli space $M_L(2; F +$*

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$\sum_{i=1}^{\rho} E_i, c_2)$, $1 \leq \rho \leq s$, *is a smooth, irreducible, rational, quasi-projective variety of dimension* $4c_2 + \rho - 3$.

Proof. As in Proposition 4.11 we only need to check the rationality of the moduli space $M_L(2; F + \sum_{i=1}^{\rho} E_i, c_2)$ and we can assume $s = \rho$.

We write $c_2 = 2n + \beta$ with $\beta \in \{0, 1\}$ and we consider the irreducible family $\mathcal{F}_{n,\beta}$ of rank 2 vector bundles E on X given by a non-trivial extension

$$(6) \qquad \epsilon : 0 \longrightarrow O_X(-D) \longrightarrow E \longrightarrow O_X\Big(D + F + \sum_{i=1}^{\rho} E_i\Big) \otimes I_Z \longrightarrow 0$$

where $D = (n + \beta - 1)F - \beta E_1$ and Z is a locally complete intersection 0-cycle of length $2n + \beta$ such that $H^0 I_Z(2D + F + \sum_{i=1}^{\rho} E_i) = 0$.

Arguing as in Proposition 4.11 we can show:

(a) $h^0 E(D) = 1$.

(b) $\dim \mathcal{F}_{n,\beta} = 4c_2 + \rho - 3$.

(c) Any $E \in \mathcal{F}_{n,\beta}$ is a simple prioritary vector bundle and it has Chern classes $(F + \sum_{i=1}^{\rho} E_i, c_2)$.

Once more, from (c) we can deduce, using Theorem 2.4 and Remark 3.6, that the moduli space $M_L(2; F + \sum_{i=1}^{\rho} E_i, c_2)$ is a smooth, irreducible, rational, quasi-projective variety of dimension $4c_2 + \rho - 3$. ∎

Gathering this results we obtain the main result of this subsection

THEOREM 4.13. *Let* $\pi : X \to X_e$ *be the blowup of a Hirzebruch surface* X_e *for some* $e \geq 0$ *and* L *any ample divisor on* X *such that* $L(K_X + F) < 0$. *For any* $c_1 \in \operatorname{Pic}(X)$ *and any integer* $c_2 \gg 0$, *the moduli space* $M_L(2; c_1, c_2)$ *is a smooth, irreducible, rational, quasi-projective variety of dimension* $4c_2 - c_1^2 - 3$.

Proof. It follows from Propositions 4.9–12. ∎

Finally, we are ready to state the main result of this work.

THEOREM A. *Let* X *be a Hirzebruch surface or the blowup* $\pi : X \to X_e$ *of a Hirzebruch surface* X_e, L *a polarization on* X *such that* $L(K_X + F) < 0$, $c_1 \in \operatorname{Pic}(X)$ *and* $0 \ll c_2 \in \mathbb{Z}$. *Then, the moduli space* $M_L(2; c_1, c_2)$ *is a smooth, irreducible, rational, quasi-projective variety of dimension* $4c_2 - c_1^2 - 3$.

Proof. It is a consequence of Theorems 4.7 and 4.13. ∎

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