

ON LIE-LIKE COMPLEX FILIFORM LEIBNIZ ALGEBRAS

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Abstract

In this paper we propose an approach to classifying a subclass of filiform Leibniz algebras. This subclass arises from the naturally graded filiform Lie algebras. We reconcile and simplify the structure constants of such a class. In the arbitrary fixed dimension case an effective algorithm to control the behavior of the structure constants under adapted transformations of basis is presented. In one particular case, the precise formulas for less than 10 dimensions are given. We provide a computer program in Maple that can be used in computations as well.

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1. Introduction

It is well known that the natural gradation of nilpotent Lie and Leibniz algebras is very helpful in investigating their structural properties. This technique is more effective when the length of the natural gradation is sufficiently large. When it is maximal the algebra is called *filiform*. For applications of this technique, see [6, 12] (for Lie algebras) and [1] (for Leibniz algebras).

The present paper deals with the filiform Leibniz algebras whose natural gradation is a filiform Lie algebra. Note that the case of algebras whose natural gradation is a non-Lie filiform Leibniz algebra was considered in [1–3, 5].

Section 2 gathers together basic definitions, notation and conventions used in this paper. Section 3 is devoted to the description of the class of non-Lie filiform Leibniz algebras whose gradation is a filiform Lie algebra. It contains the main result of the paper. Here we simplify the multiplication table and follow the behavior of the structure constants under the adapted action. Section 4 deals with the detailed description of one particular case. In this section we give precise formulas for changing of the structure constants and a simple computer program in Maple.

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2. Preliminaries

Let V be a vector space of dimension n over an algebraically closed field K ($\text{char}K = 0$). Bilinear maps $V \times V \rightarrow V$ form a vector space $\text{Hom}(V \otimes V, V)$ of dimension n^3 , which can be considered together with its natural structure of an affine algebraic variety over K and denoted by $\text{Alg}_n(K) \cong K^{n^3}$. An n -dimensional algebra L over K can be considered as an element $\lambda(L)$ of $\text{Alg}_n(K)$ via the bilinear mapping $\lambda : L \otimes L \rightarrow L$ defining a binary algebraic operation on L : let $\{e_1, e_2, \dots, e_n\}$ be a basis of the algebra L . Then the multiplication table of L is represented by point (γ_{ij}^k) of this affine space as follows:

$$\lambda(e_i, e_j) = \sum_{k=1}^n \gamma_{ij}^k e_k.$$

Here γ_{ij}^k are called *structure constants* of L . The linear reductive group $GL_n(K)$ acts on $\text{Alg}_n(K)$ by $(g * \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y)))$ ('transport of structure'). Two algebras λ_1 and λ_2 are isomorphic if and only if they belong to the same orbit under this action. It is clear that the elements of the given orbit are algebras isomorphic to each other.

Recall that an algebra L over a field F is called a *Leibniz algebra* if its composition law $[\cdot, \cdot]$ satisfies the following so-called Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]. \quad (2.1)$$

Leibniz algebras were introduced by Loday [8, 9]. (For this reason, they have also been called 'Loday algebras' [7].) A skew-symmetric Leibniz algebra is a Lie algebra. In this case (2.1) is just the Jacobi identity.

Let $LB_n(K)$ be a subvariety of $\text{Alg}_n(K)$ consisting of all n -dimensional Leibniz algebras over K . It is invariant under the above-mentioned action of $GL_n(K)$. As a subset of $\text{Alg}_n(K)$ the set $LB_n(K)$ is specified by the system of equations with respect to the structure constants γ_{ij}^k :

$$\sum_{l=1}^n (\gamma_{jk}^l \gamma_{il}^m - \gamma_{ij}^l \gamma_{lk}^m + \gamma_{ik}^l \gamma_{lj}^m) = 0.$$

In fact, theoretically, by solving the above system of equations we can get a complete classification of Leibniz algebras in a given dimension n . But in practice this approach is useless even for small values of n . Therefore one usually has to apply different methods of investigation.

If L is a Leibniz algebra we may define

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1.$$

The series

$$L^1 \supseteq L^2 \supseteq L^3 \supseteq \dots$$

is called the descending central series of L . If the series terminates for some positive n , then the Leibniz algebra L is said to be nilpotent. The termination number here is called the nilindex of L . Below we present a gradation closely related to the descending central series.

Given a nilpotent Leibniz algebra L with nilindex s , we put $L_i = L^i/L^{i+1}$, $1 \leq i \leq s-1$, and $grL = L_1 \oplus L_2 \oplus \cdots \oplus L_{s-1}$. Since $[L_i, L_j] \subseteq L_{i+j}$ the algebra grL is graded. grL is called the naturally graded Leibniz algebra.

Let us introduce the class of nilpotent Leibniz algebras, which is main object of the paper. In the case of Lie algebras, such algebras have the maximal nilindex.

DEFINITION 2.1. A Leibniz algebra L is said to be filiform if $\dim L^i = n - i$, where $n = \dim L$ and $2 \leq i \leq n$.

Clearly a filiform Leibniz algebra is nilpotent.

Let Leib_n denote the class of all n -dimensional filiform Leibniz algebras.

DEFINITION 2.2. If a Leibniz algebra G is isomorphic to a filiform naturally graded algebra grL , then G is said to be naturally graded filiform Leibniz algebra.

Later on all algebras are supposed to be over the field of complex numbers \mathbb{C} and the omitted products of basis vectors are supposed to be zero.

The following theorem summarizes the results of [1, 12].

THEOREM 2.3. Any complex $(n+1)$ -dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise nonisomorphic algebras:

$$\begin{aligned} NGF_1 &= \begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \end{cases} \\ NGF_2 &= \begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, & 2 \leq i \leq n-1, \end{cases} \\ NGF_3 &= \begin{cases} [e_i, e_0] = -[e_0, e_i] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = \alpha(-1)^{i+1}e_n, & 1 \leq i \leq n-1. \end{cases} \end{aligned}$$

$\alpha \in \{0, 1\}$ for odd n and $\alpha = 0$ for even n .

It is clear that neither NGF_1 nor NGF_2 is a Lie algebra.

The above theorem means that the natural gradation of a Leibniz algebra is in one of the classes NGF_i for $i = 1, 2, 3$.

The following result of [1] describes the class of complex filiform Leibniz algebras whose natural gradation is one of NGF_i for $i = 1, 2$.

THEOREM 2.4. Any $(n+1)$ -dimensional complex non-Lie filiform Leibniz algebra, whose naturally graded algebra is not a Lie algebra, belongs to one of the following two classes:

$$\begin{aligned}
 FLeib_{n+1} &= \begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + \dots + \alpha_{n-1} e_{n-1} + \theta e_n, \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + \dots + \alpha_{n+1-j} e_n, & 1 \leq j \leq n-2, \\ \alpha_3, \alpha_4, \dots, \alpha_n, \theta \in \mathbb{C}; \end{cases} \\
 SLeib_{n+1} &= \begin{cases} [e_0, e_0] = e_2, \\ [e_i, e_0] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + \dots + \beta_n e_n, \\ [e_1, e_1] = \gamma e_n, \\ [e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + \dots + \beta_{n+1-j} e_n, & 2 \leq j \leq n-2, \\ \beta_3, \beta_4, \dots, \beta_n, \gamma \in \mathbb{C}. \end{cases}
 \end{aligned}$$

Isomorphism criteria, classifications and invariants of $FLeib_{n+1}$ and $SLeib_{n+1}$ were investigated in [2, 3, 5, 10, 11].

3. Main result

This section is devoted to the filiform Leibniz algebras whose natural gradation is an algebra from NGF_3 . This class is denoted by $TLeib_{n+1}$. Here we clarify the multiplication table of algebras from $TLeib_{n+1}$ and investigate the behavior of structure constants under a change of basis.

PROPOSITION 3.1.

$$TLeib_{n+1} = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0, e_0] = b_{0,0} e_n, \\ [e_0, e_1] = -e_2 + b_{0,1} e_n, \\ [e_1, e_1] = b_{1,1} e_n, \\ [e_i, e_j] = a_{i,j}^1 e_{i+j+1} \\ \quad + \dots + a_{i,j}^{n-(i+j+1)} e_{n-1} + b_{i,j} e_n, & 1 \leq i < j \leq n-1, \\ [e_i, e_j] = -[e_j, e_i], & 1 \leq i < j \leq n-1, \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^i b_{i,n-i} e_n, & 1 \leq i \leq n-1, \end{cases} \tag{3.1}$$

where $a_{i,j}^k, b_{i,j} \in \mathbb{C}$ and $b_{i,n-i} = b$ whenever $1 \leq i \leq n-1, b \in \{0, 1\}$ for odd n and $b = 0$ for even n .

PROOF. Let $L \in TLeib_{n+1}$ and $\{e_0, e_1, \dots, e_n\}$ be a basis of L . Then it is clear that $[e_i, e_j] \in \langle e_{i+j+1}, \dots, e_n \rangle$ for any $i, j \neq 0$. Then

$$[e_i, e_0] = e_{i+1} + (*)e_{i+2} + \dots + (*)e_n, \quad 1 \leq i \leq n-1.$$

Putting $e'_1 = e_1, e'_0 = e_0, e'_{i+1} := [e'_i, e'_0]$, we can assume that $[e_i, e_0] = e_{i+1}, 1 \leq i \leq n - 1$.

Now consider

$$[e_0, e_i] = -e_{i+1} + \alpha_{0,i}^{i+2} e_{i+2} + \alpha_{0,i}^{i+3} e_{i+3} + \dots + \alpha_{0,i}^n e_n, \quad 1 \leq i \leq n - 1.$$

We get

$$[e_i, e_0] + [e_0, e_i] = \alpha_{0,i}^{i+2} e_{i+2} + \alpha_{0,i}^{i+3} e_{i+3} + \dots + \alpha_{0,i}^n e_n, \quad 1 \leq i \leq n - 1. \quad (3.2)$$

Note that from the Leibniz identity it follows that $[x, y] + [y, x] \in \mathfrak{R}(L)$, for any $x, y \in L$, where $\mathfrak{R}(L)$ is the right annihilator of L . Therefore, if we multiply the both sides of (3.2) from the left-hand side $(n - i - 2)$ times by e_0 we obtain $\alpha_{0,i}^{i+2} = 0$. Substituting and repeating, we get

$$\alpha_{0,i}^{i+k} = 0, \quad 2 \leq k \leq n - 1 - i.$$

Applying the above to $[e_i, e_i], 0 \leq i \leq [n/2]$, we get $[e_i, e_i] = \alpha_{i,i}^n e_n$.

The chain of equalities

$$\begin{aligned} [e_0, e_i] &= [e_0, [e_{i-1}, e_0]] = [[e_0, e_{i-1}], e_0] - [[e_0, e_0], e_{i-1}] \\ &= [-e_i + \alpha_{0,i-1}^n e_n, e_0] = -[e_i, e_0] = -e_{i+1} \end{aligned}$$

leads to $[e_i, e_0] = -[e_0, e_i] = e_{i+1}$ for $2 \leq i \leq n - 1$, that is, $[e_0, x] = -[x, e_0]$ for any $x \in L^2$. We claim that

$$[e_i, e_j] = -[e_j, e_i], \quad 1 \leq i < j \leq n. \quad (3.3)$$

The induction argument by i for any j and the chain of equalities

$$\begin{aligned} [e_i, e_{j+1}] &= [e_i, [e_j, e_0]] = [[e_i, e_j], e_0] - [[e_i, e_0], e_j] \quad (\text{since } [e_i, e_j] \in L^2) \\ &= -[e_0, [e_i, e_j]] + [[e_0, e_i] - \alpha_{0,i}^n e_n, e_j] = -[e_0, [e_i, e_j]] + [[e_0, e_i], e_j] \\ &= -[[e_0, e_i], e_j] + [[e_0, e_j], e_i] + [[e_0, e_i], e_j] \\ &= -[e_{j+1}, e_i], \quad 1 \leq j \leq n - 1, \end{aligned}$$

show (3.3).

The above observations lead to the required multiplication table of $L \in TLeib_{n+1}$. □

Note that filiform Lie algebras are in $TLeib$.

Bearing in mind Theorem 2.4 and all the foregoing, we conclude that the set of all n -dimensional filiform Leibniz algebras can be represented as a disjoint union of $FLeib_n, SLeib_n$ and $TLeib_n$.

Let $L \in TLeib_{n+1}$. Then it is easy to see that the subspace spanned by $\{e_n\}$ is an ideal of L and the quotient algebra $L/\langle e_n \rangle$ is an n -dimensional filiform Lie algebra. Therefore, later on in this case the structure constants $a_{i,j}^k$ will be considered to be satisfying the Leibniz identity. Moreover, n -dimensional filiform Lie algebras have been classified (see [4]). In order to make use of the existing classification of n -dimensional filiform Lie algebras ($n \leq 11$) from [4] we consider the structure constants $a_{i,j}^k$ in (3.1) to be fixed constants.

LEMMA 3.2. *Let $L \in T\text{Leib}_{n+1}$. Then*

$$\sum_{s=1}^{n-(i+j+k+1)} a_{j,k}^s b_{i,j+k+s} = \sum_{s=1}^{n-(i+j+k+1)} (a_{i,j}^s b_{i+j+s,k} - a_{i,k}^s b_{i+k+s,j}). \tag{3.4}$$

PROOF. The Leibniz identity for e_i, e_j and e_k gives the required relations between the structure constants

$$\begin{aligned} [e_i, [e_j, e_k]] &= \left[e_i, \sum_{s=1}^{n-(j+k+1)} a_{j,k}^s e_{j+k+s} + b_{j,k} e_n \right] \\ &= \sum_{s=1}^{n-(i+j+k+1)} a_{j,k}^s \left(\sum_{t=1}^{n-(i+j+k+s+1)} a_{i,j+k+s}^t e_{i+j+k+s+t} + b_{i,j+k+s} e_n \right), \\ [[e_i, e_j], e_k] &= \left[\sum_{s=1}^{n-(i+j+1)} a_{i,j}^s e_{i+j+s} + b_{i,j} e_n, e_k \right] \\ &= \sum_{s=1}^{n-(i+j+k+1)} a_{i,j}^s \left(\sum_{t=1}^{n-(i+j+k+s+1)} a_{i+j+s,k}^t e_{i+j+k+s+t} + b_{i+j+s,k} e_n \right), \\ [[e_i, e_k], e_j] &= \left[\sum_{s=1}^{n-(i+k+1)} a_{i,k}^s e_{i+k+s} + b_{i,k} e_n, e_j \right] \\ &= \sum_{s=1}^{n-(i+j+k+1)} a_{i,k}^s \left(\sum_{t=1}^{n-(i+j+k+s+1)} a_{i+k+s,j}^t e_{i+j+k+s+t} + b_{i+k+s,j} e_n \right), \end{aligned}$$

leading to (3.4). □

Here are several remarks regarding (3.4).

- (1) It is symmetric with respect to i, j, k (since $a_{s,t}^k = -a_{t,s}^k$ and $b_{s,t} = -b_{t,s}$ for any s and t , except for $(s, t) = (0, 0), (1, 1), (0, 1), (1, 0)$).
- (2) In the case where $(i, j, k) = (0, j, k)$ we get

$$\sum_{s=1}^{n-(j+k+1)} a_{j,k}^s b_{0,j+k+s} = \sum_{s=1}^{n-(j+k+1)} (a_{0,j}^s b_{j+s,k} - a_{0,k}^s b_{k+s,j}),$$

where $j \neq 0, k \neq 0$.

- (3) Since $a_{0,t}^s = 0$ as $s \neq 1$ and $a_{0,t}^1 = -1$, we get

$$a_{j,k}^1 b_{0,j+k+1} + a_{j,k}^2 b_{0,j+k+2} + \dots + a_{j,k}^{n-(j+k+1)} b_{0,n-1} = -b_{j+1,k} + b_{k+1,j}.$$

- (4) Since $b_{0,t} = 0$ as $t = 2, \dots, n - 2$ and $b_{0,n-1} = -1$, we get that

$$a_{j,k}^{n-(j+k+1)} = b_{j+1,k} - b_{k+1,j},$$

for $k = j + 1, j + 2, \dots, n - j - 2$ and $j = 1, 2, \dots, [(n - 3)/2]$.

LEMMA 3.3. *Let $L \in TLeib_{n+1}$. Then*

$$[e_i, e_{j+k}] = \sum_{s=0}^k (-1)^{k-s} \binom{k}{s} [e_{i+k-s}, e_j] R_{e_0}^s, \tag{3.5}$$

where $1 \leq i, j, k \leq n$ and $yR_x = [y, x]$ is the right multiplication operator on L .

PROOF. The proof will proceed by induction on k . Let $k = 1$. Then $[e_i, e_{j+1}] = [e_i, [e_j, e_0]] = -[e_{i+1}, e_j] + [[e_i, e_j], e_0]$, that is, (3.5) holds for $k = 1$. Then the following chain of equalities leads to the claim:

$$\begin{aligned} [e_i, e_{j+k+1}] &= [e_i, [e_{j+k}, e_0]] = [[e_i, e_{j+k}], e_0] - [[e_i, e_0], e_{j+k}] \\ &= \sum_{s=0}^k (-1)^{k-s} \binom{k}{s} [e_{i+k-s}, e_j] R_{e_0}^{s+1} \\ &\quad - \sum_{s=0}^k (-1)^{k-s} \binom{k}{s} [e_{i+k+1-s}, e_j] R_{e_0}^s \\ &= - \sum_{s=1}^{k+1} (-1)^{k-s} \binom{k}{s-1} [e_{i+k+1-s}, e_j] R_{e_0}^s \\ &\quad - \sum_{s=0}^k (-1)^{k-s} \binom{k}{s} [e_{i+k+1-s}, e_j] R_{e_0}^s \\ &= \sum_{s=1}^k (-1)^{k+1-s} \left(\binom{k}{s-1} + \binom{k}{s} \right) [e_{i+k+1-s}, e_j] R_{e_0}^s \\ &\quad + [e_{i+k+1}, e_j] R_{e_0}^{k+1} - (-1)^k [e_{i+1+k}, e_k] \\ &= \sum_{s=0}^{k+1} (-1)^{k+1-s} \binom{k+1}{s} [e_{i+k+1-s}, e_j] R_{e_0}^s. \quad \square \end{aligned}$$

Let $L \in TLeib_{n+1}$.

DEFINITION 3.4. A basis $\{e_0, e_1, \dots, e_n\}$ of L is said to be adapted if its multiplication table with respect to $\{e_0, e_1, \dots, e_n\}$ has the form of $TLeib_{n+1}$.

DEFINITION 3.5. Let $\{e_0, e_1, \dots, e_n\}$ be an adapted basis of L . Then a nonsingular linear transformation $f : L \rightarrow L$ is said to be adapted if the basis $\{f(e_0), f(e_1), \dots, f(e_n)\}$ is adapted.

To simplify notation, let

$$a = (b_{0,0}, b_{0,1}, b_{1,1}, a_{i,j}^1, \dots, a_{i,j}^{n-(i+j+1)}, b_{i,j})$$

stand for the algebra in $TLeib_{n+1}$, with the structure constants $b_{0,0}, b_{0,1}, b_{1,1}, a_{i,j}^1, \dots, a_{i,j}^{n-(i+j+1)}, b_{i,j}$.

Let $a' = (b'_{0,0}, b'_{0,1}, b'_{1,1}, a_{i,j}^{1'}, \dots, a_{i,j}^{n-(i+j+1)'}, b'_{i,j})$. Consider an adapted transformation f . Let

$$\begin{aligned} f(e_0) &= A_0e_0 + A_1e_1 + \dots + A_n e_n, \\ f(e_1) &= B_0e_0 + B_1e_1 + \dots + B_n e_n, \end{aligned}$$

where $A_0B_1 - A_1B_0 \neq 0$ and let $f(a) = a'$.

LEMMA 3.4. For $b'_{0,0}, b'_{0,1}$ and $b'_{1,1}$ the following hold:

$$\begin{aligned} b'_{0,0} &= \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^{n-1} B_1}, \quad B_0 = 0, \\ b'_{1,1} &= \frac{B_1 b_{1,1}}{A_0^{n-1}}, \quad b'_{0,1} = \frac{A_0 b_{0,1} + 2A_1 b_{1,1}}{A_0^{n-1}}. \end{aligned}$$

PROOF. By using multiplication on a we get

$$f(e_i) = [f(e_{i-1}), f(e_0)] = A_0^{i-2} (A_0 B_1 - A_1 B_0) e_i + \sum_{s \geq i+1} (*) e_s, \quad 2 \leq i \leq n. \tag{3.6}$$

Consider

$$[f(e_2), f(e_1)] = a_{i,j}^{1'} f(e_4) + \dots + a_{i,j}^{(n-4)'} f(e_{n-1}) + b'_{i,j} f(e_n);$$

equating the corresponding coefficients, we get $B_0(A_0 B_1 - A_1 B_0) = 0$, which implies $B_0 = 0$.

The product $[f(e_0), f(e_0)] = b'_{0,0} f(e_n)$ leads to

$$A_0^2 a_{0,0} + A_0 A_1 a_{0,1} + A_1^2 a_{1,1} = a'_{0,0} A_0^{n-1} B_1.$$

Then we get

$$b'_{0,0} = \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^{n-1} B_1}.$$

Similarly,

$$b'_{1,1} = \frac{B_1 b_{1,1}}{A_0^{n-1}}.$$

Now consider the equality

$$b'_{0,1} f(e_n) = [f(e_1), f(e_0)] + [f(e_0), f(e_1)].$$

Then $b'_{0,1} A_0^{n-1} B_1 = A_0 B_1 b_{0,1} + 2A_1 B_1 b_{1,1}$. This implies that

$$b'_{0,1} = \frac{A_0 b_{0,1} + 2A_1 b_{1,1}}{A_0^{n-1}}. \quad \square$$

DEFINITION 3.6. The following adapted transformations of L are said to be elementary:

$$\begin{aligned} \tau(a, b, c) &= \begin{cases} \tau(e_0) = ae_0 + be_1, \\ \tau(e_1) = ce_1, & ac \neq 0, \\ \tau(e_{i+1}) = [\tau(e_i), \tau(e_0)], & 1 \leq i \leq n - 1, \end{cases} \\ \sigma(a, k) &= \begin{cases} \sigma(e_0) = e_0 + ae_k, & 2 \leq k \leq n, \\ \sigma(e_1) = e_1, \\ \sigma(e_{i+1}) = [\sigma(e_i), \sigma(e_0)], & 1 \leq i \leq n - 1, \end{cases} \\ \varphi(c, k) &= \begin{cases} \varphi(e_0) = e_0, \\ \varphi(e_1) = e_1 + ce_k, & 2 \leq k \leq n, \\ \varphi(e_{i+1}) = [\varphi(e_i), \varphi(e_0)], & 1 \leq i \leq n - 1, \end{cases} \end{aligned}$$

where $a, b, c \in \mathbb{C}$; we refer to these as transformations of the first, second, and third type, respectively.

PROPOSITION 3.7. Let f be an adapted transformation of L . Then it can be represented as the composition

$$f = \varphi(B_n, n) \circ \dots \circ \varphi(B_2, 2) \circ \sigma(A_n, n) \circ \dots \circ \sigma(A_2, 2) \circ \tau(A_0, A_1, B_1).$$

PROOF. The proof is straightforward. □

4. The isomorphism criteria for truncated filiform Leibniz algebras

In this section we consider a subclass of $TLeib_{n+1}$. Let L be an algebra from $TLeib_{n+1}$ such that the quotient algebra $L/\langle e_n \rangle$ is isomorphic to an n -dimensional algebra from NGF_3 . This kind of algebra we call *truncated*. Then it is easy to see that the multiplication table of L has the form:

$$\left\{ \begin{aligned} [e_i, e_0] &= e_{i+1}, & 1 \leq i \leq n - 1, \\ [e_0, e_i] &= -e_{i+1}, & 2 \leq i \leq n - 1, \\ [e_0, e_0] &= b_{0,0}e_n, \\ [e_0, e_1] &= -e_2 + b_{0,1}e_n, \\ [e_1, e_1] &= b_{1,1}e_n, \\ [e_i, e_j] &= -[e_j, e_i] = b_{i,j}e_n, & 1 \leq i \leq n - 1, 1 \leq j \leq n - i, \\ [e_i, e_{n-i}] &= -[e_{n-i}, e_i] = (-1)^i b_{i,n-i}e_n, & 1 \leq i \leq n - 1, \end{aligned} \right.$$

where $b_{i,n-i} = b$ for $1 \leq i \leq n - 1$, and $b \in \{0, 1\}$ for odd n and $b = 0$ for even n .

One of the reasons for considering the truncated case is to give a complete classification in this particular case and thus to show how the isomorphism criterion works in low-dimensional cases. Since in this case coefficients $a_{i,j}^k = 0$, where $1 \leq i < j \leq n - 1$ and $1 \leq k \leq n - (i + j + 1)$, the multiplication table of L has a simple form. Therefore the action of changing basis can easily be managed.

We now investigate the behavior of the coefficients $b_{i,j}$ under adapted basis changing. From Lemma 3.4 we conclude that the elementary transformation of the second and third types does not change the structure constants $b_{0,0}$, $b_{0,1}$ and $b_{1,1}$.

Consider

$$\tau(a, b, c) = \begin{cases} \tau(e_0) = ae_0 + be_1, \\ \tau(e_1) = ce_1, \\ \tau(e_{i+1}) = [\tau(e_i), \tau(e_0)], \end{cases} \quad \begin{matrix} ac \neq 0, \\ 1 \leq i \leq n - 1. \end{matrix}$$

We get

$$\begin{aligned} \tau(e_i) &= [\tau(e_{i-1}), \tau(e_0)] = a^{i-1}ce_i + a^{i-2}bcb_{i-1,1}e_n, \\ 2 \leq i \leq n - 1, \tau(e_n) &= a^{n-1}ce_n. \end{aligned}$$

The chain of equalities

$$\begin{aligned} a^{n-1}cb'_{i,j}e_n &= b'_{i,j}\tau(e_n) = [\tau(e_i), \tau(e_j)] \\ &= [a^{i-1}ce_i + a^{i-2}bcb_{i-1,1}e_n, a^{j-1}ce_j + a^{j-2}bcb_{j-1,1}e_n] \\ &= a^{i+j-2}c^2[e_i, e_j] = a^{i+j-2}c^2b_{i,j}e_n, \end{aligned}$$

yields

$$b'_{i,j} = a^{i+j-1-n}cb_{i,j}. \tag{4.1}$$

Now consider

$$\sigma(a, k) = \begin{cases} \sigma(e_0) = e_0 + ae_k, & 2 \leq k \leq n - 1, \\ \sigma(e_1) = e_1, \\ \sigma(e_{i+1}) = [\sigma(e_i), \sigma(e_0)], & 1 \leq i \leq n - 1. \end{cases}$$

Using induction it is easy to see that

$$\sigma(e_i) = e_i + ab_{i-1,k}e_n, \quad 2 \leq i \leq n - 1, \quad \sigma(e_n) = e_n.$$

Then $[\sigma(e_i), \sigma(e_j)] = [e_i, e_j]$. So $b'_{i,j} = b_{i,j}$, that is the elementary transformations of second type do not affect $b_{i,j}$.

LEMMA 4.1.

$$b_{i+t,t} = (-1)^{s_0}b_{i+t-s_0,t+s_0} = \begin{cases} 0, & i = 2s_0, \\ (-1)^{s_0}b_{i+t-s_0,t+s_0}, & i = 2s_0 + 1. \end{cases}$$

PROOF. Since $a_{i,j}^k = 0$, due to (4) we get $b_{i+1,j} = -b_{i,j+1}$, $b_{j+2,j} = b_{j,j+2} = 0$. A sequential application of this completes the proof. \square

In Lemma 4.1, letting $t = 1$ and $i = 2k$, we get $b_{2k+1,1} = 0$, and so $b_{1,2k+1} = 0$. Consider the third type of transformation:

$$\varphi(c_k, k) = \begin{cases} \varphi(e_0) = e_0, \\ \varphi(e_1) = e_1 + c_k e_k, & 2 \leq k \leq n, \\ \varphi(e_{i+1}) = [\varphi(e_i), \varphi(e_0)], & 1 \leq i \leq n - 1. \end{cases}$$

Then $\varphi(e_i) = e_i + c_k e_{k+i-1}$, $2 \leq i \leq n + 1 - k$, and $\varphi(e_i) = e_i$, $n + 2 - k \leq i \leq n$. Now the chain of equalities

$$\begin{aligned} b'_{i,j} e_n &= b'_{i,j} \varphi(e_n) = [\varphi(e_i), \varphi(e_j)] = [e_i + c_k e_{k+i-1}, e_j + c_k e_{k+j-1}] \\ &= [e_i, e_j] + c_k [e_{k+i-1}, e_j] + c_k [e_i, e_{k+j-1}] + c_k^2 [e_{k+i-1}, e_{k+j-1}] \end{aligned}$$

yields

$$\begin{aligned} b'_{i,j} &= b_{i,j}, & k + i + j - 1 &\geq n + 1, \\ b'_{i,j} &= b_{i,j} + c_k (b_{k+i-1,j} + b_{i,k+j-1}), & k + i + j - 1 &\leq n, \\ b'_{i,j} &= b_{i,j} + c_k (b_{k+i-1,j} + b_{i,k+j-1}) + c_k^2 b_{k+i-1,k+j-1}, & n + 1 &\leq 2k + i + j - 2, \\ & & 2k + i + j - 2 &\leq n. \end{aligned}$$

Note that Lemma 4.1 implies the equalities

$$b_{k+i-1,j} = (-1)^{k-1} b_{i,k+j-1}.$$

As a result, for $b'_{i,j}$ one has the following form:

$$\begin{aligned} b'_{i,j} &= b_{i,j}, & n + 2 - (i + j) &\leq k, \\ b'_{i,j} &= b_{i,j} + c_k (1 + (-1)^{k-1}) b_{i,j+k-1}, \\ & \left[\frac{n + 1 - (i + j)}{2} \right] + 1 \leq k \leq n + 1 - (i + j), \\ b'_{i,j} &= b_{i,j} + c_k (1 + (-1)^{k-1}) b_{i,j+k-1} \\ & + c_k^2 (-1)^{k-1} b_{i,j+2k-2}, & k &\leq \left[\frac{n - (i + j)}{2} \right] + 1. \end{aligned}$$

Applying sequentially the third type transformations $\varphi(c_k, k)$ for $k = 2, 3, \dots, n$, we get the iteration

$$b_{i,j}^k = b_{i,j}^{k-1} + c_k (1 + (-1)^{k-1}) b_{i,j+k-1}^{k-1} + (-1)^{k-1} c_k^2 b_{i,j+2k-2}^{k-1}, \tag{4.2}$$

where the $b_{s,t}^{k-1}$ are supposed to be zero at $s + t \geq n + 1$ and $b_{i,j}^1 = b_{i,j}$.

Moreover, due to $b_{1,2t+1} = 0$ and $b_{i,j} = (-1)^{i-1} b_{1,i+j-1}$ (see Lemma 4.1), we can suppose, without loss of generality, that n is even and the iteration can be reduced to

$$\begin{aligned} b_{1,2i}^k &= b_{1,2i}^{k-1} + c_k (1 + (-1)^{k-1}) b_{1,2i+k-1}^{k-1} + (-1)^{k-1} c_k^2 b_{1,2i+2(k-1)}^{k-1}, \\ & k \geq 2, i = 1, \dots, n/2, \end{aligned} \tag{4.3}$$

where the $b_{1,t}^{k-1}$ are supposed to be zero at $t \geq n$ and $b_{1,2i}^1 = b_{1,2i}$.

Here is a computer program in Maple which performs the above iterative algorithm.

Calling Sequence:

`Iterate(Structure constants, Sequence, Dimension)`

Parameters:

`Structure constants` — initial structure constants.

`Sequence` — sequence of elementary transformations.

`Dimension` — The dimension of the given algebra — 1.

```

SC := Vector(n, symbol = b);
φ := Vector(n, symbol = B);
for i from 1 to n do
    if type(i, odd) then Bi := 0 fi;
    if i > n - 2 then SCi := 0 fi;
od;
Iterate := proc(SC, φ, n)
    local i;
    for i to trunc(n/2) do
        if 2i + k - 1 ≤ n then
            SC2i := SC2i + Bk((-1)k-1 + 1)SC2i+k-1;
        if 2i + 2k - 2 ≤ n then
            SC2i := SC2i + Bk2(-1)k-1SC2i+2k-2;
        fi;
    fi;
od;
end proc;
for k from 2 to n do Iterate(SC, φ, n) od;
print(SC);

```

4.1. Isomorphism criteria In this section, as an application of Sections 3 and 4, we give isomorphism criteria for the truncated filiform Leibniz algebras for some low-dimensional cases. Here those structure constants $b_{i,j}$ that are zero will be omitted.

(1) Two truncated filiform Leibniz algebras $a = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2})$ and $a' = (b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2})$ from $TLeib_5$ are isomorphic if and only if there exist $A_0, A_1, B_1 \in \mathbb{C}$ such that $A_0B_1 \neq 0$ and the following equalities hold:

$$\begin{aligned}
 b'_{0,0} &= \frac{A_0^2 b_{0,0} + A_0 A_1 b_{0,1} + A_1^2 b_{1,1}}{A_0^3 B_1}, & b'_{1,1} &= \frac{B_1 b_{1,1}}{A_0^3}, \\
 b'_{0,1} &= \frac{A_0 b_{0,1} + 2A_1 b_{1,1}}{A_0^3}, & b'_{1,2} &= \frac{B_1}{A_0^2} b_{1,2}.
 \end{aligned}$$

(2) Two truncated filiform Leibniz algebras $a = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2})$ and $a' = (b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2})$ from $TLeib_6$ are isomorphic if and only if there exist

$A_0, A_1, B_1 \in \mathbb{C}$ such that $A_0B_1 \neq 0$ and the following equalities hold:

$$b'_{0,0} = \frac{A_0^2b_{0,0} + A_0A_1b_{0,1} + A_1^2b_{1,1}}{A_0^4B_1}, \quad b'_{1,1} = \frac{B_1b_{1,1}}{A_0^4},$$

$$b'_{0,1} = \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0^4}, \quad b'_{1,2} = \frac{B_1}{A_0^3}b_{1,2}.$$

(3) Two truncated filiform Leibniz algebras $a = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4})$ and $a' = (b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}, b'_{1,4})$ from $TLeib_7$ are isomorphic if and only if there exist $A_0, A_1, B_1 \in \mathbb{C}$ such that $A_0B_1 \neq 0$ and the following equalities hold:

$$b'_{0,0} = \frac{A_0^2b_{0,0} + A_0A_1b_{0,1} + A_1^2b_{1,1}}{A_0^5B_1}, \quad b'_{1,1} = \frac{B_1b_{1,1}}{A_0^5},$$

$$b'_{0,1} = \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0^5}, \quad b'_{1,2} = \frac{B_1}{A_0^4}(b_{1,2} - (B_2^2 - 2B_3))b_{1,4}, \quad b'_{1,4} = \frac{B_1}{A_0^4}b_{1,4}.$$

(4) Two truncated filiform Leibniz algebras $a = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4})$ and $a' = (b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}, b'_{1,4})$ from $TLeib_8$ are isomorphic if and only if there exist $A_0, A_1, B_1 \in \mathbb{C}$ such that $A_0B_1 \neq 0$ and the following equalities hold:

$$b'_{0,0} = \frac{A_0^2b_{0,0} + A_0A_1b_{0,1} + A_1^2b_{1,1}}{A_0^6B_1}, \quad b'_{1,1} = \frac{B_1b_{1,1}}{A_0^6},$$

$$b'_{0,1} = \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0^6}, \quad b'_{1,2} = \frac{B_1}{A_0^5}(b_{1,2} - (B_2^2 - 2B_3))b_{1,4}, \quad b'_{1,4} = \frac{B_1}{A_0^5}b_{1,4}.$$

(5) Two truncated filiform Leibniz algebras $a = (b_{0,0}, b_{0,1}, b_{1,1}, b_{1,2}, b_{1,4}, b_{1,6})$ and $a' = (b'_{0,0}, b'_{0,1}, b'_{1,1}, b'_{1,2}, b'_{1,4}, b'_{1,6})$ from $TLeib_9$ are isomorphic if and only if there exist $A_0, A_1, B_1 \in \mathbb{C}$ such that $A_0B_1 \neq 0$ and the following equalities hold:

$$b'_{0,0} = \frac{A_0^2b_{0,0} + A_0A_1b_{0,1} + A_1^2b_{1,1}}{A_0^7B_1}, \quad b'_{1,1} = \frac{B_1b_{1,1}}{A_0^7},$$

$$b'_{0,1} = \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0^7},$$

$$b'_{1,2} = \frac{B_1}{A_0^6}(b_{1,2} - (B_2^2 - 2B_3)b_{1,4} + (B_3^2 - 2B_2^2B_3 + 2B_5)b_{1,6}),$$

$$b'_{1,4} = \frac{B_1}{A_0^6}(b_{1,4} - (B_2^2 - 2B_3))b_{1,6}, \quad b'_{1,6} = \frac{B_1}{A_0^6}b_{1,6}.$$

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