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EULERIAN GRAPHS AND POLYNOMIAL IDENTITIES FOR SKEW-SYMMETRIC MATRICES

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1. Introduction. Let the standard identity of degree m be given by

$$(1) \quad [x_1, x_2, \ldots, x_m] \equiv \sum_{\sigma \in S_m} \operatorname{sgn} \sigma \, x_{\sigma 1} \ldots x_{\sigma m} = 0.$$

Then we shall show that the set of all $n \times n$ skew-symmetric matrices over a field of characteristic 0 satisfies the standard identity of degree at least $2n - 2$; specifically, we shall prove the following.

THEOREM 1. *If $A_1, \ldots, A_m$ are $n \times n$ skew-symmetric matrices over a field of characteristic* 0 *and if $m \geqq 2n - 2$, then $[A_1, \ldots, A_m] = 0$.*

In fact, the standard identity of degree $2n - 2$ is the minimum standard identity which the set of skew-symmetric matrices satisfies, for we shall also prove the following.

THEOREM 2. *If $m < 2n - 2$, then there are m $n \times n$ skew-symmetric matrices, $A_1, \ldots, A_m$, over a field of characteristic* 0, *such that $[A_1, \ldots, A_m] \neq 0$.*

Amitsur and Levitzki proved by algebraic means that the polynomial identity of minimal degree, satisfied by all $n \times n$ matrices over a field of characteristic 0, was the standard identity of degree $2n$ [**1**]. Bertram Kostant also considered the same question and using cohomology theory, showed that the Amitsur-Levitzki result was equivalent to two other established results. Furthermore, he obtained the result that the Lie algebra of all $n \times n$ complex valued, skew-symmetric matrices satisfies the standard identity of degree m if n is even and if $m \geqq 2n - 2$ [**3**]. Richard Swan gave another proof of the Amitsur-Levitzki result by recognizing the essentially combinatorial nature of their proof and by translating the problem into an equivalent one in graph theory [**7**; **8**].

K. C. Smith and H. J. Kumin have conjectured that Kostant's result for skew-symmetric matrices is true for n both even and odd and is the best possible result [**4**]. The purpose of this paper is to demonstrate the validity of their conjectures (Theorems 1 and 2). Like Swan, we shall translate the problem into one concerning Eulerian graphs and show that the appropriate

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result holds for these graphs. An introduction to and an outline of the proof of Theorems 1 and 2 will appear in [**2**], and some details found in the latter work will be omitted here.

Independent proofs of Smith and Kumin's conjecture have also been announced in the last months by Louis H. Rowen [**6**] and by Frank Owens [**5**]. Copies of their work have been received by the author.

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2. The graph-theoretic problem. To establish Theorem 1, it is sufficient to prove the statement for a basis of the skew-symmetric matrices since the bracket function as defined in line 1 is multilinear. We choose as a basis for the skew-symmetric matrices those matrices which are all 0's except for two entries a_{ij} and a_{ji} with values ± 1. Then given a set of m basis matrices, we construct a directed graph with n vertices and m edges which has an edge directed from vertex i to vertex j for every matrix in our given set with $a_{ij} = +1$. We shall study properties of graphs arising in this way. Similarly, to establish Theorem 2, we shall exhibit appropriate directed graphs on n vertices and m edges, and from these construct m $n \times n$ skew-symmetric matrices which do not satisfy the standard identity of degree m.

In the course of this work, we shall also have occasion to consider partially directed graphs, and thus we make the following definitions in this more general context.

Definition 1. An *E path* on a directed or partially directed graph G is a path on G which traverses every edge of G once and only once and which may travel in either direction on an edge e of G, regardless of the orientation of e.

If the path forms a circuit, we may call it an E circuit. Thus an E path is an Eulerian path on the underlying, undirected graph which we think of as living on G. In fact, we shall find that it is the study of these Eulerian paths on Eulerian graphs which is at the center of the problem.

Definition 2. The *orientation coefficient* of an E path on a directed or partially directed graph is $(-1)^z$ where z is the number of edges traversed in the direction opposite to their orientation.

We shall use $\mathrm{OC}(P)$ to denote the orientation coefficient of an E path, P.

Definition 3. The *sign of an E path* on a directed or partially directed graph with labelled edges is the number $\operatorname{sgn} \sigma(-1)^z$ where σ is the associated permutation of the edges of the graph and where $(-1)^z$ is the orientation coefficient of the path.

Definition 4. A *positive* (respectively negative) E path is an E path whose sign is $+1$ (respectively -1).

We shall say that a set of paths *cancels* if there are an equal number of positive and negative paths in the set.

Definition 5. Two sets of paths will be said to be *isomorphic* if there is a one-to-one, sign-consistent correspondence between the two sets. A *sign-consistent* correspondence is a correspondence for which corresponding paths always have the same sign or always have the opposite sign. In the former case we will call the correspondence *sign preserving*, in the latter case, *sign reversing*. Often to demonstrate that a set S of paths cancels, we shall show that S is isomorphic to some set S' of paths which cancels, and thus the paths of S will cancel as well.

Definition 6. A vertex V of a directed or a partially directed graph is called *null* if the set of all E paths at V (i.e., which begin at V) cancels. A graph is *null* if all its vertices are null.

The transition to graph theory from the original matrix problem can now be made by observing that if $A_1, \ldots, A_m$ are $n \times n$ skew-symmetric basis matrices, and if G is the associated directed graph with n vertices and m edges, then the $n \times n$ matrix $[A_1, \ldots, A_m]$ has as its (i, j)th entry the number of positive E paths, going from vertex i to vertex j on G, minus the number of negative E paths from i to j. Thus Theorem 1 is equivalent to the following.

THEOREM 1′. *If G is a directed graph with n vertices and m edges and if $m \geqq 2n - 2$, then G is null.*

Similarly Theorem 2 becomes the following.

THEOREM 2′. *Given $n \geqq 2$ and $m < 2n - 2$, there is a directed graph with n vertices and m edges which is not null.*

In fact, we can reduce the work in proving Theorems 1′ and 2′ by making the following remarks which are either self-evident or their proof can be found in [**2**]. Let G be a directed or partially directed graph with n vertices and m edges.

1. If the underlying, undirected graph of G is not Eulerian (i.e. does not have 0 or 2 vertices of odd degree), then G is null.
2. If G is null with one labelling of its m edges, then it is null with any labelling of its edges.
3. If G is null with one orientation of its edges, then it is null with any orientation of its edges (provided directed edges remain directed and undirected edges remain undirected).
4. If G contains two vertices which are joined either by two or more directed edges or two or more undirected edges, then G is null.
5. If $m \geqq 2n$, then G is null. (This result follows from Swan's main theorem in [**7**].)
6. If G contains two vertices of odd degree, then G is null if and only if one of these vertices is null.

Definition 7. $C(n, m, p)$ is defined to be the set of all directed and partially directed graphs G on n vertices and m edges, of which precisely p edges are undirected, such that

(1) G is connected and contains precisely 0 or 2 vertices of odd degree;
(2) G may contain multiple edges;
(3) G may contain undirected loops, but no directed loops.

In [**2**] it is shown that all graphs, not in $C(n, m, p)$ for some value of n, m, and p, are either automatically null or do not arise when translating the algebra problem to graph theory. Furthermore, because of Remark 5 it follows that to prove Theorem 1′, it is sufficient to prove that if $G \in C(n, m, 0)$ where $m = 2n - 2$ or $2n - 1$, then G is null. We shall prove two Theorems (3 and 4) which will then give us Theorem 1′.

3. The main theorems. We begin with a few lemmas. Here and in the proof of Theorem 3 we shall defer the proofs of the more technical lemmas to the final section of the paper.

Suppose $G \in C(n, m, p)$ has all vertices of even degree. We define an equivalence relation on the E circuits at a fixed vertex V of G as follows. Let C and C' be two different circuits at V. Let C have the associated permutation $(x_1, x_2, \ldots, x_m)$ and C' $(y_1, y_2, \ldots, y_m)$. Then C and C' will be in the same equivalence class if and only if

$$(y_1, y_2, \ldots, y_m) = (x_i, \ldots, x_m, x_1, \ldots, x_{i-1})$$

for some $i \in \{2, 3, \ldots, m\}$. We shall refer to these equivalence classes as the rotation classes of E circuits at a vertex V.

LEMMA 1. *If $G \in C(n, m, p)$ where m is odd and where G has all vertices of even degree, then a vertex Y of G is null if and only if the rotation classes of E circuits at Y have a set of representatives, one from each rotation class, which cancels.*

Proof. If the valence of Y is $2p$, then each equivalence class of E circuits at Y has p elements, all of the same sign, and the lemma follows.

LEMMA 2. *If $G \in C(n, m, p)$ has all vertices of even degree and if m is an odd integer, then G is null if and only if G has at least one null vertex.*

Proof. If G is null, by definition all vertices are null. Thus we suppose G has one null vertex Y, and let Z be another vertex of G.

Notice that we may pair the E circuits at Z by pairing two circuits where one is obtained by reversing the other. If $m \equiv 0$ or $3 \pmod 4$ and p is odd or if $m \equiv 1$ or $2 \pmod 4$ and p is even, such paired circuits always have opposite signs. Since all circuits at Z can be associated in such cancelling pairs, Z is a null vertex as we wished to show. Thus we shall assume that either $m \equiv 0$ or $3 \pmod 4$ and p is even or $m \equiv 1$ or $2 \pmod 4$ and p is odd. For these values of m and p, two circuits, paired by reversing, will have the same signs.

By Lemma 1 we know that the set R of representatives of the circuits at the null vertex Y cancel where R is the set of those circuits which either begin or end on a fixed edge e at Y. $R = R_1 \cup R_2$ where R_1 (respectively R_2) is the set of those circuits of R which begin on (respectively end on) e. From the discussion of the last paragraph we know that there is a sign preserving isomorphism between R_1 and R_2 and therefore the circuits of R_1 cancel.

Let S_1 (respectively S_2) be the set of E circuits at Z with respect to which the initial vertex of e is (respectively is not) Y. Again there is a one-to-one sign preserving correspondence between S_1 and S_2 so that Z is null if and only if S_1 cancels. By Lemma 1 the paths of S_1 cancel if and only if a set of representatives cancels, and we choose this set, T, to be all those circuits of S_1 which traverse edge e before returning to Z for the first time. But now we see that the sets R_1 and T are isomorphic since an element of R_1 can be cyclically rotated to a unique E circuit in T and conversely. The signs of the corresponding elements are the same so that T cancels as do then S_1 and S_2. The vertex Z is null then and since Z was arbitrary, G is null.

One other important lemma requires a bit of notation which we now consider.

Definition 8. Given $G \in C(n, m, p)$, Y a vertex of G, and (e_i, e_j, e_k), an ordered triple of edges of G with e_i adjacent to e_j and e_j adjacent to e_k, $S(e_i, e_j, e_k, Y)$, $i < k$, is defined to be the set of all E paths on G which begin at Y and which are of the form

$$p_1e_ie_je_kp_2 \quad \text{or} \quad p_3e_ke_je_ip_4$$

where p_i, $i = 1, \ldots, 4$, are paths of G (maybe of length 0).

Suppose e_i joins vertices S_i and S_j, edge e_j joins S_j and S_k, and e_k joins S_k and S_l. Let $G(e_i, e_j, e_k) = G - \{e_i, e_j, e_k\} + \{f\}$ where f joins S_i and S_l and is directed (respectively undirected) if an even (respectively odd) number of the edges $\{e_i, e_j, e_k\}$ are directed.

If $S_i = S_l$ and an odd number of $\{e_i, e_j, e_k\}$ are directed, define $S'(e_i, e_j, e_k, Y)$, $i < k$, to be those paths of $S(e_i, e_j, e_k, Y)$ of the form

$$p_1e_ie_je_kp_2, \quad i < k.$$

LEMMA 3. *Given $G \in C(n, m, p)$ and a non-empty set of paths $S(e_i, e_j, e_k, Y)$ on G,*

(1) *If $S_i = S_l$ and an even number of $\{e_i, e_j, e_k\}$ are directed, the paths of $S(e_i, e_j, e_k, Y)$ cancel.*

(2) *Otherwise if $S_i = S_l$, the paths of $S(e_i, e_j, e_k, Y)$ cancel if and only if the paths of $S'(e_i, e_j, e_k, Y)$ cancel. Furthermore, there is an isomorphism between $S'(e_i, e_j, e_k, Y)$ and the set of all E paths which begin at vertex Y on $G(e_i, e_j, e_k)$.*

(3) *If $S_i \neq S_l$, there is an isomorphism between $S(e_i, e_j, e_k, Y)$ and the set of all E paths which begin at vertex Y on $G(e_i, e_j, e_k)$.*

The proof of this lemma is included at the end of the paper.

THEOREM 3. *If $G \in C(n, 2n - 1, i)$, $i = 0, 1$, then G is null.*

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Proof. Suppose the theorem is not true. Let

$$S(m) = \{G \in C(m, 2m - 1, i), i = 0, 1 | G \text{ is not null}\}$$

and let $n = \min \{m | S(m) \neq \emptyset\}$. We shall call an element of $S(n)$ a minimal graph.

Observe that $n > 2$. Every element of $C(2, 3, i)$, $i = 0, 1$, has multiple directed edges and hence is null by Remark 4.

We shall study the properties of minimal graphs, that is, of elements of $S(n)$, and we will show that, in fact, $S(n)$ is empty. To clarify the proof, we give now a general outline. A graph $G \in S(n)$ has either all vertices of even degree or precisely two vertices of odd degree. In the former case we shall see such a graph is not in $S(n)$ by using the minimality of n. In the latter case, suppose the two vertices of odd degree of G are V and W. G must fall into precisely one of the following four categories. (We denote the valence of a vertex Z by $\rho(Z)$.)

(2) I $\min \{\rho(V), \rho(W)\} = 1$

II $\min \{\rho(V), \rho(W)\} = 3$

III $\min \{\rho(V), \rho(W)\} \geqq 5$ and if n is odd, $G \in C(n, 2n - 1, 1)$ or if n is even $G \in C(n, 2n - 1, 0)$

IV $\min \{\rho(V), \rho(W)\} \geqq 5$ and if n is even, $G \in C(n, 2n - 1, 1)$ or if n is odd, $G \in C(n, 2n - 1, 0)$.

We shall check each case and find that in each category, there can be no minimal graph. Therefore $S(n) = \emptyset$, a contradiction, and thus the theorem will be proved.

We begin by supposing G is a minimal graph with all vertices of even degree. We observe that G has a vertex of degree 2 since there are $2n - 1$ edges in G. Suppose V, a vertex of degree 2, has incident edges e and e' and is adjacent to vertices A and B (maybe $A = B$). We consider $G' = G - \{V, e, e'\}$, a graph also with either 0 or 2 vertices of odd degree. Thus $G' \in C(n - 1, 2n - 3, i)$, $i = 0, 1$, and therefore is null by the minimality of n. There is an isomorphism between the set of all E circuits on G which begin at V and the set containing the union of the E paths on G' beginning at A and those beginning at B. (3) Since A and B are null vertices of G', V is a null vertex of G. By Lemma 2, G is null, and we see that no minimal graph can have all vertices of even degree.

Therefore, all minimal graphs must have two vertices of odd degree. If $G \in S(n)$, then G belongs to precisely one of the 4 cases mentioned earlier (see line 2).

Case I. No minimal G can have a vertex V of degree 1, for suppose G were such a graph and suppose e is the edge joining V to a vertex Y. We consider $G' = G - \{V, e\}$ and since $G' \in C(n - 1, 2n - 2, i)$, $i = 0, 1$, G' is null by Remark 5. Since there is an isomorphism between the set of all E paths on G, beginning at V, with the set of all E paths on G', beginning at Y, V is a null vertex of G, and G is null by Remark 6.

Case II. We shall show next that no minimal G can have a vertex V of degree 3. Let $G \in C(n, 2n-1, i)$, $i = 0, 1$, let G have a vertex V of degree 3 and let W be the other vertex of odd degree. Then there are either 2 or 3 distinct edges incident at V. If $G \in C(n, 2n-1, 1)$, there may be a loop l at V as well as an edge e, joining V to Y. In this case we consider $G' = G - \{V, e, l\}$, and since $G' \in C(n-1, 2n-3, 0)$, G' is null by the minimality of n. There is an isomorphism between the E paths on G, beginning at V, with the E paths on G', beginning at Y, whence V is a null vertex and G is a null graph by Remark 6.

Otherwise there must be three distinct edges, e_i, $i = 1, 2, 3$, incident at V and let e_i join vertex S_i to V (the S_i need not all be distinct). Let $G' = G + \{l\}$, G with an undirected loop added at V. $G' \in C(n, 2n, i)$, $i = 1, 2$, and by Remark 5 is null. The set of all E paths from V to W on G' is the disjoint union of two sets P_1 and P_2 where P_1 is the set of all paths from V to W on G' of the form $le_ip_1e_je_kp_2$ and where P_2 is the set of all paths of the form $e_ip_3e_jle_kp_4$ where (i, j, k) is a permutation of $\{1, 2, 3\}$ and where p_i, $i = 1, \ldots, 4$, are paths on G (maybe of length 0).

We first will see that the paths of P_2 cancel with the aid of Lemma 3. $P_2 = S(e_1, l, e_2, V) \cup S(e_1, l, e_3, V) \cup S(e_2, l, e_3, V)$ and we shall see that each of these three sets cancels. To simplify notation we shall prove that the paths of $S(e_1, l, e_2, V)$ cancel.

By Lemma 3 we know that if $S_1 = S_2$ and both e_1 and e_2 are directed, then the paths of $S(e_1, l, e_2, V)$ cancel. If $S_1 = S_2$ and only one of the edges e_1 or e_2 is directed or if $S_1 \neq S_2$, we consider $S'(e_1, l, e_2, V)$ or $S(e_1, l, e_2, V)$ respectively. By Lemma 3 each is isomorphic to the set of all E paths from V to W on $G(e_1, l, e_2)$. $G(e_1, l, e_2) \in C(n, 2n-2, i)$, $i = 0, 1$, and contains a vertex of degree 1, namely V. Let $\tilde{G} = G(e_1, l, e_2) - \{V, e_3\}$. Then $\tilde{G} \in C(n-1, 2n-3, i)$, $i = 0, 1$, which is null by our assumption of the minimality of n. Then $G(e_1, l, e_2)$ is also null, implying that the paths of $S'(e_1, l, e_2, V)$ and $S(e_1, l, e_2, V)$ cancel.

The proof that the sets $S(e_1, l, e_3, V)$ and $S(e_2, l, e_3, V)$ cancel is identical, and therefore we see that the paths of P_2 cancel.

Since G' is null, the set of all E paths from V to W on G' cancels, implying that the paths of P_1 must cancel also. But there is an isomorphism between the paths of P_1 and the set of all E paths from V to W on G whence V is a null vertex of G. By Remark 6, G is null and is not a minimal graph.

We now know that a minimal graph G has two vertices of odd degree $\geqq 5$.

Case III. Let G be a minimal graph. Suppose n is odd and $G \in C(n, 2n-1, 1)$ or n is even and $G \in C(n, 2n-1, 0)$. We shall then see that G is null, contradicting the fact that G is minimal.

Let V and W be the two vertices of odd degree and let $\rho(V) = k$. Let $G' = G + \{e, l\}$ where e is an edge directed from W to V and where l is an undirected loop at V. (See Figure 1.) Then $G' \in C(n, 2n+1, 1)$ if n is even and $G' \in (n, 2n+1, 2)$ if n is odd. In either case G' is null by Remark 5. In particular, W is a null vertex of G'.

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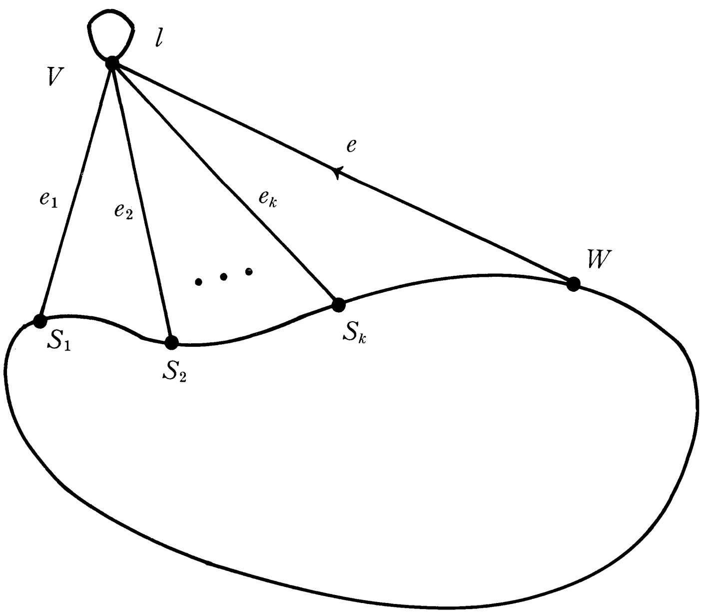

FIGURE 1

The set of E circuits at W is the disjoint union of two sets C_1 and C_2 where C_1 is the set of circuits in which l either immediately precedes or immediately succeeds edge e, and where C_2 is the set of circuits in which l and e are not traversed consecutively.

We shall first show that the circuits of C_2 cancel. Now the circuits of C_2 are all of the form $p_1ep_2e_ile_jp_3$ or $p_4e_ile_jp_5ep_6$ where p_i, $i = 1, \ldots, 6$, are paths on G', and where e_i and e_j are two distinct edges, other than e, incident with vertex V.

Thus

$$C_2 = \bigcup_{\substack{i,j \in \{1,\ldots,k\} \\ i<j}} S(e_i, l, e_j, W).$$

We apply Lemma 3 to each set $S(e_i, l, e_j, W)$ and see that either the set cancels immediately or the set is isomorphic to the set of all E paths on $G(e_i, l, e_j)$ which begin at W. Since $G(e_i, l, e_j) \in C(n, 2n-1, i)$, $i = 0, 1$, and $G(e_i, l, e_j)$ has all vertices of even degree, we know that $G(e_i, l, e_j)$ is not a minimal graph and hence is null. Therefore the paths of each $S(e_i, l, e_j, W)$ cancel and so do those of C_2.

Since the graph G' is null, the paths of C_1 cancel. As in Lemma 1 we divide the elements of C_1 into rotation classes and we have by that lemma that the set R of representatives of the rotation classes cancels, where we pick R to be the set of E circuits at W of the form elp or ple, p an E path between V and W on G.

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There is a natural pairing of the elements of R: those which begin with el being paired with those which end with le, namely by associating with elp, the reversed circuit $p^{-1}le$. A routine check shows that two such associated paths have the same sign.

Since the paths of R cancel, the subset S_e of those circuits which begin with el already cancel. There is an isomorphism between S_e and the set of all E paths from V to W on the original G whence V is a null vertex of G. Thus G is null and we see that no minimal graph can meet the conditions of case III.

Case IV. If there is a minimal graph G, it must have two vertices of odd degree $\geqq 5$, and if n is even $G \in C(n, 2n - 1, 1)$ or if n is odd, $G \in C(n, 2n - 1, 0)$.

Let the two vertices of G of odd degree be U and Y. Let G' be the graph obtained from G by adding an edge e', directed from Y to A, a new vertex, and an edge e, directed from A to U. $G' \in C(n + 1, 2n + 1, i)$, $i = 0, 1$, and G' has all vertices having even degree. We wish to show that G' is null for the following reason: If G' is null, then A is a null vertex of G'. The E circuits at A in G' may be paired, epe' with $e'p^{-1}e$, one circuit obtained by reversing the other and such paired paths have the same sign.

Thus if A is a null vertex of G', then even the set S_e of E circuits at A which begin with edge e cancel. There is an isomorphism between S_e and the set of E paths from U to Y on the original G whence U is a null vertex, G is null by Remark 6 and therefore is not a minimal graph as claimed. (4)

We shall now see that G' is null, and since G' has an odd number of edges, by Lemma 2 G' is null if it has a null vertex. (The question of whether n is even or odd plays no part here.) Notice that if G' has a vertex Z of degree 2 which is adjacent to only one other vertex, then G' is null, for $\tilde{G} = G - \{Z \,\&\, 2 \text{ edges at } Z\}$ has all vertices of even degree, $\tilde{G} \in C(n, 2n - 1, i)$, $i = 0, 1$. Thus $\tilde{G}$ is null, and Z is a null vertex of G'. By Lemma 2 G' is null and hence we shall assume G' contains no such vertex Z. (5).

We now invoke the following lemma which will be proved in the next section.

LEMMA 4. *If Γ is a graph with p vertices and $2p - 1$ edges which has all vertices of even degree, then either Γ contains two adjacent vertices of degree 2 or a vertex of degree $2d$, $d > 1$, which is adjacent to at least $2d - 3$ vertices of degree 2.*

We may apply this lemma to the graph G'. If G' contains two adjacent vertices of degree 2, label them B and V. If not, consider the set X of vertices of G', where $U \in X$ if and only if U is adjacent to at least $2d - 3$ vertices of degree 2 when $\rho(U) = 2d$. Let $V \in X$ be such that

$$\rho(V) = \min \{\rho(U_i), U_i \in X\}$$

and let B be a vertex of degree 2, adjacent to V. By the assumption of line 5, we have B adjacent to V and W with $V \neq W$.

Let $G'' = G' - \{B \,\&\, 2 \text{ edges at } B\}$. Then $G'' \in C(n, 2n-1, i)$, $i = 0, 1$. We wish to show that the E paths from V to W on G'' cancel, for then G'' is a null graph. As we saw in line 3, B is then a null vertex of G' (6)
and G' is also null by Lemma 2 as desired.

Consider the E paths on G'' from V to W. If the valence of V in G'' is 1 or 3, we know by cases I and II that G'' is not a minimal graph, i.e. G'' is null. Otherwise vertex V has valence $2d - 1$ in G'', $d \geqq 3$, and there are at least $2d - 4$ vertices of degree 2 adjacent to V in G''. (See Figure 2.)

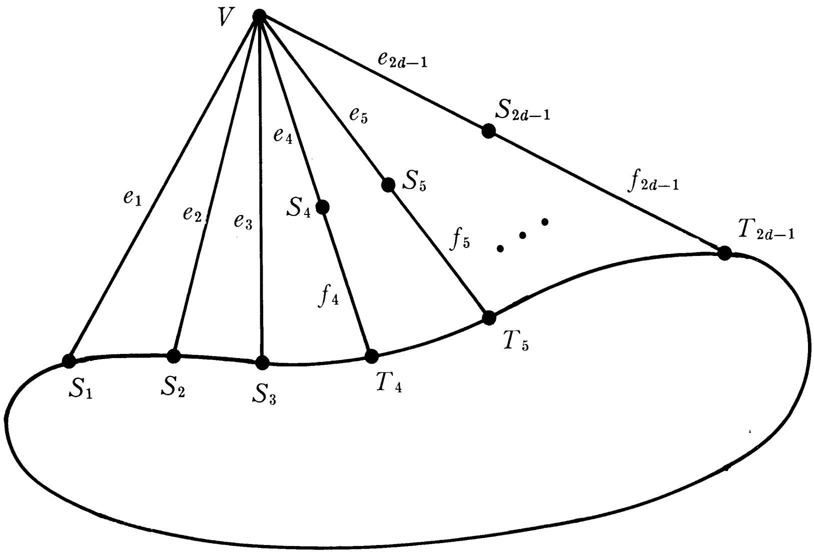

FIGURE 2. Some of the vertices, other than S_i, $i = 4, \ldots, 2d-1$, may coincide, but note that $T_j \neq V$, $j = 4, \ldots, 2d-1$ by the assumption of line 5. Also notice that by our choice of V and B in G'', no S_i and S_j are adjacent, $i, j = 4, \ldots, 2d-1$, and $\rho(T_i) \geqq 5$, $i = 4, \ldots, 2d-1$. In fact, $\rho(T_i) \geqq 6$ unless $T_i = W$.

The E paths from V to W on G'' can be divided into $2d - 1$ disjoint sets Q_i, where Q_i is the set of those E paths which begin with edge e_i. Consider Q_i, $i \geqq 4$. Let R_i be the set of all E paths from T_i to W (maybe $T_i = W$) (7)
on $G_i = G'' - \{S_i, e_i, f_i\}$. Since $G_i \in C(n-1, 2n-3, i)$, $i = 0, 1$, we know that G_i is null. Thus the paths of R_i cancel and since there is an isomorphism between Q_i and R_i, the paths of Q_i cancel, $i = 4, \ldots, 2d-1$. (8)

Let $Q = \bigcup_{i=1}^{3} Q_i$. Every element q of Q_i, $i \leqq 3$, partitions (9)
the set of indices $\{1, 2, \ldots, 2d-1\} - \{i\}$ into unordered pairs where j and k are paired if and only if $q = p_1 e_j e_k p_2$ or $q = p_3 e_k e_j p_4$ where p_i, $i = 1, \ldots, 4$, are subpaths of q. For this reason we define a *Type I partition* to be a partition of $\{1, 2, \ldots, 2d-1\}$ into $d - 1$ pairs of elements and one singleton, j, such that $j \leqq 3$ and the two elements $\{1, 2, 3\} - \{j\}$ are not paired together. A

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Type II partition is a partition of $\{1, 2, \ldots, 2d-1\}$ into $d-1$ pairs and one singleton, j, such that $j \leqq 3$ and the two elements $\{1, 2, 3\} - \{j\}$ are paired together. Thus associated with every $q \in Q$ we have a Type I or II partition and $Q = Q(I) \cup Q(II)$ where $Q(I)$ (respectively $Q(II)$) is the set of those paths of Q whose associated partition of $\{1, 2, \ldots, 2d-1\}$ is of Type I (respectively II).

LEMMA 5. *Let $H \in C(n, 2n-1, i)$, $i = 0, 1$, have two vertices of odd degree, V and W, such that $\rho(V) = 2d - 1 \geqq 5$ and V is adjacent to at least $2d-4$ nonadjacent vertices of degree 2. Suppose these vertices of degree 2 are each adjacent to another vertex besides V. Then the set P of all E paths on H which induce a partition of Type I on $\{1, 2, \ldots, 2d-1\}$ cancels.*

This lemma says precisely that the paths of $Q(I)$ cancel.

We now consider the paths of $Q(II)$. For example, if $q \in Q$ begins on edge e_1, then $q \in Q(II)$ if and only if q traverses the edges e_2 and e_3 consecutively.

LEMMA 6. *Let $H \in C(n, 2n-1, i)$, $i = 0, 1$, satisfy the same hypotheses as in Lemma 5. Then the set P of all E paths on H which induce a partition of Type II on $\{1, 2, \ldots, 2d-1\}$ cancels.*

The proofs of Lemmas 5 and 6 are included in the final section of the paper.

With the aid of these lemmas we see that the set Q cancels (see line 9). Then so do all E paths from V to W on G'' (see lines 7 and 8), and by Remark 6 G'' is null.

To review why we are done now, recall that B is therefore a null vertex of G' (see line 6), and G' is a null graph. Thus A, a vertex of G', is null and (by line 4) we see that our original G is a null graph and therefore not minimal.

Earlier we saw that a minimal graph must fall into one of four cases, but then we saw that there could be no minimal graph in each of the four categories. Hence there are no minimal graphs; $S(n) = \emptyset$, contradicting our original assumption; and Theorem 3 is true.

THEOREM 4. *If $G \in C(n, 2n-2, 0)$, then G is null.*

Proof. Suppose G has all vertices of even degree. We wish to show that an arbitrary vertex V is null. Let $G' = G + \{l\}$, where l is a loop attached to V. Thus $G'' \in C(n, 2n-1, 1)$ and hence is null by Theorem 3. Consider the rotation classes of E circuits at V. By Lemma 1, the set R of representatives of the E circuits at V on G'' cancels where we pick as our set R, the set of circuits at V which begin by traversing the loop l. There is an isomorphism between R and the set of all E circuits at V on G. Thus V is a null vertex of G and G is a null graph.

Suppose G has two vertices, V and W, of odd degree. Let $G' = G + \{e\}$ where e is an edge, directed from W to V. $G' \in C(n, 2n-1, 0)$ and hence is null. Consider the E circuits at W. The set of E circuits at W which either begin or end with edge e are a set R of representatives of the rotation classes and hence cancel by Lemma 1. $R = C_1 \cup C_2$ where C_1 is the set of those circuits which begin with edge e and C_2 is the set of those circuits which end with edge e. Because of Remark 2 we may assume that edge e has the label "$2n - 1$". Then there is a one-to-one, sign preserving correspondence between C_1 and the set S_{VW} of all E paths from V to W on G, and there is a one-to-one, sign reversing correspondence between C_2 and the set S_{WV} of all E paths from W to V on G. If $-S_{WV}$ denotes the set S_{WV} with all elements having sign opposite to their sign as E paths from W to V, we see that $S_{VW} \cup -S_{WV}$ is isomorphic to R and hence cancels.

Now let $G' = G + \{e\}$ where e is an undirected edge joining V to W. $G' \in C(n, 2n - 1, 1)$ and is null by Theorem 3. As above, we consider the E circuits in G' at W and see that there is a one-to-one, sign preserving correspondence between the paths of C_1 (respectively C_2) and the paths of S_{VW} (respectively S_{WV}). Therefore $S_{VW} \cup S_{WV}$ is a set of paths which cancels. Thus both S_{VW} and S_{WV} must cancel and G is null.

Theorems 3 and 4 together with Remark 5 give us Theorem 1′ which we have asserted then gives us Theorem 1 as desired.

We wish to mention one other related result since the same technique as used in the proof of Theorem 4 can give us the following.

THEOREM 5. *If n is odd and $G \in C(n, 2n - 2, 1)$, then G is null.*

This result, together with Theorem 3, tells us that if we are given $m - 1$ $n \times n$ skew-symmetric matrices and one symmetric matrix, if n is odd and $m \geqq 2n - 2$ or if n is even and $m \geqq 2n - 1$, then the bracket of these matrices is 0. We shall see that these results are also best possible for a set of skew-symmetric matrices and one symmetric matrix (see Theorem 8). These results have also been obtained by L. Rowen [**6**].

4. Examples of non-null graphs. We shall now exhibit a number of non-null directed graphs with n vertices and m edges which will demonstrate the validity of Theorem 2′.

THEOREM 6. *Given $n \geqq 2$ and $m = 2n - 3$, there is a directed graph with n vertices and m edges which is not null.*

Proof. Consider the graph H in Figure 3. We shall show that the vertex A is not a null vertex of H.

Notice that there are $n - 1$ paths joining vertices A and B, $n - 2$ of length 2 and one of length 1. Thus there are $(n - 1)!$ E paths on H which begin at A. If n is odd, these E paths also end at A, but if n is even, the paths end at the vertex B.

Notice also that if P is an E path beginning at A, then $(-1)^z$ as defined in line 1, is $+ 1$ if and only if the edge e is traversed from A to B by P. Also if P has the associated permutation $\sigma \in S_m$, then sgn $\sigma = (-1)^x$ where x is

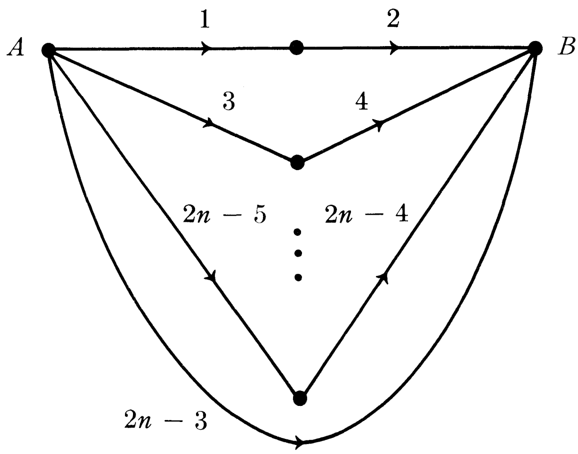

FIGURE 3

the number of paths of length 2 which are used in P when travelling from B to A.

Let S be the set of all E paths on H which begin at A. We shall see that all elements of S are positive if $n \equiv 1, 2$ (modulo 4) and all are negative if $n \equiv 0, 3$ (modulo 4). Let S_1 be the set of all paths in S which traverse edge e from A to B and let S_2 be the remaining paths of S (which therefore traverse e from B to A). As mentioned before, given P an E path, $(-1)^z = +1$ if and only if $P \in S_1$. Thus for each P we must determine sgn σ to establish whether P is positive or negative.

Suppose n is even. Then in an E path P, $n/2$ trips are made from vertex A to vertex B and $(n-2)/2$ are made from B to A. For the paths of S_1, $(n-2)/2$ subpaths of length 2 are used in travelling from B to A. Thus the sign of the permutation corresponding to $P \in S_1$ is

$$(-1)^{(n-2)/2} = \begin{cases} -1 & \text{if } n \equiv 0 \pmod 4 \\ +1 & \text{if } n \equiv 2 \pmod 4). \end{cases}$$

For the paths of S_2, $(n-4)/2$ paths of length 2 are used in travelling from B to A and have permutation signs

$$(-1)^{(n-4)/2} = \begin{cases} +1 & \text{if } n \equiv 0 \pmod 4 \\ -1 & \text{if } n \equiv 2 \pmod 4). \end{cases}$$

Then remembering to multiply by the appropriate value of $(-1)^z$, we see that the paths of S_1 and S_2 have signs as E paths given by

$$\begin{cases} -1 & \text{if } n \equiv 0 \pmod 4 \\ +1 & \text{if } n \equiv 2 \pmod 4). \end{cases}$$

If n is odd, $(n-1)/2$ paths are traversed from A to B and $(n-1)/2$ from B to A. For the paths of S_1 (defined as before) we have the permutation signs and E path signs given by

$$(-1)^{(n-1)/2} = \begin{cases} +1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4). \end{cases}$$

For the paths of S_2 we have $(n-3)/2$ paths of length 2 traversed from B to A whence we have associated permutation signs

$$(-1)^{(n-3)/2} = \begin{cases} -1 & \text{if } n \equiv 1 \pmod 4 \\ +1 & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

and by multiplying by $(-1)^z = -1$, we have signs of E paths given by

$$\begin{cases} +1 & \text{if } n \equiv 1 \pmod 4 \\ -1 & \text{if } n \equiv 3 \pmod 4). \end{cases}$$

Thus regardless of n, we have either all positive or all negative E paths beginning at A. A cannot then be null.

We can now use the graph H of Figure 3 to obtain the necessary examples to complete the proof of Theorem 2′ for $m < 2n - 3$. Consider the connected directed graph on p vertices and $p - 1$ edges which has one vertex Y with out-degree 1 and in-degree 0, one vertex Z with out-degree 0 and in-degree 1, and the remaining $p - 2$ vertices with both in-degree and out-degree 1. We shall call this graph the directed p-chain from Y to Z. Then the following result is clear.

LEMMA 7. *Let G be a directed graph and V a vertex of G. Let G' be the directed graph obtained by adding the directed p-chain from Y to Z on at the vertex V* (i.e. by identifying vertices Z and V). *Then Y is not a null vertex of G' if and only if V is not a null vertex of G.*

THEOREM 7. *Given $n \geqq 2$ and $m < 2n - 3$, there is a directed graph with n vertices and m edges which is not null.*

Proof. Suppose $n - 1 < m < 2n - 3$. Then construct a graph G as in Figure 3 with $m + 3 - n$ vertices and $2n + 3 - 2n$ edges. We know that A is not a null vertex of G by Theorem 6. (Notice that $m + 3 - n > 2$ since $m > n - 1$.) If we attach the directed p-chain from Y to Z on to G at vertex A where $p = 2n - m - 3$, we obtain a new graph G' with n vertices and m edges. By Lemma 7, the vertex Y of G' is not null.

Suppose $m \leqq n - 1$. Then the directed m-chain plus $n - m$ isolated vertices is not null, for if Y is one of the two vertices of degree 1, Y is not null since precisely one E path begins at Y.

Theorems 6 and 7 together give us Theorem 2′ which we have asserted is equivalent to Theorem 2. In fact, given n and $m < 2n - 2$, we can explicitly describe a set of m $n \times n$ skew-symmetric matrices $A_1, \ldots, A_m$ for which $[A_1, \ldots, A_m] \neq 0$. Namely, given such an n and m, draw and label the appropriate graph G with these parameters as given in Theorems 6 and 7. Then for each edge e_k, $k = 1, \ldots, m$, of G which is directed from vertex i to vertex j,

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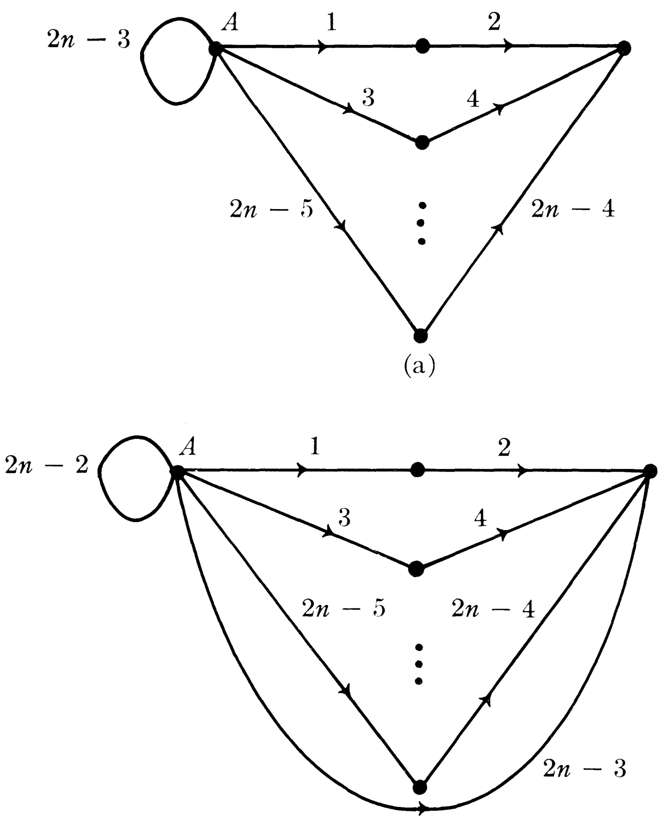
let the matrix A_k be the matrix of all 0's except for $a_{ij} = +1$ and $a_{ji} = -1$. Then for these A_k's, $[A_1, \ldots, A_m] \neq 0$. In this way, we obtain Theorem 2.

We mentioned one additional result in Theorem 5 which we can now see is also the best possible result for partially directed graphs with one undirected edge.

THEOREM 8. *If n is odd and $m < 2n - 2$ or if n is even and $m < 2n - 1$, then there is a partially directed graph G with n vertices and m edges, of which precisely one is undirected, such that G is not null.*

Sketch of the proof. Using the same techniques as in the proof of Theorem 6, it can be shown that the vertex A in each of the graphs, shown in Figure 4, is not null. Then for $n - 1 < m < 2n - 3$, a graph as in Figure 4a with $m + 3 - n$ vertices and $2m + 3 - 2n$ edges and with a directed p-chain added on at A $(p = 2n - m - 3)$ will be a suitable non-null graph. For $m \leqq n - 1$, an m-chain with one undirected edge will do.



(b) n even.

FIGURE 4

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5. Technical lemmas.

Proof of Lemma 3. If $S_i = S_l$, then the paths of $S(e_i, e_j, e_k, Y)$ can be paired

$$p_1e_ie_je_kp_2 \quad \text{with} \quad p_1e_ke_je_ip_2$$

where p_1 and p_2 are paths on G.

(1) If an even number of edges $\{e_i, e_j, e_k\}$ are directed, these paired elements have opposite signs. In this case $S(e_i, e_j, e_k, Y)$ is the union of such cancelling pairs and therefore cancels.

(2) If $S_i = S_l$ and an odd number of the edges $\{e_i, e_j, e_k\}$ are directed, then the paths, paired as above, are both of the same sign. Therefore $S(e_i, e_j, e_k, Y)$ cancels if and only if $S'(e_i, e_j, e_k, Y)$ cancels.

$G(e_i, e_j, e_k) = G - \{e_i, e_j, e_k\} + \{f\}$ where f is an undirected loop at vertex S_i. There is then a one-to-one correspondence between $S'(e_i, e_j, e_k, Y)$ and the E paths at Y on $G(e_i, e_j, e_k)$ and we claim that the correspondence is also sign-consistent. Because of Remark 2 we may assume that the edge e_j has some label $h \in \{1, 2, \ldots, m\}$, e_k the label $h + 1$, and we give f the label of e_i. Consider two corresponding paths $P_1 = p_1e_ie_je_kp_2$, $i < k$ and $P_2 = p_1fp_2$ where f is a loop. The signs of the corresponding permutations are the same and the corresponding orientation coefficients are either always the same or always opposite. In either case the correspondence is sign-consistent as claimed in (2).

(3) If $S_i \neq S_l$, $G(e_i, e_j, e_k) = G - \{e_i, e_j, e_k\} + \{f\}$ where f is an edge (not a loop) between S_i and S_l. Then there is a one-to-one correspondence between $S(e_i, e_j, e_k, Y)$ and the set of all E paths at Y on $G(e_i, e_j, e_k)$, obtained by associating an element $P = p_1e_ie_je_kp_2$, $i < k$, (respectively $Q = p_3e_ke_je_ip_4$) with the E path $P' = p_1fp_2$ (respectively $Q' = p_3fp_4$) on $G(e_i, e_j, e_k)$. Conversely, given an E path on $G(e_i, e_j, e_k)$ $P' = pfp'$, we obtain the element $P = pe_ie_je_kp'$ (respectively $Q = pe_ke_je_ip'$) of $S(e_i, e_j, e_k, Y)$ if f is traversed from S_i to S_l (respectively from S_l to S_i) in P'. We claim that the correspondence is sign-consistent; that is, we must show that the correspondence of paths P with P' and Q with Q' is always sign preserving or always sign reversing for our given graph. We may assume that the edge e_j has some label h, $h \in \{1, 2, \ldots, m\}$, and that e_k has the label $h + 1$. Let f receive the label of e_i.

Then the permutations corresponding to P and P' have the same signs whereas those corresponding to Q and Q' differ from each other. Suppose $\mathrm{OC}(P) = \mathrm{OC}(P')$. Then a routine check shows that $\mathrm{OC}(Q) = -\mathrm{OC}(Q')$, and in these cases we have that the correspondence is sign preserving. If $\mathrm{OC}(P) = -\mathrm{OC}(P')$, then similarly $\mathrm{OC}(Q) = \mathrm{OC}(Q')$ and we see that the correspondence is sign reversing. Thus in all cases there is an isomorphism as claimed in (3).

Proof of Lemma 4 (due to Herbert S. Wilf). Let us construct the $p \times p$ symmetric adjacency matrix, A, of Γ in the following way. Let ν_j be the number of vertices of degree j in Γ. Then in A we list the ν_2 vertices of degree 2 in the first ν_2 rows and columns; in the next ν_4 rows and columns we list the vertices of degree 4, etc. Let $B_{2,2}$ stand for the $\nu_2 \times \nu_2$ submatrix in the first ν_2 rows and ν_2 columns; let $B_{4,2}$ stand for the $\nu_4 \times \nu_2$ submatrix of A in the first ν_2 columns and in rows $\nu_2 + 1$ through $\nu_2 + \nu_4$; and in general let $B_{2j,2}$, $j \geqq 1$, stand for the $\nu_{2j} \times \nu_2$ submatrix of A in the first ν_2 columns and in rows $k + 1$ through $k + \nu_{2j}$, where $k = \sum_{i=1}^{j-1} \nu_{2i}$. Let C_{22} stand for the $\nu_2 \times (p - \nu_2)$ submatrix of A in the first ν_2 rows and in columns $\nu_2 + 1$ through p.

Suppose that no two vertices of degree 2 are adjacent. $B_{2,2}$ is then a $\nu_2 \times \nu_2$ zero matrix.

Suppose also that every vertex of degree $2d$ is adjacent to at most $2d - 4$ vertices of degree 2. Thus every vertex of degree 4 is adjacent to no vertices of degree 2; hence $B_{4,2}$ is also a zero matrix of dimension $\nu_4 \times \nu_2$. Furthermore since each vertex of degree 6 is adjacent to at most 2 vertices of degree 2, each row sum of $B_{6,2}$ is at most 2. Similarly each row sum of $B_{8,2}$ is at most 4, etc.

Then the sum of the entries in $B_{4,2} + B_{6,2} + B_{8,2} + \ldots$ is at most

$$\begin{aligned} 0 + 2\nu_6 + 4\nu_8 + \ldots &= \sum \nu_{2d}(2d - 4) \\ &= \sum (\rho(v) - 4), \end{aligned}$$

this last sum being over all vertices of G of degree at least 4.

Since A is symmetric, the sum of the entries in $B_{4,2} + B_{6,2} + \ldots$ is equal to the sum of the entries in C_{22} and each row sum of C_{22} is precisely 2 since $B_{2,2} = 0$. Thus the sum of the entries of C_{22} is $2\nu_2$ and

$$2\nu_2 \leqq \sum_{\rho(v) \geqq 4} (\rho(v) - 4) = \sum_{v} (\rho(v) - 4) - (-2\nu_2),$$

with the last sum now over all vertices of the graph. Thus

$$\sum_{v} (\rho(v) - 4) \geqq 0,$$

or equivalently $2(2p - 1) \geqq 4p$, a contradiction.

Proof of Lemma 5. $P = \bigcup_{i=1}^{3} P_i$ where P_i is the set of those paths of P which begin on edge e_i. (See Figure 2.) We shall see that each P_i cancels, but to simplify notation, consider P_1.

$P_1 = \bigcup S(\pi)$ where $S(\pi)$ is the set of all paths of P_1 whose associated Type I partition is π, and where the union is taken over all Type I partitions in which the singleton is $\{1\}$. We shall see that each $S(\pi)$ cancels, but to simplify notation, suppose π is the partition

$$\{1\}, \{2, 4\}, \{3, 5\}, \{6, 7\}, \ldots, \{2d - 2, 2d - 1\}.$$

We shall apply Lemma 3 several times. All the paths of $S(\pi)$ traverse the edges e_2, e_4, and f_4 consecutively. Therefore we see by parts 1 and 2 of Lemma 3 that if $S_2 = T_4$ either the paths of $S(\pi)$ cancel or the paths of $S(\pi)$ cancel if and only if the paths of $S'(\pi)$ cancel where $S'(\pi)$ contains those paths of $S(\pi)$ of the form $p_1e_2e_4f_4p_2$.

Then applying parts 2 and 3 of the lemma, there is an isomorphism between the paths of $S'(\pi)$ or $S(\pi)$ respectively and the set of paths $S(\pi')$, all E paths from V to W on $H_{24} = H - \{e_2, e_4, f_4\} + \{d_2\}$, (where d_2 joins S_2 and T_4 and is directed or not as dictated by the lemma) which induce on the set $\{1, 3, 5, 6, \ldots, 2d - 1\}$ the partition π' where

$$\pi' = \{1\}, \{3, 5\}, \{2i, 2i + 1\}, \quad i = 3, \ldots, d - 1.$$

$H_{24} \in C(n - 1, 2n - 3, p)$ where $p = 0, 1, 2$. (The number of undirected edges may have increased.)

We apply the lemma again to the triple (e_3, e_5, f_5) in the graph H_{24}, and we see again that either $S(\pi')$ cancels or there is an appropriate isomorphism of these paths with those of $S(\pi'')$, the set of all E paths from V to W on $H_{2435} = H_{24} - \{e_3, e_5, f_5\} + \{d_3\}$ (where d_3 joins S_3 and T_5) which partition the set $\{1, 6, 7, \ldots, 2d - 1\}$ into the sets $\{1\}$, $\{2i, 2i + 1\}$ $i = 3, \ldots, d - 1$. $H_{2435} \in C(n - 2, 2n - 5, p)$ where $p = 0, 1, 2$ or 3.

We apply Lemma 3 $d - 3$ more times, once for each of the triples $(e_{2i}, e_{2i+1}, f_{2i+1})$ for $i = 3, \ldots, d - 1$, and we find eventually that either the paths of $S(\pi)$ cancel or $S(\pi)$ is isomorphic to the set of all E paths from V to W on

$$\tilde{H} = H - \{e_i, i = 2, \ldots, 2d - 1, f_4, f_{2j+1}, j = 2, \ldots, d - 1\} \\ + \{d_3, d_{2n}, n = 2, \ldots, d - 1\}$$

where d_i joins S_i to T_{i+1} and may be directed or not. Thus

$$\tilde{H} \in C(n - d + 1, 2n - 2d + 1, p),$$

$p \in \{0, 1, \ldots, d - 1\}$, and has a vertex of degree 1, namely V, since all edges except for e_1 have been removed. But then $K = \tilde{H} - \{V, e\}$ is null by Remark 5 since $K \in C(n - d, 2n - 2d, p)$. Thus $\tilde{H}$ is null so that the paths of $S(\pi)$ cancel.

In the same way the paths of $S(\pi)$ cancel for any Type I partition π and thus so do the paths of P_i, $i = 1, 2, 3$ and hence P.

Proof of Lemma 6. (See Figure 2.) The proof is by induction on m and p. We know that the result is true for $p = 2$ and for every $m \leqq n$, for when $p = 2$, $\rho(V) = 3$ and in this case we are discussing the set of all E paths on H which begin at V. By the minimality of n and by the discussion of case II, we know that such a set of E paths cancels since the graph H is null.

We assume the lemma is true for every pair (q, k) $q < p$ and $k < m$ and we wish to prove the lemma for the parameters p and $m \leqq n$.

We may write $P = \bigcup S(\pi)$ where $S(\pi)$ is the set of all E paths on H with associated partition π of Type II and where the union is taken over all partitions of Type II of $\{1, 2, \ldots, 2p - 1\}$. To simplify notation, let $z = 2p - 1$. In every Type II partition π, z is paired with some element w where $4 \leqq w < z$, and we may write

$$P = \bigcup S(\pi) = \bigcup_{j=4}^{2p-2} \bigcup_{\pi'} S(j, z, \pi')$$

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where $S(j, z, \pi')$ is the set of those paths of $S(\pi)$ in which z and j are paired and where π' is the Type II partition of $\{1, 2, \ldots, 2d - 2\} - \{j\}$, obtained by removing the pair $\{j, z\}$ from π.

For a fixed j, consider $\bigcup_{\pi'} S(j, z, \pi')$ and to simplify notation, assume $j = z - 1 = 2p - 2$. Thus we consider the set $D = \bigcup_{\pi'} S(z - 1, z, \pi')$, the union being over all Type II partitions π' of $\{1, 2, \ldots, 2d - 3\}$.

Notice that if $T_z = W$, then there may be some paths in D which are of the form $pf_{z-1}e_{z-1}e_zf_z$, i.e. paths which end at T_z just after having traversed edge f_z. Let the set of all such paths of D be denoted by D' and let $D'' = D - D'$. Thus D'' contains those paths of D which are either of the form

$$p_1f_{z-1}e_{z-1}e_zf_zp_2 \quad (p_2 \neq 0) \quad \text{or} \quad p_3f_ze_ze_{z-1}f_{z-1}p_4,$$

where p_i, $i = 1, \ldots, 4$, are subpaths.

We shall see first that the paths of D' cancel, for there is an isomorphism between the paths of D' and the set R of all E paths from V to T_{z-1} on $G' = G - \{f_{z-1}, e_{z-1}, e_z, f_z\}$ which induce a Type II partition on $\{1, 2, \ldots, 2d - 3\}$. $G' \in C(m - 2, 2m - 5, i)$, $i = 0, 1$; vertex V in G' has valence $2p - 3$ and is adjacent to at least $2p - 6$ vertices of degree 2. Therefore by induction the paths of R cancel as do the paths of D'.

Notice that in a path of D'' f_z has either a predecessor or successor which is an edge at T_z. If $\rho(T_z) = k + 1$, suppose the edges, other than f_z, are labelled g_i, $i = 1, 2, \ldots, k$, where g_i joins T_z and U_i. (The U_i may not all be distinct and it may happen that $U_i = T_j$ or S_k for some values of i, j or k.)

We now apply Lemma 3 twice, first with the triple of edges (f_{z-1}, e_{z-1}, e_z). Note that every path of D'' travels on these three edges consecutively and so we replace them by an edge f joining T_{z-1} and S_z and which is suitably directed or not. (Notice that by assumption $T_{z-1} \neq S_z$.) Thus we have that the set D'' is isomorphic to the set R of all E paths from V to W on $H' = H - \{f_{z-1}, e_{z-1}, e_z\} + \{f\}$ which have an associated Type II partition $\{1, 2, \ldots, 2d - 3\}$ and which are of the form $q_1ff_zq_2$ $(q_2 \neq 0)$ or $q_3f_zfq_4$ where q_i, $i = 1, \ldots, 4$, are sub-paths. $H' \in C(m - 1, 2m - 3, j)$, $j = 0, 1, 2$. (The number of undirected edges may have increased.)

Now notice that $R = \bigcup_{i=1}^{k} R_i$ where R_i is the set of those paths of R which are of the form $q_1ff_zg_iq_2$ or $q_3g_if_zfq_4$ $i \in \{1, \ldots, k\}$, where g_i is the successor or predecessor of f_z, other than f. Now we apply Lemma 3 again to each triple (f, f_z, g_i) to see that either the set R_i cancels immediately or there is an appropriate isomorphism between R_i and the set X_i of all E paths from V to W on $H_i = H' - \{f, f_z, g_i\} + \{d_i\}$ where d_i joins T_{z-1} and U_i and is appropriately directed or not and where the paths must induce a Type II partition of $\{1, 2, \ldots, 2d - 3\}$.

The important thing to notice is that $H_i \in C(m - 2, 2m - 5, j)$, $j = 0, 1$, for every $i = 1, 2, \ldots, k$ even though H' may have had two undirected edges. (A routine check will verify this.) The graph H_i and the set X_i satisfy our inductive hypothesis. Y in H_i has valence $2p - 3$ and is adjacent to at least

$2p - 6$ vertices of degree 2, and X_i is the set of all E paths at Y on H_i which induce a Type II partition on $\{1, 2, \ldots, 2d - 3\}$. Thus the paths of X_i cancel for every i, implying that the paths of R_i and R cancel. Thus D'' cancels and the set $D = \bigcup_{\pi'} S(z - 1, z, \pi')$, cancels. In the same way so does the set $\bigcup_{\pi'} S(j, z, \pi')$, $j = 4, \ldots, 2p - 3$. Thus the paths of P cancel as claimed.

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