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SIGNED SUMS OF RECIPROCALS, I

Dedicated to George Szekeres on his 65th birthday

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Abstract

The author investigates $M(n) = \min|\Sigma \eta_k k^{-1}|$ where the minimum is over all sets of signs $\eta_j = \pm 1$ and shows $M(n) < n^{\frac{1}{2}-(1-\varepsilon)\log_2 n}$.

R. R. Hall recently suggested the problem of finding an upper bound for

$$M(n) = \min \left| \sum_{1 \leq k \leq n} \frac{\eta_k}{k} \right|$$

where the minimum is taken over all sets $\eta_1, \cdots, \eta_n$ with each η_i being ± 1. Trivially, on writing all terms $1/k$ as rationals with denominator l.c.m. $(1, 2, \cdots, n)$ the numerator of $\Sigma \eta_k / k$ is odd so $M(n)$ is certainly non-zero. On the other hand it is easily shown by induction that $M(n) < 1/n$. We show here that this can be improved considerably.

THEOREM. *For real $\varepsilon > 0$ there exists real $N(\varepsilon)$ such that*

$$M(n) < 1/n^{\frac{1}{2}(1-\varepsilon)\log_2 n}$$

for $n > N(\varepsilon)$, where $\log_2$ denotes the base 2 logarithm.

The essential part of the proof is a variant of the mean value theorem of calculus. Before giving this we introduce some notation. For a given function $f = f(x)$ define $H_1(f)$ by

$$H_1(f)(x) = f(x+1) - f(x).$$

Further functions $H_2(f), \cdots, H_r(f), \cdots$ are defined inductively by

$$H_n(f)(x) = H_{n-1}(g)(\tfrac{1}{2}x)$$

where

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$$g(x) = H_1(f)(2x).$$

The following lemma gives some inkling of the relevance of the above to the problem being considered.

LEMMA 1. *Let* $\varepsilon_j = (-1)^{d(j)}$ *where* $d(j)$ *denotes the sum of the digits of the binary expansion of* j. *Then*

$$H_n(f)(x) = (-1)^n \sum_{0 \leq j \leq 2^n - 1} \varepsilon_j f(x+j).$$

PROOF. The lemma is easily checked in the case $n = 1$. Now suppose it is true in the case $n = t$. Then

$$H_{t+1}(f)(x) = H_t(g)(\tfrac{1}{2}x) = (-1)^t \sum_{0 \leq j \leq 2^t - 1} \varepsilon_j g(\tfrac{1}{2}x + j)$$

$$= (-1)^t \sum_{0 \leq j \leq 2^t - 1} \varepsilon_j (f(2(\tfrac{1}{2}x + j) + 1) - f(2(\tfrac{1}{2}x + j)))$$

$$= (-1)^{t+1} \left\{ \sum_{0 \leq j \leq 2^t - 1} f(x+2j)\varepsilon_j - \sum_{0 \leq j \leq 2^t - 1} f(x + 2j + 1)\varepsilon_j \right\}$$

$$= (-1)^{t+1} \sum_{0 \leq j \leq 2^{t+1} - 1} \varepsilon_j f(x+j)$$

since $\varepsilon_{2j} = \varepsilon_j$ and $\varepsilon_{2j+1} = -\varepsilon_j$. This completes the proof of the lemma by induction.

We now obtain an estimate for $H_n(f)(t)$. It is possible to give a series expansion for $H_n(f)(a)$ of the type

$$H_n(f)(a) = 2^{n(n-1)/2} f^{(n)}(a + \tfrac{1}{2}(2^n - 1)) + c_{n+1} f^{(n+1)}(a + \tfrac{1}{2}(2^n - 1)) + \cdots$$

and estimate the coefficients $c_{n+1}, c_{n+2}, \cdots$, but this does not seem to give any better result when applied to the problem in hand than the following, suggested by R. R. Hall.

LEMMA 2. *Let* f *be a function with derivative* $f^{(n)}$ *of order* n *existing on* $(a, a + 2^n - 1)$ *and* $f^{(n-1)}$ *continuous on* $[a, a + 2^n - 1]$. *Then*

$$H_n(f)(a) = 2^{n(n-1)/2} f^{(n)}(a + \theta)$$

for some $\theta \in (0, 2^n - 1)$.

PROOF. The case $n = 1$ is just the mean value theorem of calculus. Now suppose the lemma is true for $n = t$ and that the conditions of the lemma hold for $n = t + 1$. Then for g the conditions hold at $\tfrac{1}{2}a$ with $n = t$ so

$$H_{t+1}(f)(a) = H_t(g)(\tfrac{1}{2}a) = 2^{t(t-1)/2} g^{(t)}(\tfrac{1}{2}a + \theta_1)$$

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where $\theta_1 \in (0, 2^t - 1)$. But

$$g^{(t)}(x) = 2^t(f^{(t)}(2x+1) - f^{(t)}(2x))$$

for all $x \in (\frac{1}{2}a, \frac{1}{2}a + 2^t - 1)$, so by the mean value theorem

$$g^{(t)}(\tfrac{1}{2}a + \theta_1) = 2^t(f^{(t+1)}(a + 2\theta_1 + \phi_1))$$

where $\phi_1 \in (0,1)$. Hence

$$H_{t+1}(f)(a) = 2^{(t+1)(t)/2} f^{(t+1)}(a + \theta)$$

where $\theta = 2\theta_1 + \phi_1 \in (0, 2^{t+1} - 1)$, completing the proof of the lemma by induction.

In order to apply lemma 2 to the sum in question we need to show that the relatively large initial terms can be ignored.

Lemma 3. *If* $1 \leqq k \leqq \frac{1}{2}n + 1$ *then* $M(n) \leqq M_k(n)$, *where*

$$M_k(n) = \min \left| \sum_{k \leqq j \leqq n} \frac{\eta_j}{j} \right|,$$

the minimum being over all sets $\eta_k, \cdots, \eta_n$ *with each* η_j *being* ± 1.

Proof. The case $k = 1$ is trivial. That $M_k(n) \leqq M_{k+1}(n)$ for $k + 1 \leqq \frac{1}{2}n + 1$ follows trivially on observing that the substitution

$$\eta_{2k}/2k = \eta_{2k}/k - \eta_{2k}/2k$$

converts a sum $\Sigma_{k+1 \leqq j \leqq n}\, \eta_j/j$ to one of the form $\Sigma_{k \leqq j \leqq n}\, \eta_j/j$, so the lemma follows by induction.

Proof of the theorem. Let m be such that $n/2 < 2^{m+2} \leqq n$ and set $l = [n/2^{m+1}] + 1$. It is easily seen that

$$\begin{aligned} &| \pm H_m(f)(n+1-2^m) \pm H_m(f)(n+1-2\cdot 2^m) \pm \cdots \pm H_m(f)(n+1-l\cdot 2^m)| \\ &\qquad \leqq \max_{1 \leqq j \leqq l} |H_m(f)(n+1-j2^m)| \end{aligned}$$

for suitable choices of sign. Taking $f(x) = 1/x$, and using lemmas 1 and 3 the left side of the above inequality is at least as big as $M(n)$ while the right side is, by lemma 2, bounded above by $2^{m(m-1)/2} m!/(\frac{1}{4}n)^{m+1}$. Hence

$$M(n) \leqq 2^{m(m-1)/2} m!/(\tfrac{1}{4}n)^{m+1}.$$

Using the inequality $\log_2 n - 3 < m \leqq \log_2 n - 2$ we have

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$$2^{m(m-1)/2} \leqq \left(\frac{n}{4}\right)^{1/2(\log_2 n - 3)} < 8n^{1/2\log_2 n}$$

$$m! \leqq (\log_2 n)^{\log_2 n} = n^{\log_2 \log_2 n},$$

$$4^{m+1} < n^2 \quad \text{and} \quad n^{m+1} > n^{\log_2 n - 2}.$$

Thus

$$M(n) < n^{-\frac{1}{2}\log_2 n}\,(8n^{4+\log_2\log_2 n})$$

which clearly gives the theorem.

Erdos (1972) has stated the problem: Let T_n denote the fractional part of $\frac{1}{2} + \frac{1}{3} + \cdots + 1/n$. Does there exist $n \geqq 5$ for which $T_n = 1/[2, 3, \cdots, n]$ where $[2, 3, \cdots, n]$ denotes the least common multiple of $2, 3, \cdots, n$. Along similar lines is the problem: Does there exist $n \geqq 5$ for which $M(n) = 1/[2, 3, \cdots, n]$?

Reference

P. Erdos (1972), *Elemente der Mathematik* **27**, 68.

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