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ON REDUCTIVE LIE ADMISSIBLE ALGEBRAS

ARTHUR A. SAGLE

1. Introduction. A Lie admissible algebra is a non-associative algebra A such that A^- is a Lie algebra where A^- denotes the anti-commutative algebra with vector space A and with commutation $[X, Y] = XY - YX$ as multiplication; see [**1**; **2**; **5**]. Next let $L^-(X): A^- \to A^-: Y \to [X, Y]$ and $H = \{L^-(X): X \in A^-\}$; then, since A^- is a Lie algebra, we see that H is contained in the derivation algebra of A^- and consequently the direct sum $\mathfrak{g} = A^- \oplus H$ can be naturally made into a Lie algebra with multiplication $[PQ]$ given by: $P = X + L^-(U)$, $Q = Y + L^-(V) \in \mathfrak{g}$, then

$$[PQ] = [X, Y] + L^-(U)Y - L^-(V)X + L^-([U, V]) + L^-([X, Y])$$

and note that for any P, $[PP] = 0$ so that $[PQ] = -[QP]$ and the Jacobi identity for $\mathfrak{g}$ follows from the fact that A^- is Lie. In particular, $[L^-(U)Y] = -[YL^-(U)] = L^-(U)Y$ and $[XY] = [X, Y] + L^-([X, Y])$; thus $\mathfrak{g} = A^- \oplus H$ is a reductive Lie algebra according to the following definition.

Definition 1. Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}$ be a subalgebra; then the pair $(\mathfrak{g}, \mathfrak{h})$ is called a *reductive pair* if there is in $\mathfrak{g}$ a subspace $\mathfrak{m}$ with $\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$ (subspace direct sum) and $[\mathfrak{h}\mathfrak{m}] \subset \mathfrak{m}$. In this case we shall frequently say $\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$ *is a reductive Lie algebra.*

For example, if $\mathfrak{g}$ and $\mathfrak{h}$ are finite-dimensional and semi-simple over a field of characteristic zero, then since the Killing form, K, of $\mathfrak{g}$ restricted to $\mathfrak{h} \times \mathfrak{h}$ is non-degenerate, we can write $\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$ with $\mathfrak{m} = \mathfrak{h}^\perp$ the orthogonal complement of $\mathfrak{h}$ relative to K. For $X \in \mathfrak{m}$ and $U, V \in \mathfrak{h}$ we have

$$K([XU], V) = K(X, [UV]) = 0$$

so that $[\mathfrak{m}\mathfrak{h}] \subset \mathfrak{h}^\perp = \mathfrak{m}$ and consequently $(\mathfrak{g}, \mathfrak{h})$ is a reductive pair [**9**].

Let A be a non-associative algebra over the field F with identity element 1, then the algebra A^- has $F1$ as a set of absolute divisors of zero. Thus, when considering problems relating the simplicity of A with that of A^- (see [**5**]) it is perhaps more natural to use the algebra $A^0 = A^-/F1$. We can relate A^- and A^0 to Lie algebras as follows.

Definition 2. A non-associative algebra A is *reductive Lie admissible* if there exists a Lie subalgebra H (or H^0) of the derivation algebra of A^- (or A^0) so that $\mathfrak{g} = A^- \oplus H$ (or $\mathfrak{g}^0 = A^0 \oplus H^0$) is a reductive Lie algebra with multiplication, $[PQ]$, satisfying: for $X, Y \in A^-$ (or A^0) and $D, D' \in H$ (or H^0)

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we have $[XD] = -[DX] = D(X) \in A^-$ (or A^0), $[DD'] = DD' - D'D$ and $[XY] = [X, Y] + D(X, Y)$, where $[X, Y]$ is the product in A^- (or A^0) and $D(X, Y)$ is a suitable element in H (or H^0).

Note that if A contains an identity 1 and $\mathfrak{g} = A^- \oplus H$ is a reductive Lie algebra as above, then $\mathfrak{g} = F1 \dotplus \tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{g}^0$ above. This follows since $A^- = F1 \dotplus B$, where B is isomorphic to A^0, and since $D \in H$ is such that $D(1) = 0$, we see that D induces a derivation $D^0 \in H^0$. Thus we could define reductive Lie admissibility in terms of $\mathfrak{g} = A^- \oplus H$ and then pass to the algebra $\mathfrak{g}^0 = A^0 \oplus H^0$. But it is frequently easier to use A^0 when considering simplicity of algebras and easier to use A^- when considering identities of algebras.

As an example let A be an alternative algebra, then A^- is a Malcev algebra (a Lie algebra if A is associative). From the identities for a Malcev algebra it was noted in [**10**] that for the inner derivations $D(X, Y) = [L(X), L(Y)] + L([X, Y])$ in H (= derivation algebra of A^-), the set $\mathfrak{g} = A^- \oplus H$ is a reductive Lie algebra with the product as in the above definition. But if $1 \in A$, then $\mathfrak{g}$ or A^- is not simple. For example, if A is the 8-dimensional split Cayley-Dickson algebra, then $A^0 = A^-/F1$ is the split simple 7-dimensional Malcev algebra and we discuss this and the corresponding simple Lie algebra $\mathfrak{g}^0$ later. Thus from this case, since A^- is not a Lie algebra, the class of reductive Lie admissible algebras is larger than the class of Lie admissible algebras; also see [**7**; **8**].

In this paper we start with the reductive pair $(\mathfrak{g}, \mathfrak{h})$ with fixed decomposition $\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$ and construct a reductive Lie admissible algebra relative to $\mathfrak{m}$. First, for $X, Y \in \mathfrak{m}$ let $[XY] = X \circ Y + h(X, Y)$, where $X \circ Y = [XY]_{\mathfrak{m}}$ ($h(X, Y)$) is the projection of $[XY]$ in $\mathfrak{g}$ into $\mathfrak{m}$ ($\mathfrak{h}$); then we have the following identities for $X, Y, Z \in \mathfrak{m}$ and $h \in \mathfrak{h}$:

(1) $X \circ Y = -Y \circ X$ (bilinear);

(2) $h(X, Y) = -h(Y, X)$ (bilinear);

(3) $[h(X, Y)Z] + [h(Y, Z)X] + [h(Z, X)Y] =$
$$X \circ (Y \circ Z) + Y \circ (Z \circ X) + Z \circ (X \circ Y);$$

(4) $h(X \circ Y, Z) + h(Y \circ Z, X) + h(Z \circ X, Y) = 0$;

(5) $[h\,h(X, Y)] = h([hX], Y) + h(X, [hY])$;

(6) $[hX \circ Y] = [hX] \circ Y + X \circ [hY]$.

In particular, we see from (6) that the map $D(h)\colon \mathfrak{m} \to \mathfrak{m}\colon X \to [hX]$ is a derivation of the anti-commutative algebra $\mathfrak{m}$ with multiplication $X \circ Y = [XY]_{\mathfrak{m}}$; see [**7**; **8**; **9**; **10**; **11**]. Let $D(\mathfrak{h}) = \{D(h)\colon h \in \mathfrak{h}\}$.

Next, by Ado's theorem we can represent the reductive Lie algebra by a reductive Lie algebra of endomorphisms $\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$. We form the associative enveloping algebra $\mathfrak{g}^*$ and assume that it contains an identity element 1. Let $\mathfrak{a} = F1 + \mathfrak{m}$ and decompose $\mathfrak{g}^* = \mathfrak{a} \dotplus \mathfrak{k}$ into subspaces; then we discuss the reductive Lie admissible algebras formed from $\mathfrak{a}$ with the product $P * Q$ obtained from the projection of the product PQ in $\mathfrak{g}^*$ into $\mathfrak{a}$. We see that $\mathfrak{a}^-$ or $\mathfrak{a}^0$ is isomorphic to the algebra $\mathfrak{m}$ with multiplication $X \circ Y$ and relate the simplicity of the algebras $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{a}^-$, and $\mathfrak{a}^0$. Finally, we indicate how the split 7-dimensional simple Malcev algebra can be considered as a space $\mathfrak{m}$ with multiplication $X \circ Y = [XY]_{\mathfrak{m}}$ and use this process to construct the split 8-dimensional Cayley-Dickson algebra. All algebras in this paper are finite-dimensional over an algebraically closed field F of characteristic zero.

In [**4**] there is considered the opposite process to the above; namely starting with an associative algebra K with subalgebra B, decompose $K = A \dot{+} B$ and use the projection multiplication in A. This is analogous to constructing the anti-commutative algebra $\mathfrak{m}$ from the reductive Lie algebra $\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}$.

2. The construction. Let $\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}$ (fixed decomposition) be a reductive Lie algebra of endomorphisms with commutation $[X, Y] = XY - YX$ as multiplication and let $\mathfrak{g}^*$ be the enveloping associate algebra of endomorphisms generated by $\mathfrak{g}$ [**3**]. We shall assume that $\mathfrak{g}^*$ has an identity element 1; adjoin 1 if necessary. Let $\mathfrak{a} = F1 + \mathfrak{m}$ and let $\mathfrak{g}^* = \mathfrak{a} \dot{+} \mathfrak{k}$ be a fixed subspace decomposition for a suitable subspace $\mathfrak{k}$. For example, if $D(\mathfrak{h})$ is completely reducible in $\mathfrak{g}$, then since $\mathfrak{a}$ is $D(\mathfrak{h})$-invariant, choose $\mathfrak{k}$ to be a $D(\mathfrak{h})$-invariant complement. We now define a multiplication $*$ on $\mathfrak{a}$ which will give the reductive Lie admissible algebras as follows. Let $P = \alpha 1 + X$ and $Q = \beta 1 + Y$ be in $\mathfrak{a}$ and form the product PQ in $\mathfrak{g}^*$ and let $P * Q = (PQ)_{\mathfrak{a}}$ which is the projection of PQ in $\mathfrak{g}^*$ into $\mathfrak{a}$ relative to the fixed decomposition $\mathfrak{g}^* = \mathfrak{a} \dot{+} \mathfrak{k}$. We shall see in Remark (1) that this yields a reductive Lie admissible algebra but we first consider the following special case.

The usual situation for our construction will be when $\mathfrak{g}$ is a semi-simple Lie algebra and $\mathfrak{h}$ is a semi-simple subalgebra. Then we can write $\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}$, where $\mathfrak{m} = \mathfrak{h}^{\perp}$ which is the orthogonal complement of $\mathfrak{h}$ relative to the Killing form, K, of $\mathfrak{g}$ and note that $[\mathfrak{m}\mathfrak{h}] \subset \mathfrak{m}$. Thus $\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}$ is a reductive Lie algebra and in particular if $\mathfrak{g}$ is simple, then $\mathfrak{m}$ with the multiplication $[X, Y]_{\mathfrak{m}}$ is the zero algebra (i.e. $[\mathfrak{m}, \mathfrak{m}]_{\mathfrak{m}} \equiv 0$) or $\mathfrak{m}$ is a simple algebra [**9**].

Now assume that $\mathfrak{g}$ is simple as above; then it is completely reducible so that the associative algebra $\mathfrak{g}^*$ is semi-simple with identity 1. Thus the form $\tau(U, V) = \text{trace } UV$ is a non-degenerate invariant (or associative) form on $\mathfrak{g}^*$ [**3**, p. 69]. But since $\mathfrak{g}$ is a simple Lie algebra of endomorphisms, $\tau|\mathfrak{g} \times \mathfrak{g}$ is a non-degenerate invariant form on $\mathfrak{g}$. Thus, since the field F is algebraically closed, $\tau(U, V) = \lambda K(U, V)$ for all U, V in $\mathfrak{g}$, where $\lambda \in F$; in particular, $\tau|\mathfrak{m} \times \mathfrak{m}$ is non-degenerate. Let $\mathfrak{a} = F1 \dot{+} \mathfrak{m}$ be the subspace of $\mathfrak{g}^*$ spanned by 1 and $\mathfrak{m}$ (note that $1 \notin \mathfrak{g}$ since $\mathfrak{g}$ is simple so that $1 \notin \mathfrak{m}$). Then since $\tau(1, 1) \neq 0$ we see that $\tau|\mathfrak{a} \times \mathfrak{a}$ is non-degenerate and we can decompose $\mathfrak{g}^* = \mathfrak{a} \dot{+} \mathfrak{k}$, where $\mathfrak{k} = \mathfrak{a}^{\perp}$ is the orthogonal complement of $\mathfrak{a}$ relative to τ. $\mathfrak{k}$ is usually not a subalgebra of $\mathfrak{g}^*$ but is $D(\mathfrak{h})$-invariant since τ is an invariant form and $D(\mathfrak{h})\mathfrak{a} \subset \mathfrak{a}$. Now, relative to this decomposition $\mathfrak{g}^* = \mathfrak{a} \dot{+} \mathfrak{k}$ we define the multiplication $P * Q$ as before to make $\mathfrak{a}$ into an algebra which we denote, in general, by $(\mathfrak{a}, *)$.

Remark 1. The algebra $(\mathfrak{a}, *)$ is reductive Lie admissible as follows. Let $P = \alpha 1 + X$ and $Q = \beta 1 + Y$ be in $\mathfrak{a} = F1 + \mathfrak{m}$; then $PQ = \alpha\beta 1 + \alpha Y + \beta X + XY$ in $\mathfrak{g}^*$. Thus $P * Q = \alpha\beta 1 + \alpha Y + \beta X + X * Y$ and consequently

$$P * Q - Q * P = X * Y - Y * X = (XY)_{\mathfrak{a}} - (YX)_{\mathfrak{a}}$$

in $\mathfrak{g}^*$. First assume that $1 \in \mathfrak{m}$; then $\mathfrak{m} = \mathfrak{a}$ and from $XY = (XY)_{\mathfrak{a}} + (XY)_{\mathfrak{k}} = (XY)_{\mathfrak{m}} + (XY)_{\mathfrak{k}}$ in $\mathfrak{g}^*$ we see that

$$\begin{aligned}[X, Y]_{\mathfrak{m}} &= (XY - YX)_{\mathfrak{a}} \\ &= [(XY)_{\mathfrak{a}} + (XY)_{\mathfrak{k}} - (YX)_{\mathfrak{a}} - (YX)_{\mathfrak{k}}]_{\mathfrak{a}} \\ &= (XY)_{\mathfrak{a}} - (YX)_{\mathfrak{a}} \\ &= P * Q - Q * P.\end{aligned}$$

Thus in the algebra $\mathfrak{a}^-$ the commutator is the product in the anti-commutative algebra $\mathfrak{m}$. Consequently, $H = D(\mathfrak{h})$ is contained in the derivation algebra of $\mathfrak{a}^-$ and $\mathfrak{a}^- \oplus H = \tilde{\mathfrak{g}}$ with the obvious operations becomes a reductive Lie algebra which is a homomorphic image of $\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$.

Next assume that $1 \notin \mathfrak{m}$; then for $P = \alpha 1 + X, Q = \beta 1 + Y \in \mathfrak{a} = F1 \dotplus \mathfrak{m}$ we have in $\mathfrak{g}^* = \mathfrak{a} \dotplus \mathfrak{k}$,

$$XY = (XY)_{\mathfrak{a}} + (XY)_{\mathfrak{k}} = (XY)_{\mathfrak{m}} + \lambda(XY)1 + (XY)_{\mathfrak{k}},$$

where $(XY)_{\mathfrak{m}} \in \mathfrak{m}$ and $\lambda(XY) \in F$. Thus

$$\begin{aligned}[X, Y]_{\mathfrak{m}} &= (XY - YX)_{\mathfrak{m}} \\ &= (XY)_{\mathfrak{m}} - (YX)_{\mathfrak{m}} \\ &= (XY)_{\mathfrak{a}} - (YX)_{\mathfrak{a}} - [\lambda(XY) - \lambda(YX)]1 \\ &= P * Q - Q * P + \lambda 1\end{aligned}$$

with $\lambda \in F$. Now extend the derivations $D(h)$ of $\mathfrak{m}$ to $\mathfrak{a}^-$ by setting $D(h)1 = [1, h] = 0$; then $D(\mathfrak{h})$ induces a derivation algebra H on $\mathfrak{a}^0 = \mathfrak{a}^-/F1$. In this case $\mathfrak{g}^0 = \mathfrak{a}^0 \oplus H^0$ becomes a reductive Lie algebra.

Example 1. Let $\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$ be a reductive Lie algebra where $\mathfrak{g}$ is of type G_2, $\mathfrak{h}$ of type A_2, and let $\mathfrak{g}$ be represented by derivations of the 7-dimensional simple split Malcev algebra [**6**, p. 455]. Then using the notation of [**6**], the elements of $\mathfrak{g}$ have the matrix representation

$$D = \begin{bmatrix} 0 & 2d_2 & 2d_3 & 2d_4 & -2d_5 & -2d_6 & -2d_7 \\ d_5 & d_8 & d_9 & d_{10} & 0 & d_4 & -d_3 \\ d_6 & d_{11} & d_{12} & d_{13} & -d_4 & 0 & d_2 \\ d_7 & d_{14} & d_{15} & -d_8 - d_{12} & d_3 & -d_2 & 0 \\ -d_2 & 0 & -d_7 & d_6 & -d_8 & -d_{11} & -d_{14} \\ -d_3 & d_7 & 0 & d_5 & -d_9 & -d_{12} & -d_{15} \\ -d_4 & -d_6 & d_5 & 0 & d_{10} & -d_{13} & d_8 + d_{12} \end{bmatrix}.$$

Now if D_i denotes the matrix with $d_i = 1$ and $d_j = 0$ for $i \neq j$, then for the reductive decomposition of $\mathfrak{g}$ we let $\mathfrak{m}$ have basis $\{D_i\colon 2 \leqq i \leqq 7\}$ and $\mathfrak{h}$ have basis $\{D_i\colon 8 \leqq i \leqq 15\}$. From this we easily see that $\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$ is actually

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a reductive decomposition and $\mathfrak{h}$ is of type A_2. For $\mathfrak{m}$ with multiplication $X \circ Y = [X, Y]_{\mathfrak{m}}$ as given in the introduction, we have the following multiplication table.

	D_2	D_3	D_4	D_5	D_6	D_7
D_2	0	$2D_7$	$-2D_6$	0	0	0
D_3	$-2D_7$	0	$2D_5$	0	0	0
D_4	$2D_6$	$-2D_5$	0	0	0	0
D_5	0	0	0	0	$-2D_4$	$2D_3$
D_6	0	0	0	$2D_4$	0	$-2D_2$
D_7	0	0	0	$-2D_3$	$2D_2$	0

It is easy to check that $\mathfrak{m}$ is a simple algebra (also see [**9**]). Now to compute $\mathfrak{a}$ we let $\mathfrak{k} = \mathfrak{a}^{\perp}$ as previously explained and obtain the following multiplication table for $\mathfrak{a} = IF \dot{+} \mathfrak{m}$.

	I	D_2	D_3	D_4	D_5	D_6	D_7
I	I	D_2	D_3	D_4	D_5	D_6	D_7
D_2	D_2	0	D_7	$-D_6$	0	0	0
D_3	D_3	$-D_7$	0	D_5	0	0	0
D_4	D_4	D_6	$-D_5$	0	0	0	0
D_5	D_5	0	0	0	0	$-D_4$	D_3
D_6	D_6	0	0	0	D_4	0	$-D_2$
D_7	D_7	0	0	0	$-D_3$	D_2	0

Notice that $2[D_i, D_j]_{\mathfrak{m}} = D_i * D_j - D_j * D_i$ so that $\mathfrak{m} \cong \mathfrak{a}^0 = \mathfrak{a}^-/IF$ and also notice that the subspace $\mathfrak{m} \subset \mathfrak{a}$ is an ideal of $\mathfrak{a}$. This leads to the following theorem; cf. [**2**; **5**].

THEOREM. *Let* $\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}$ *be a reductive Lie algebra of endomorphisms and let* $\mathfrak{g}^*$ *be its enveloping algebra which we assume contains an identity element* 1. *Let* $\mathfrak{a} = F1 + \mathfrak{m}$ *be the algebra with multiplication* $P * Q$ *as defined in Remark* 1 *relative to a fixed decomposition* $\mathfrak{g}^* = \mathfrak{a} \dot{+} \mathfrak{k}$. *Then*

(1) *If* $1 \in \mathfrak{m}$, *then* $\mathfrak{a}^-$ *is isomorphic to* $\mathfrak{m}$ *as algebras;*

(2) *If* $1 \notin \mathfrak{m}$, *then* $\mathfrak{a}^0$ *is isomorphic to* $\mathfrak{m}$ *as algebras;*

(3) *If* $\mathfrak{m}$ *is a simple anti-commutative algebra and* $\mathfrak{b}$ *is a proper ideal of* $\mathfrak{a}$, *then* $\mathfrak{b}^0$ *is isomorphic to* $\mathfrak{m}$ *as algebras and* $\mathfrak{b}$ *is the only proper ideal of* $\mathfrak{a}$. *That is, if* $\mathfrak{a} = F1 \dot{+} \mathfrak{m}$ *is not simple, then it can have only one ideal.*

Proof. Parts (1) and (2) follow from Remark 1. Next suppose that $\mathfrak{m}$ with multiplication $[X, Y]_{\mathfrak{m}}$ is a simple anti-commutative algebra and suppose that $\mathfrak{b}$ is a proper ideal in the algebra $\mathfrak{a}$. Since $\mathfrak{m}$ is simple, $1 \notin \mathfrak{m}$ because $[1, \mathfrak{m}]_{\mathfrak{m}} = 0$ implies that $F1$ is an ideal of $\mathfrak{m}$; therefore $\mathfrak{m} \cong \mathfrak{a}^0$ as algebras. The ideal $\mathfrak{b}$ of $\mathfrak{a}$ yields an ideal $\mathfrak{b}^0 = \mathfrak{b} + F1$ of $\mathfrak{a}^0$ and since $\mathfrak{a}^0 \cong \mathfrak{m}$ is simple, $\mathfrak{b}^0 = \mathfrak{a}^0$ or $\mathfrak{b}^0 = 0$. If $\mathfrak{b}^0 = 0$, then $\mathfrak{b} = 1F$ which is not an ideal of $\mathfrak{a}$; thus $\mathfrak{b}^0 = \mathfrak{a}^0 \cong \mathfrak{m}$.

Next note that since $\mathfrak{b}$ is a proper ideal of $\mathfrak{a}$, $1 \notin \mathfrak{b}$ so that $\dim \mathfrak{b}^0 = \dim \mathfrak{b}$; thus we have $\dim \mathfrak{m} = \dim \mathfrak{a}^0 = \dim \mathfrak{b}^0 = \dim \mathfrak{b}$. Thus let $\mathfrak{b}_1$ be any other proper ideal of $\mathfrak{a}$ and consider the ideal $\mathfrak{b} \cap \mathfrak{b}_1$. If $\mathfrak{b} \cap \mathfrak{b}_1 \neq 0$, then from the above dimension results applied to the proper ideals $\mathfrak{b}$, $\mathfrak{b}_1$ and $\mathfrak{b} \cap \mathfrak{b}_1$ we have $\dim \mathfrak{b} = \dim \mathfrak{b}_1 = \dim(\mathfrak{b} \cap \mathfrak{b}_1)$ since they all equal $\dim \mathfrak{m}$. Thus $\mathfrak{b} \cap \mathfrak{b}_1 \subset \mathfrak{b}$ implies that $\mathfrak{b} = \mathfrak{b} \cap \mathfrak{b}_1$ and similarly $\mathfrak{b}_1 = \mathfrak{b} \cap \mathfrak{b}_1$ so that $\mathfrak{b} = \mathfrak{b}_1$. Next, if $\mathfrak{b} \cap \mathfrak{b}_1 = 0$, then we have $\dim \mathfrak{b} = \dim \mathfrak{b}_1 = \dim \mathfrak{a} - 1$. Thus since $\dim \mathfrak{b}$ or $\dim \mathfrak{b}_1$ is at least 1, we have $\mathfrak{a} = \mathfrak{b} + \mathfrak{b}_1$ and actually $\mathfrak{a} = \mathfrak{b} \dotplus \mathfrak{b}_1$ since $\mathfrak{b}_1 \cap \mathfrak{b} = 0$. Therefore

$$\dim \mathfrak{m} + 1 = \dim \mathfrak{a} = \dim(\mathfrak{b} \dotplus \mathfrak{b}_1) = \dim \mathfrak{b} + \dim \mathfrak{b}_1 = 2 \dim \mathfrak{m}.$$

Thus $\dim \mathfrak{m} = 1$, a contradiction to the simplicity of $\mathfrak{m}$. These results show that $\mathfrak{b}$ is the only proper ideal in $\mathfrak{a}$.

Remark 2. (i) Part (3) of the Theorem is illustrated by the preceding example; that is, $\mathfrak{a} = F1 \dotplus \mathfrak{m}$ can have an ideal even though $\mathfrak{m}$ is simple. However, the simple 8-dimensional split Cayley-Dickson algebra $\mathfrak{a}$ is of the form $\mathfrak{a} = F1 \dotplus \mathfrak{m}$ and $\mathfrak{a}^0 = \mathfrak{a}^-/F1 \cong \mathfrak{m}$ is a simple 7-dimensional Malcev algebra; that is, $\mathfrak{a} = F1 \dotplus \mathfrak{m}$ is simple where $\mathfrak{m}$ is simple.

(ii) As noted in the beginning of this section, the hypothesis that $\mathfrak{m}$ be a simple algebra is satisfied in the case that $\mathfrak{g}$ is simple, $\mathfrak{h}$ is semi-simple, and $[\mathfrak{m}, \mathfrak{m}]_\mathfrak{m} \neq 0$. Thus many examples can easily be formed.

Example 2. We can use this construction to determine the split Cayley-Dickson algebra from the corresponding Malcev algebra and associative algebra $\mathfrak{g}^*$. Thus, let A be the split simple 7-dimensional Malcev algebra as given in [**6**, p. 434]. In [**10**] it was shown that there exists a reductive Lie algebra $\mathfrak{g} = \mathfrak{m} \dotplus \mathfrak{h}$, where $\mathfrak{g}$ is of type B_3 and $\mathfrak{h}$ is of type G_2 so that the Malcev algebra A is given by the subspace $\mathfrak{m}$ with multiplication $[X, Y]_\mathfrak{m}$ and we identify A with $\mathfrak{m}$. Briefly, the construction is that for $X, Y \in A$ and XY the product in A we have, from the identities in [**6**],

$$(*) \qquad [L(X), L(Y)] = -L(XY) + D(X, Y),$$

where $D(X, Y)$ is an inner derivation of A (and all derivations of A are sums of inner derivations). Next, since no derivations of A are of the form $L(Z)$ with $Z \neq 0$ [**6**], we have the direct sum

$$\mathfrak{g} = L(A) \dotplus D(A),$$

where $D(A)$ is the derivation algebra of A. Using equation $(*)$ above, we see that $\mathfrak{g}$ is a reductive Lie algebra of endomorphisms and with $\mathfrak{m} = L(A)$, $\mathfrak{h} = D(A)$ we see that the map $\phi: A \to \mathfrak{m}: X \to -L(X)$ is an algebra isomorphism since

$$\phi(XY) = -L(XY) = [L(X), L(Y)]_\mathfrak{m} = [\phi(X), \phi(Y)]_\mathfrak{m}.$$

Next, by choosing a suitable basis in A, the system of roots were computed in [**11**] to obtain $\mathfrak{g}$ and $\mathfrak{h}$ of type B_3 and G_2, respectively. Note that $\mathfrak{g}$ is the Lie algebra generated by $L(A)$.

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Now with $\mathfrak{g} = \mathfrak{m} \dot{+} \mathfrak{h}$ as above, note that $\mathfrak{g}^* = \operatorname{Hom}(A, A)$ $(= 7 \times 7$ matrix algebra) and for $\mathfrak{a} = F1 \dot{+} \mathfrak{m}$ let $\mathfrak{g}^* = \mathfrak{a} \dot{+} \mathfrak{k}$, where $\mathfrak{k}$ is the orthogonal complement as previously discussed. Thus with the multiplication on $\mathfrak{a}$ defined by $P * Q = (PQ)_{\mathfrak{a}}$ as in Remark 1, we see that $\mathfrak{a}$ is a reductive Lie admissible algebra with $\mathfrak{a}^0$ isomorphic to $\mathfrak{m}$ and therefore isomorphic to the Malcev algebra A. Also, $\mathfrak{a}$ is isomorphic to the split Cayley-Dickson algebra $\mathfrak{A}$ as follows. Choose the basis $\{e_i\}$ of A as given in [**6**] and let $E_i = \frac{1}{2}L(e_i)$; then a straightforward computation using the decomposition $\mathfrak{g}^* = \mathfrak{a} \dot{+} \mathfrak{k}$ yields the following multiplication table which shows that $\mathfrak{a}$ is isomorphic to the split 8-dimensional Cayley-Dickson algebra as indicated in [**6**, p. 434].

	I	E_1	E_2	E_3	E_4	E_5	E_6	E_7
I	I	E_1	E_2	E_3	E_4	E_5	E_6	E_7
E_1	E_1	0	E_2	E_3	E_4	$-E_5$	$-E_6$	$-E_7$
E_2	E_2	$-E_2$	0	E_7	$-E_6$	u	0	0
E_3	E_3	$-E_3$	$-E_7$	0	E_5	0	u	0
E_4	E_4	$-E_4$	E_6	$-E_5$	0	0	0	u
E_5	E_5	E_5	$-u$	0	0	0	$-E_4$	E_3
E_6	E_6	E_6	0	$-u$	0	E_4	0	$-E_2$
E_7	E_7	E_7	0	0	$-u$	$-E_3$	E_2	0

where $u = \frac{1}{2}(I - E_1)$. Thus we may recover the split Cayley-Dickson algebra $\mathfrak{A}$ from the corresponding Malcev algebra A.

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University of Minnesota,
Minneapolis, Minnesota