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Perturbations of random matrix products in a reducible case

YURI KIFER AND ERIC SLUD

Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel; and Department of Mathematics, University of Maryland, College Park, Maryland 20742, *USA*

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Dedicated to the memory of V. M. Alexeyev

Abstract. It is known that for any sequence $X_1, X_2 \dots,$ of identically distributed independent random matrices with a common distribution μ the limit

$$\Lambda(\mu) = \lim_{n\to\infty} n^{-1} \log \|X_n \cdots X_1\|$$

exists with probability 1. We study some conditions under which $\Lambda(\mu_k) \to \Lambda(\mu)$ provided $\mu_k \to \mu$ in the weak sense.

1. *Introduction*

Let $X_1, X_2, \dots$ be a sequence of identically distributed independent random $m \times m$ real matrices with common distribution μ on the unimodular group SL (m, R). Under the assumption that

$$\int \log \|g\| \mu(dg) < \infty \tag{1.1}$$

Furstenberg and Kesten [**6**] showed that

$$\Lambda(\mu) = \lim_{n\to\infty} n^{-1} \log \|X_n \cdots X_1\| \tag{1.2}$$

exists with probability 1 and is almost surely (a.s.) constant.

Because of the applications of random matrix products to physical and to population processes (e.g., see [**3**] and [**8**]), it is of interest to understand when $\Lambda(\mu)$ is stable under perturbations of μ, say, in the weak topology of measures.

In the case when the support of μ is irreducible (in the sense that the minimal closed subgroup of SL (m, R) containing the support of μ leaves no proper subspace of R^m invariant) Kifer [**9**] has applied Furstenberg's formula [**7**] to show that if μ_k converges weakly to μ ($\mu_k \overset{w}{\to} \mu$) then $\Lambda(\mu_k) \to \Lambda(\mu)$ as $k \to \infty$.

In the present paper, we consider the quite different case of μ supported on a reducible subgroup of SL (m, R) and prove under certain assumptions on μ and μ_k that $\mu_k \overset{w}{\to} \mu$ implies $\Lambda(\mu_k) \to \Lambda(\mu)$. Slud [**12**] had previously shown $\Lambda(\mu_k) \to \Lambda(\mu)$

in the special case when $m=2$, μ has support on two diagonal matrices, and μ_k is the convolution of μ with the random rotation uniformly distributed in the orthogonal group SO $(2, \mathbb{R})$ on the $1/k$-neighbourhood of the identity.

The counterexample of § 2 in [9] shows that in general the convergence $\Lambda(\mu_k)\rightarrow\Lambda(\mu)$ does not take place and so in the reducible case some assumptions are needed. In § 2 we formulate our conditions on measures and the main result of this paper. In § 3 we prove auxiliary lemmas and in § 4 we establish our theorem. Finally, in § 5 we indicate some classes of examples fulfilling our hypotheses. In particular, our assumptions on μ are satisfied if μ is supported on a commutative subgroup of SL (m, R).

Our result, which extends the work begun in [9] and [12], also has some connection with the problem considered by Ruelle in [11].

2. *Assumptions and the main theorem*

Let μ be a Borel probability measure on SL (m, R) with a compact support satisfying the following properties.

(A_1). There exist two subspaces $\Gamma_{\max}$ and $\Gamma_{\min}$ left invariant by all matrices from the support of μ (and so by all matrices of the smallest closed subgroup G_μ containing supp μ) such that

$$R^m=\Gamma_{\max}\oplus\Gamma_{\min}.$$

(A_2). For any $\delta>0$,

$$\lim_{n\rightarrow\infty}\sup_{x\in\Gamma_{\max}\cap S}P\{\log\|X_n\cdots X_1x\|<(\Lambda(\mu)-\delta)n\}=0,$$

where $X_1, X_2, \ldots$ are identically distributed random matrices with the common distribution μ, $P\{\cdot\}$ is the probability of an event in brackets, $S=\{z\in R^m: \|z\|=1\}$ and the norm $\|\cdot\|$ is Euclidean.

(A_3). There exists $\gamma>0$ such that

$$\liminf_{n\rightarrow\infty} n^{-1}E\log\|\Pi_{\min}X_n\cdots X_1\|<\Lambda(\mu)-3\gamma,$$

where E denotes the expectation and $\Pi_{\max}$ and $\Pi_{\min}$ are the projection operators on $\Gamma_{\max}$ and $\Gamma_{\min}$ so that for any $y\in R^m$ one has

$$\Pi_{\max}y\in\Gamma_{\max},\quad \Pi_{\min}y\in\Gamma_{\min}\quad\text{and}\quad y=\Pi_{\max}y+\Pi_{\min}y.$$

The counterexample of [9] shows that the conditions (A_1)–(A_3) are not enough to obtain the desired convergence. So we make the following additional assumptions on the 'perturbed' measures μ_k, $k=1, 2, \ldots$:

(B_1).
$$\sup_{h\in\operatorname{supp}\mu_k}\ \inf_{g\in\operatorname{supp}\mu}\|h-g\|\leq\frac{1}{k},$$

(B_2).
$$\mu_k\xrightarrow{w}\mu\ \text{as}\ k\rightarrow\infty.$$

(B_3). There exist α, β, R_1, $R_2>0$ such that for any $\varepsilon>0$ and $z\in S$ satisfying the property

$$\|\Pi_{\max}z\|/\|\Pi_{\min}z\|\leq\beta$$

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the following holds:

$$P\Big\{\text{there exists } n \le R_1 \log k \text{ so that}$$
$$\frac{\|\Pi_{\max} X_n^{(k)} \cdots X_1^{(k)} z\|}{\|\Pi_{\min} X_n^{(k)} \cdots X_1^{(k)} z\|} - \frac{\|\Pi_{\max} z\|}{\|\Pi_{\min} z\|} \ge \frac{1}{kR_2}\Big\} \ge \alpha,$$

where $X_1^{(k)}, X_2^{(k)}, \ldots$ are identically distributed independent random matrices with the common distribution μ_k on $\mathrm{SL}(m, R)$.

We show in § 5 that the assumptions (A_1)–(A_3) are satisfied, for instance, if $\operatorname{supp} \mu$ is contained in a commutative subgroup. The assumption (B_3) is unwieldy, so we discuss it here to show that it is actually a rather weak regularity condition. For example, (B_3) is satisfied if the measures μ_k are convolutions of μ with measures η_k which are concentrated in $1/k$-neighbourhoods of the identity matrix and have positive density $p_k(g)$ with respect to the Haar measure on a compact subgroup of $\mathrm{SL}(m, R)$ acting transitively on the sphere S, provided for any $g_1, g_2 \in \operatorname{supp} \eta_k$ and some constant $\tilde{c} > 0$

$$\tilde{c}^{-1} \le p_k(g_1)/p_k(g_2) \le \tilde{c} < \infty.$$

Indeed, in this case one can write $X_1^{(k)} = U^{(k)} \cdot X_1$, where X_1 and $U^{(k)}$ are independent and have distributions μ and η_k, respectively. Define

$$A_{z,R} = \left\{ g \in \mathrm{SL}(m, \mathbb{R}) \colon \frac{\|\Pi_{\max} gz\|}{\|\Pi_{\min} gz\|} - \frac{\|\Pi_{\max} z\|}{\|\Pi_{\min} z\|} \ge \frac{1}{kR} \right\}.$$

One can see that for some positive R_2 and $\tilde{\alpha}$ independent of z, the intersection of A_{z,R_2} and $1/k$-neighbourhood of the identity matrix has Haar measure bigger than $\tilde{\alpha}/k$. Thus by the definition of the measure η_k one has

$$P\{U^{(k)} \in A_{z,R_2}\} \ge \alpha$$

for some α independent of z and k. Therefore the same is true if we replace z by gz for any $g \in \mathrm{SL}(m, \mathbb{R})$. Since X_1 and $U^{(k)}$ are independent we can write also $X_1 z$ in place of gz to obtain

$$P\left\{ \frac{\|\Pi_{\max} X_1^{(k)} z\|}{\|\Pi_{\min} X_1^{(k)} z\|} - \frac{\|\Pi_{\max} z\|}{\|\Pi_{\min} z\|} \ge \frac{1}{kR_2} \right\} \ge \alpha$$

which is a special case of (B_3) with $n = 1$.

Our main result is the following

THEOREM. *Let μ and $\{\mu_k\}_{k=1}^{\infty}$ satisfy* (A_1)–(A_3) *and* (B_1)–(B_3), *respectively. Then*

$$\Lambda(\mu_k) \to \Lambda(\mu) \quad \textit{as } k \to \infty.$$

Remark. It is well known (cf. [**5**]) that (B_2) together with proposition 1 and theorem 2 of [**5**] also imply that independent identically distributed sequences $\{X_i\}$ with law μ and $\{X_i^{(k)}\}$ with law μ_k, $k = 1, 2, \ldots$ can be constructed on the same probability space with

$$P\{\|X_i - X_i^{(k)}\| \ge \alpha(k)\} \le \beta(k),$$

where

$$\lim_{k \to \infty} \alpha(k) = \lim_{k \to \infty} \beta(k) = 0.$$

Without loss of generality we assume that

$$P\left\{\|X_i - X_i^{(k)}\| \ge \frac{1}{k}\right\} \le \frac{1}{k} \tag{2.1}$$

since otherwise one can pass to a subsequence.

Conjecture. The theorem remains true without the assumption (B_3).

3. *Auxiliary lemmas*

LEMMA 1. *If $\mu_k \xrightarrow{w} \mu$ and the supports of all measures μ_k are contained in one compact set then*

$$\limsup_{k\to\infty} \Lambda(\mu_k) \le \Lambda(\mu). \tag{3.1}$$

The proof is very easy and can be found in the introduction of [9].

For any probability measure η on SL(m, R) and each probability measure ν on S we define the measure $\eta * \nu$ on S by the formula

$$\int f(z)\eta * \nu(dz) = \iint f(gz/\|gz\|)\eta(dg)\nu(dz) \tag{3.2}$$

which holds for any Borel function f on S. Here and in what follows we omit the space of integration if it is the whole sphere S or the space SL(m, R). From now on we assume that (A_1)–(A_3) and (B_1)–(B_3) are satisfied.

LEMMA 2. *If $\mu * \nu = \nu$ then*

$$\nu((\Gamma_{\max} \cup \Gamma_{\min}) \cap S) = 1. \tag{3.3}$$

Proof. If $\Gamma_{\min}$ is trivial i.e. $\Gamma_{\max} = R^m$, then (3.3) is trivially true. Thus we assume for the proof of this lemma that $\Gamma_{\min}$ is not the zero subspace of R^m.

By (A_2) and (A_3) it follows that

$$\lim_{n\to\infty} \sup_{z\notin\Gamma_{\max}\cup\Gamma_{\min}} P\{W(X_n\cdots X_1 z) - W(z) < 2\gamma n\} = 0 \tag{3.4}$$

where for any $z \ne 0$

$$W(z) = \begin{cases} \log \dfrac{\|\Pi_{\max} z\|}{\|\Pi_{\min} z\|}, & \text{if } z \notin \Gamma_{\max} \cup \Gamma_{\min}; \\ \infty, & \text{if } z \in \Gamma_{\max}; \\ -\infty, & \text{if } z \in \Gamma_{\min}. \end{cases} \tag{3.5}$$

Indeed, clearly

$$\|\Pi_{\min} X_n \cdots X_1\| \cdot \|\Pi_{\min} z\| \ge \|\Pi_{\min} X_n \cdots X_1 z\|. \tag{3.6}$$

Therefore

$$W(X_n\cdots X_1 z) - W(z) \ge \log(\|\Pi_{\max} X_n \cdots X_1 z\|/\|\Pi_{\max} z\|) \\ - \log\|\Pi_{\min} X_n \cdots X_1\|. \tag{3.7}$$

But results of [6] applied to the product of matrices $X_1, X_2, \ldots$ restricted to $\Gamma_{\min}$ together with (A_3) imply that with probability 1,

$$\lim_{n\to\infty} n^{-1} \log\|\Pi_{\min} X_n \cdots X_1\| < \Lambda(\mu) - 3\gamma. \tag{3.8}$$

Now (A_2), (3.7) and (3.8) yield (3.4).

Therefore if Q is a compact subset of S such that

$$Q \cap (\Gamma_{\max} \cup \Gamma_{\min}) = \emptyset$$

then for any $z \notin \Gamma_{\max} \cup \Gamma_{\min}$

$$\lim_{n\to\infty} P\{X_n \cdots X_1 z / \|X_n \cdots X_1 z\| \in Q\} = 0. \tag{3.9}$$

But if $\mu * \nu = \nu$ then also

$$\mu^{*n} * \nu = \nu \tag{3.10}$$

where μ^{*n} is n-fold convolution $\mu * \cdots * \mu$. Therefore

$$\nu(Q) = \int_S \nu(dz) P\left\{\frac{X_n \cdots X_1 z}{\|X_n \cdots X_1 z\|} \in Q\right\}$$
$$= \int_{S \setminus (\Gamma_{\max} \cup \Gamma_{\min})} \nu(dz) P\left\{\frac{X_n \cdots X_1 z}{\|X_n \cdots X_1 z\|} \in Q\right\}, \tag{3.11}$$

where we have used (A_1) to say that if

$$X_n \cdots X_1 z \notin \Gamma_{\max} \cup \Gamma_{\min}$$

then $z \notin \Gamma_{\max} \cup \Gamma_{\min}$. Letting $n \to \infty$ in (3.11) one gets from (3.9) that $\nu(Q) = 0$ and so (3.3) is true.

Define the family of Markov chains $\{Z_n^{(k)}\}$ on S by

$$Z_n^{(k)} = \frac{X_n^{(k)} Z_{n-1}^{(k)}}{\|X_n^{(k)} Z_{n-1}^{(k)}\|} = X_n^{(k)} \cdots X_1^{(k)} Z_0^{(k)} (\|X_n^{(k)} \cdots X_1^{(k)} Z_0^{(k)}\|)^{-1} \tag{3.12}$$

where $Z_0^{(k)}$ is chosen to be independent of $X_1^{(k)}, X_2^{(k)}, \ldots$. Substituting here X_i in the place of $X_i^{(k)}$ and Z_0 in the place of $Z_0^{(k)}$ we define analogously the Markov chain $\{Z_n\}$. Let $q_k(x, \Gamma)$ be the transition function of $\{Z_n^{(k)}\}$ and $q(x, \Gamma)$ be the transition function of Y_n, i.e.,

$$q_k(x, \Gamma) = P\left\{\frac{X_1^{(k)} x}{\|X_1^{(k)} x\|} \in \Gamma\right\} \quad \text{and} \quad q(x, \Gamma) = P\left\{\frac{X_1 x}{\|X_1 x\|} \in \Gamma\right\}, \tag{3.13}$$

Since S is compact, the Markov chains $\{Z_n^{(k)}\}$ have invariant measures ν_k (see [**4**]) i.e. measures satisfying

$$\nu_k(\Gamma) = \int_S \nu_k(dx) q_k(x, \Gamma) \tag{3.14}$$

or, equivalently,

$$\mu_k * \nu_k = \nu_k. \tag{3.15}$$

LEMMA 3. *Let* $\nu_{k_i} \xrightarrow{w} \nu$ *as* $k_i \to \infty$ *and* $\mu_{k_i} * \nu_{k_i} = \nu_{k_i}$ *for all* $i = 1, 2, \ldots$. *If* $\nu(\Gamma_{\max} \cap S) = 1$ *then* $\Lambda(\mu_{k_i}) \to \Lambda(\mu)$ *as* $k_i \to \infty$.

Proof. Assume that ν_{k_i} is an ergodic invariant measure of the Markov chain $\{Z_n^{(k_i)}\}$ and $Z_0^{(k_i)}$ is a random point of S with the law ν_{k_i} and is independent of $X_1^{(k_i)}, X_2^{(k_i)}, \ldots$. Then $\{Z_n^{(k_i)}\}$ is an ergodic stationary process and by the Birkhoff ergodic theorem with probability 1,

$$\lim_{n\to\infty} n^{-1}\log\|X_n^{(k_i)}\cdots X_1^{(k_i)}Z_0^{(k_i)}\| = \lim_{n\to\infty} n^{-1}\sum_{j=1}^{n}\log\|X_j^{(k_i)}Z_{j-1}^{(k_i)}\|$$

$$= \iint \log\|gz\|\mu_{k_i}(dg)\nu_{k_i}(dz). \tag{3.16}$$

But the left hand side of (3.16) is less than or equal to $\Lambda(\mu_{k_i})$. Therefore

$$\Lambda(\mu_{k_i}) \geq \iint \log\|gz\|\mu_{k_i}(dg)\nu_{k_i}(dz) \tag{3.17}$$

for any invariant ergodic measure ν_{k_i}. Thus (3.17) holds for any invariant measure ν_{k_i} since the ergodic invariant measures are extremals of the convex set of all invariant probability measures.

Since $\mu_{k_i} \overset{w}{\to} \mu$ and $\nu_{k_i} \overset{w}{\to} \nu$, also (see [**1**, theorem 3.2]) $\mu_{k_i}\times\nu_{k_i}\to\mu\times\nu$ and letting $i\to\infty$ in (3.17) one obtains

$$\liminf_{i\to\infty}\Lambda(\mu_{k_i}) \geq \iint \log\|gz\|\mu(dg)\nu(dz) \tag{3.18}$$

where we make use of the compact support of μ and the inclusion

$$\text{supp}\,\mu_{k_i} \subset \{g : \text{dist}\,(g, \text{supp}\,\mu) \leq k_i^{-1}\} \tag{3.19}$$

which follows from (B_1).

If $\mu_{k_i} * \nu_{k_i} = \nu_{k_i}$ then also $\mu_{k_i}^{*n} * \nu_{k_i} = \nu_{k_i}$ and in the same way as above one can prove

$$\liminf_{i\to\infty}\Lambda(\mu_{k_i}^{*n}) \geq \iint \log\|g_n\cdots g_1 z\|\mu(dg_1)\cdots\mu(dg_n)\nu(dz)$$

$$= \int_{\Gamma_{\max}\cap S} E\log\|X_n\cdots X_1 z\|\nu(dz), \tag{3.20}$$

since we assume $\nu(\Gamma_{\max}\cap S) = 1$.

It is easy to see that

$$\Lambda(\mu_{k_i}^{*n}) = n\Lambda(\mu_{k_i}). \tag{3.21}$$

Then by (3.20) and (3.21),

$$\liminf_{i\to\infty}\Lambda(\mu_{k_i}) \geq \lim_{n\to\infty} n^{-1}\int_{\Gamma_{\max}\cap S} E\log\|X_n\cdots X_1 z\|\nu(dz). \tag{3.22}$$

But

$$\lim_{n\to\infty} n^{-1}E\log\|X_n\cdots X_1 z\| = \Lambda(\mu) \tag{3.23}$$

boundedly for any $z\in\Gamma_{\max}\cap S$. Indeed, since supp μ is compact, there exists a constant $M>0$ such that with probability 1

$$-nM \leq \log\|X_n\cdots X_1 y\| \leq nM \quad \text{for all } y\in S \text{ and } n\geq 1 \tag{3.24}$$

and for $z\in\Gamma_{\max}\cup S$ by (1.2) and (A_2),

$$n^{-1}\log\|X_n\cdots X_1 z\|$$

converges in probability to $\Lambda(\mu)$ as $n\to\infty$. By the dominated convergence theorem, (3.23) follows.

By hypothesis $\nu(\Gamma_{\max} \cap S) = 1$, and then (3.22)–(3.24) imply

$$\liminf_{i\to\infty} \Lambda(\mu_{k_i}) \geq \Lambda(\mu),$$

which together with (3.1) proves lemma 3. □

Define the set $\mathcal{M}$ of probability measures on S by

$$\mathcal{M} = \{\nu : \exists\{k_i\} \text{ such that } \nu_{k_i} \xrightarrow{w} \nu, \text{ where } \nu_{k_i} \text{ satisfy (3.15)}\}.$$

COROLLARY. *If all $\nu \in \mathcal{M}$ have the property $\nu(\Gamma_{\max} \cap S) = 1$ then*

$$\Lambda(\mu_k) \to \Lambda(\mu) \text{ as } k \to \infty.$$

Proof. Suppose that for some subsequence $\{k_i\}$

$$\lim_{i\to\infty} \Lambda(\mu_{k_i}) = \lambda \neq \Lambda(\mu). \tag{3.25}$$

The sequence of measures ν_{k_i} on the compact set S is compact and so there is a subsequence $\{k_{i_j}\}$ such that

$$\nu_{k_{i_j}} \xrightarrow{w} \tilde{\nu} \quad \text{as } j \to \infty,$$

where $\tilde{\nu} \in \mathcal{M}$ by the definition of $\mathcal{M}$. By hypothesis $\tilde{\nu}(\Gamma_{\max} \cap S) = 1$ and lemma 3 implies

$$\Lambda(\mu_{k_{i_j}}) \to \Lambda(\mu) \quad \text{as } j \to \infty$$

contradicting (3.25) and proving the corollary. □

Remark. It is well known in the present setting (see e.g. [**9**, formulae (1.19), (1.20)]), and follows easily from

$$\mu_{k_i} \times \nu_{k_i} \xrightarrow{w} \mu \times \nu,$$

that $\nu \in \mathcal{M}$, $\nu_{k_i} \xrightarrow{w} \nu$ and (3.15) together imply

$$\mu * \nu = \nu. \tag{3.26}$$

4. *Proof of theorem*

By the corollary of the previous section it suffices to prove that if $\nu \in \mathcal{M}$ then

$$\nu(\Gamma_{\max} \cap S) = 1. \tag{4.1}$$

This is, clearly, true when $\Gamma_{\min}$ is trivial i.e.,

$$\Gamma_{\max} = R^m,$$

and so it remains to consider the case of non-trivial $\Gamma_{\min}$.

Take now an arbitrary $\nu \in \mathcal{M}$. This means that there exists a subsequence $\{k_i\}$ such that

$$\nu_{k_i} \xrightarrow{w} \nu \quad \text{as } i \to \infty \tag{4.2}$$

and all ν_{k_i} satisfy the property

$$\mu_{k_i} * \nu_{k_i} = \nu_{k_i}. \tag{4.3}$$

By (3.4) one can find $N>0$ such that

$$\sup_{z\notin\Gamma_{\max}\cup\Gamma_{\min}} P\{W(X_N\cdots X_1z)-W(z)<2\gamma N\}=p, \tag{4.4}$$

where $p>0$ satisfies the property

$$2(2p)^{\gamma/\gamma+3M}\equiv\rho<1, \tag{4.5}$$

$W(z)$ is defined by (3.5) and M is the same as in (3.24).

From (3.24) it follows easily that

$$-2nM\leq W(X_n\cdots X_1z)-W(z)\leq 2nM \text{ (a.s.)} \tag{4.6}$$

for any $z\notin\Gamma_{\max}\cup\Gamma_{\min}$ and all $n=1,2,\ldots$.

By (4.4) and (4.6), employing (B_1), (B_2) and

$$\|X_n^{(k)}\cdots X_1^{(k)}-X_n\cdots X_1\|\leq e^{2Mn}n/k,$$

one can see that there exist $k_0>0$ and $\mathcal{D}>0$ so large that if $k\geq k_0$ then

$$-3NM\leq W(X_n^{(k)}\cdots X_1^{(k)}z)-W(z)\leq 3NM \text{ (a.s.)} \tag{4.7}$$

for all $n=1,\ldots,N$ and

$$P\{W(X_N^{(k)}\cdots X_1^{(k)}z)-W(z)<\gamma N\}\leq 2p \tag{4.8}$$

for any $z\neq 0$ belonging to the domain

$$\mathcal{U}_1(k)=\{z:|W(z)|\leq|\log\mathcal{D}/k|\}. \tag{4.9}$$

In what follows we shall assume that k is big enough so that (4.7) and (4.8) hold for all $z\in\mathcal{U}_1(k)$.

Let $P_z^{(k)}\{\cdot\}$ be the probability of an event in brackets under the condition $Z_0^{(k)}=z$. Then we can rewrite (4.8) as follows

$$P_z^{(k)}\{W(Z_N^{(k)})-W(z)<\gamma N\}\leq 2p \quad \text{for any } z\in\mathcal{U}_1(k). \tag{4.10}$$

We need the following technical result.

Lemma 4. *For any number L, integer $n>0$ and point $z\in\mathcal{U}_1(k)$,*

$$\begin{aligned} Q(z,n,L)\equiv P_z^{(k)}\{W(Z_n^{(k)})-W(z)\leq L \text{ and } Z_j^{(k)}\in\mathcal{U}_1(k) \\ \text{for all } j=0,1,\ldots,n-1\} \\ \leq 2\rho^{[n/N]}(\rho/2)^{-(L+3NM)/\gamma N}, \end{aligned} \tag{4.11}$$

where ρ is given by (4.5) and $[a]$ denotes the integral part of a number a.

Proof. Let $l=[n/N]$ and

$$n=lN+d, \tag{4.12}$$

where $d\geq 0$ is an integer less than N. Let $j_1<j_2<\cdots<j_s<l$ be the random numbers such that

$$W(Z_{(j_i+1)N}^{(k)})-W(Z_{j_iN}^{(k)})<\gamma N \tag{4.13}$$

for all $i=1,\ldots,s$ and

$$W(Z_{(j+1)N}^{(k)})-W(Z_{jN}^{(k)})\geq\gamma N \tag{4.14}$$

if $0\leq j<l$ and $j\neq j_i$, $i=1,\ldots,s$.

Since we assume that

$$L > W(Z_n^{(k)}) - W(z)$$
$$= W(Z_n^{(k)}) - W(Z_{n-d}^{(k)}) + \sum_{j=0}^{l-1} (W(Z_{(j+1)N}^{(k)}) - W(Z_{jN}^{(k)})), \quad (4.15)$$

where $Z_0^{(k)} = z$, then by (4.7),

$$-3NM - 3NMs + \gamma N(l-s) \le L$$

and so

$$s \ge \frac{l\gamma}{\gamma + 3M} - \frac{L + 3NM}{N(\gamma + 3M)} \equiv r. \quad (4.16)$$

For any integers (non-random) $j_1, j_2, \ldots, j_s$ such that $0 \le j_1 < j_2 < \cdots < j_s < l$ define

$$Q_{j_1,\ldots,j_s}(z) \equiv P_z^{(k)}\{W(Z_{(j_i+1)N}^{(k)}) - W(Z_{j_iN}^{(k)}) < \gamma N$$
$$\text{and } Z_{j_iN}^{(k)} \in \mathcal{U}_1(k) \text{ for all } i = 1, \ldots, s\}. \quad (4.17)$$

Then by (4.16) and the definition of $Q(z, n, L)$ one can easily see that

$$Q(z, n, L) \le \sum_{l > s \ge r} \sum_{0 \le j_1 < \cdots < j_s < l} Q_{j_1,\ldots,j_s}(z). \quad (4.18)$$

Employing s times the Markov property of the process $\{Z_j^{(k)}\}$ in the expression (4.17) one has by (4.10) for all $z \in \mathcal{U}_1(k)$

$$Q_{j_1,\ldots,j_s}(z) \le (2p)^s. \quad (4.19)$$

Finally, (4.5), (4.16), (4.18) and (4.19) yield

$$Q(z, n, L) \le 2^l (2p)^r (1-2p)^{-1} \le 2\rho^l \left(\frac{\rho}{2}\right)^{-(L+3NM)/(\gamma N)} \quad (4.20)$$

since from (4.5) it follows that $p < \frac{1}{4}$. This completes the proof of lemma 4.

Next, define the following domains:

$$\mathcal{U}_2(k) = \{z : W(z) \le \log \mathcal{D}/k\};$$
$$\mathcal{U}_3(k) = \{z : W(z) \ge \log k/\mathcal{D}\};$$
$$\mathcal{U}_4(\delta) = \{z : W(z) < \log \delta\}; \text{ and}$$
$$\mathcal{U}(k, C) = \{z : W(z) < \log C/k\};$$

where constants δ and C will be chosen in (4.27) and (4.34) below.

Defining the Markov times

$$\tau(\rho) = \inf\{n : \exp W(Z_n^{(k)}) - \exp W(Z_0^{(k)}) \ge \rho\} \quad (4.21)$$

we can rewrite the condition (B_3) of § 2 as

$$P_z^{(k)}\left\{\tau\left(\frac{1}{kR_2}\right) \le [R_1 \log k]\right\} \ge \alpha \quad (4.22)$$

for any z satisfying $\exp W(z) \le \beta$. Set

$$l_0(k) = 2[R_1 \log k]([R_2 C] + 1). \quad (4.23)$$

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By the strong Markov property of the process $\{Z_n^{(k)}\}$ one can obtain from (4.22) that

$$P_z^{(k)}\{\tau(2C/k)\le l_0(k)\}\ge \alpha^{2([R_2C]+1)}, \tag{4.24}$$

for any $z\in \mathcal{U}(k, C)$.

Fix now a small number $\varepsilon>0$. By (4.24) and the strong Markov property, there exists $K_\varepsilon>0$ sufficiently large, depending only on ε, that

$$P_z^{(k)}\{\tau(2C/k)>K_\varepsilon l_0(k)\}<\varepsilon, \tag{4.25}$$

for any $z\in\mathcal{U}(k, C)$.

If the Markov time θ_1 is defined by

$$\theta_1=\inf\{n: Z_n^{(k)}\in S\backslash\mathcal{U}(k, C)\}$$

then clearly $\theta_1\le\tau(2C/k)$ if $Z_0^{(k)}\in\mathcal{U}(k, C)$ and k is big enough. Hence by (4.25)

$$P_z^{(k)}\{\theta_1>K_\varepsilon l_0(k)\}<\varepsilon \tag{4.26}$$

for any $z\in\mathcal{U}(k, C)$.

In order to complete the proof of the theorem we need the following.

LEMMA 5. *For any $z\in S$ and sufficiently large k,*

$$P_z^{(k)}\{Z_{l_1(k)}^{(k)}\in\mathcal{U}_4(\delta)\}<4\varepsilon, \tag{4.27}$$

where

$$l_1(k)=[K_\varepsilon \log k]l_0(k), \tag{4.28}$$

$l_0(k)$ is given by (4.23), ε is the same as in (4.25) and δ is small enough but independent of ε.

Proof. Define the Markov times

$$\theta_2=\inf\{n: Z_n^{(k)}\in\mathcal{U}_2(k)\}$$

and

$$\theta_3=\inf\{n: Z_n^{(k)}\in\mathcal{U}_3(k)\}.$$

By (4.11), for any $z\notin\mathcal{U}_3(k)\cup\mathcal{U}(k, C)$ and any integer n such that

$$\tfrac{1}{2}l_1(k)\le n\le l_1(k) \tag{4.29}$$

one obtains

$$P_z^{(k)}\{\theta_2\wedge\theta_3\ge n \text{ and } Z_n^{(k)}\in\mathcal{U}_4(\delta)\}\le 2\rho^{[l_1(k)/2N]}\left(\frac{\rho}{2}\right)^{-(\log(\delta k/C)+3NM)/\gamma N}, \tag{4.30}$$

since if $z\notin\mathcal{U}(k, C)$ and $Z_n^{(k)}\in\mathcal{U}_4(\delta)$ then

$$W(Z_n^{(k)})-W(z)\le\log\delta k/C.$$

Here $a\wedge b\equiv\min(a, b)$.

Next, if $z\notin\mathcal{U}(k, C)$ and $Z_n^{(k)}\in\mathcal{U}_2(k)$ then

$$W(Z_n^{(k)})-W(z)\le\log\mathcal{D}/C$$

and so by (4.11) for such z it follows

$$P_z^{(k)}\{\theta_2\wedge\theta_3=\theta_2\}\le\sum_{n\ge1}Q(z,n,\log\mathcal{D}/C)$$
$$\le 2\left(\frac{\rho}{2}\right)^{(\log C/\mathcal{D})/\gamma N}\left(\frac{\rho}{2}\right)^{-3M/\gamma}(1-\rho^{1/N})^{-1}\rho^{-1}$$
$$=\tilde{K}_1\left(\frac{\rho}{2}\right)^{(\log C/\mathcal{D})/\gamma N} \tag{4.31}$$

where $\tilde{K}_1>0$ is independent of z, k and C.

Next, we have to estimate

$$P_z^{(k)}\{Z_n^{(k)}\in\mathcal{U}_4(\delta)\}\quad\text{for } z\in\mathcal{U}_3(k).$$

Since supp μ is compact it is easy to see by (A_1) and (B_1) of § 1 that there exists a constant $\tilde{M}>0$ such that for any $z\in\mathcal{U}_3(k)$,

$$W(X_j^{(k)}z)-W(z)\ge-\tilde{M}\ \text{(a.s.)}$$

for all $j=1,2,\dots$ and k big enough.

Therefore if $Z_n^{(k)}\in\mathcal{U}_4(\delta)$ then there exist two positive integers $i<j\le n$ such that

$$\log k/\mathcal{D}\ge W(Z_i^{(k)})\ge\log k/\mathcal{D}-\tilde{M},\ Z_j^{(k)}\in\mathcal{U}_4(\delta)$$

and $Z_l^{(k)}\in\mathcal{U}_1(k)$ for all $l=i,i+1,\dots,j-1$.

Hence employing the Markov property we get by (4.11) for any $z\in\mathcal{U}_3(k)$ that

$$P_z^{(k)}\{Z_n^{(k)}\in\mathcal{U}_4(\delta)\}\le\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}P_z^{(k)}\{W(Z_j^{(k)})-W(Z_i^{(k)})\le\tilde{M}+\log\delta\mathcal{D}/k$$
$$\text{and } Z_l^{(k)}\in\mathcal{U}_1(k)\text{ for all } l=i,i+1,\dots,j\}$$
$$=\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}E_z^{(k)}P_{Z_i^{(k)}}^{(k)}\{W(Z_j^{(k)})-W(Z_i^{(k)})\le\tilde{M}+\log\delta\mathcal{D}/k\text{ and}$$
$$Z_l^{(k)}\in\mathcal{U}_1(k)\text{ for all } l=i,i+1,\dots,j\}$$
$$\le n\tilde{K}_2\left(\frac{\rho}{2}\right)^{(\log k/\delta\mathcal{D})/\gamma N}$$

where $E_z^{(k)}$ is the expectation under the condition $Z_0^{(k)}=z$ and $\tilde{K}_2$ is a constant independent of k and n.

Now for any $z\notin\mathcal{U}(k,C)\cup\mathcal{U}_3(k)$ and each integer n satisfying (4.29) one obtains by (4.30)–(4.32) and the strong Markov property that

$$P_z^{(k)}\{Z_n^{(k)}\in\mathcal{U}_4(\delta)\}$$
$$\le P_z^{(k)}\{\theta_2\wedge\theta_3\ge n\text{ and } Z_n^{(k)}\in\mathcal{U}_4(\delta)\}+P_z^{(k)}\{\theta_2\wedge\theta_3=\theta_2\}$$
$$+P_z^{(k)}\{\theta_2\wedge\theta_3=\theta_3<n\text{ and } Z_n^{(k)}\in\mathcal{U}_4(\delta)\}$$
$$=P_z^{(k)}\{\theta_2\wedge\theta_3\ge n\text{ and } Z_n^{(k)}\in\mathcal{U}_4(\delta)\}+P_z^{(k)}\{\theta_2\wedge\theta_3=\theta_2\}$$
$$+E_z^{(k)}\chi_{\{\theta_2\wedge\theta_3=\theta_3<n\}}P_{Z_{\theta_3}^{(k)}}\{Z_{n-\theta_3}^{(k)}\in\mathcal{U}_4(\delta)\}$$
$$\le 2\rho^{[l_1(k)/2N]}\left(\frac{\rho}{2}\right)^{-(\log\delta k/C+3NM)/\gamma N}+\tilde{K}_1\left(\frac{\rho}{2}\right)^{(\log C/\mathcal{D})/\gamma N}+l_1(k)\tilde{K}_2\left(\frac{\rho}{2}\right)^{(\log k/\delta\mathcal{D})/\gamma N.} \tag{4.33}$$

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Taking C big enough such that

$$\tilde{K}_1\left(\frac{\rho}{2}\right)^{(\log C/\mathcal{D})/\gamma N} < \varepsilon \tag{4.34}$$

and then k big enough so that

$$2\rho^{[l_1(k)/2N]}\left(\frac{\rho}{2}\right)^{-(\log \delta k/C+3NM)/\gamma N} < \varepsilon \tag{4.35}$$

and

$$l_1(k)\tilde{K}_2\left(\frac{\rho}{2}\right)^{(\log k/\delta\mathcal{D})/\gamma N} < \varepsilon \tag{4.36}$$

one obtains (4.27) for $z \notin \mathcal{U}(k, C)$ by (4.32)–(4.36).

At last, for $z \in \mathcal{U}(k, C)$ we have by (4.26), (4.32)–(4.36) and the strong Markov property that

$$\begin{aligned} P_z^{(k)}\{Z_{l_1(k)}^{(k)} \in \mathcal{U}_4(\delta)\} &\le P_z^{(k)}\{\theta_1 > K_\varepsilon l_0(k)\} + P_z^{(k)}\{\theta_1 \le K_\varepsilon l_0(k) \text{ and } Z_{l_1(k)}^{(k)} \in \mathcal{U}_4(\delta)\} \\ &= P_z^{(k)}\{\theta_1 > K_\varepsilon l_0(k))\} \\ &\quad + E_z^{(k)}\chi_{\{\theta_1 \le K_\varepsilon l_0(k)\}} P_{Z_{\theta_1}^{(k)}}\{Z_{l_1(k)-\theta_1}^{(k)} \in \mathcal{U}_4(\delta)\} < 4\varepsilon, \end{aligned} \tag{4.37}$$

provided k is big enough so that

$$l_1(k) - K_\varepsilon l_0(k) > \tfrac{1}{2} l_1(k).$$

That completes the proof of (4.27). □

Now we are able to prove the theorem. By (4.3) it follows (see also (3.14)–(3.16)) that

$$\nu_{k_i}(\mathcal{U}_4(\delta)) = \int \nu_{k_i}(dz) P_z^{(k_i)}\{Z_{l_1(k_i)}^{(k_i)} \in \mathcal{U}_4(\delta)\} \tag{4.38}$$

and so by (4.27),

$$\nu_{k_i}(\mathcal{U}_4(\delta)) < 4\varepsilon \tag{4.39}$$

for all sufficiently large k_i.

Since $\mathcal{U}_4(\delta)$ is an open set, (4.2) implies (see theorem 2.1 of [**1**])

$$\liminf_{i\to\infty} \nu_{k_i}(\mathcal{U}_4(\delta)) \ge \nu(\mathcal{U}_4(\delta)).$$

Therefore by (4.39),

$$\nu(\mathcal{U}_4(\delta)) \le 4\varepsilon$$

and since ε is arbitrarily small,

$$\nu(\mathcal{U}_4(\delta)) = 0. \tag{4.40}$$

Now (4.40) and (3.3) give (4.1), completing the proof of our theorem. □

5. *Discussion and examples*

First we discuss assumptions (B_1)–(B_3) on perturbations. The counterexample of § 2 in [**9**] shows that some such assumption as (B_1) is necessary to make the perturbations local. We have already discussed condition (B_3) in § 2. One can see that it is not necessary for the assertion of our theorem but some such assumption

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is necessary to prove that from (4.2) and (4.3) follows (4.1), i.e. that all limits of invariant measures of the process $Z_n^{(k)}$ are concentrated on $\Gamma_{\max} \cap S$.

Now we consider certain examples of measures μ satisfying (A_1)–(A_3).

PROPOSITION 1. *Let the support of μ be compact and the minimal closed subgroup G_μ containing* supp μ *be commutative. Then* (A_1)–(A_3) *are satisfied.*

Proof. Since G_μ is a commutative subgroup of SL (m, R), it is known (see [**2**, chapter 1, § 4]) that there exists a matrix $A \in$ SL (m, R) such that for any $g \in G_\mu$,

$$AgA^{-1} = g^{u}g^{d}, \tag{5.1}$$

where g^u is an unipotent matrix, i.e. an upper triangular matrix with the diagonal elements all equal to one and g^d is a diagonal matrix. The representation (5.1) is unique and all g_1^u, g_1^d, g_2^u, g_2^d commute for any $g_1, g_2 \in G_\mu$.

Let $X_1, X_2, \ldots$ be independent random matrices with the common distribution μ and

$$X_i = A^{-1}X_i^{u}X_i^{d}A \tag{5.2}$$

be the unique decomposition (5.1) for X_i.

Since this decomposition is unique, the matrices X_i^u, $i = 1, 2, \ldots$ are independent with the same distribution μ^u, and also X_i^d, $i = 1, 2, \ldots$ are independent with common distribution μ^d.

Denote by $d_i^{(j)}$ the j-th diagonal element of X_i. Without loss of generality we suppose that

$$E \log |d_1^{(1)}| = \cdots = E \log |d_1^{(l)}| > E \log |d_1^{(l+1)}| \geq \cdots \geq E \log |d_1^{(m)}|. \tag{5.3}$$

Let $\tilde{\Gamma}_{\max}$ be the subspace of R^m generated by all vectors having last $(m-l)$ coordinates equal to zero and $\tilde{\Gamma}_{\min}$ the subspace of all vectors with first l coordinates equal to zero.

We need

LEMMA 5. *The matrix*

$$\mathscr{D} = \begin{pmatrix} E \log |d_1^{(1)}| & & 0 \\ & \ddots & \\ 0 & & E \log |d_1^{(m)}| \end{pmatrix}$$

commutes with all matrices from supp μ^u, *where μ^u is the distribution of X_i^u in* (5.2).

Proof. Let

$$g^d = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_m \end{pmatrix} \in \text{supp}\, \mu^d$$

and

$$g^u = \begin{pmatrix} 1 & & (b_{ij}) \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in \text{supp}\, \mu^u$$

then $g^u g^d = g^d g^u$ iff $a_i b_{ij} = a_j b_{ij}$. Hence if $g^u g^d = g^d g^u$ then also $g^u \log |g^d| = (\log |g^d|) g^u$, where

$$\log |g^d| = \begin{pmatrix} \log |a_1| & & 0 \\ & \ddots & \\ 0 & & \log |a_m| \end{pmatrix}.$$

This implies the assertion of lemma 5.

From (5.3) we see that $\tilde{\Gamma}_{\max}$ and $\tilde{\Gamma}_{\min}$ are spectral invariant subspaces of the matrix $\mathscr{D}$. By lemma 5 we conclude that $\tilde{\Gamma}_{\max}$ and $\tilde{\Gamma}_{\min}$ are invariant with respect to all matrices from supp μ^u and, of course, from supp μ^d.

Set $\Gamma_{\max} = A^{-1}\tilde{\Gamma}_{\max}$ and $\Gamma_{\min} = A^{-1}\tilde{\Gamma}_{\min}$. Then (5.1) implies that $\Gamma_{\max}$ and $\Gamma_{\min}$ are invariant with respect to any $g \in G_\mu$.

Now we check (A_2). Let $z \in \Gamma_{\max}$ and $\|z\| = 1$. Then $y = Az \in \tilde{\Gamma}_{\max}$ and by (5.2),

$$\|X_n \cdots X_1 z\| = \|A^{-1} X_n^u \cdots X_1^u X_n^d \cdots X_1^d y\| \geq$$
$$\|A\|^{-1} \|\tilde{\Pi}_{\max}(X_n^u)^{-1} \cdots (X_1^u)^{-1}\|^{-1} \|\tilde{\Pi}_{\max}(X_n^d)^{-1} \cdots (X_1^d)^{-1}\|^{-1} \cdot \|A^{-1}\|^{-1}, \quad (5.5)$$

where $\tilde{\Pi}_{\max} = A \Pi_{\max} A^{-1}$.

From [**10**] it follows that

$$\lim_{n\to\infty} n^{-1} \log \|(X_n^u)^{-1} \cdots (X_1^u)^{-1}\| = 0. \quad (5.6)$$

By the strong law of large numbers and (5.3),

$$\lim_{n\to\infty} n^{-1} \log \|\tilde{\Pi}_{\max}(X_n^d)^{-1} \cdots (X_1^d)^{-1}\| = -E \log |d_1^{(1)}|. \quad (5.7)$$

Since the right hand side of (5.5) does not depend on z, (5.5)–(5.7) imply (A_2) with $\Lambda(\mu) = E \log |d_1^{(1)}|$.

In the same way as above (A_3) follows using inequality

$$\|\Pi_{\min} X_n \cdots X_1\| \leq \|A^{-1}\| \cdot \|\tilde{\Pi}_{\min} X_n^u \cdots X_1^u\| \cdot \|\tilde{\Pi}_{\min} X_n^d \cdots X_1^d\| \cdot \|A\|. \quad (5.8)$$

Indeed, by [**10**]

$$\lim_{n\to\infty} n^{-1} \log \|\tilde{\Pi}_{\min} X_n^u \cdots X_1^u\| = 0 \quad (5.9)$$

and by the strong law of large numbers

$$\lim_{n\to\infty} n^{-1} \log \|\tilde{\Pi}_{\min} X_n^d \cdots X_1^d\| = E \log |d_1^{(l+1)}| < \Lambda(\mu), \quad (5.10)$$

gives (A_3) and completes the proof of proposition 1. □

Remark. The condition (A_2) also follows from the stronger assumption

$$\lim_{n\to\infty} n^{-1} \log \|\Pi_{\max} X_1^{-1} \cdots X_n^{-1}\| = -\Lambda(\mu),$$

since

$$\|\Pi_{\max} X_1^{-1} \cdots X_n^{-1}\| \cdot \|\Pi_{\max} X_n \cdots X_1 z\| \geq \|\Pi_{\max} z\|.$$

Another relationship among our conditions is provided in the following statement.

PROPOSITION 2. *Suppose* (A_1) *and* (A_3) *hold and the distribution* $\mu_{\max}$ *of* $\Pi_{\max} X_1$ *on* GL (dim ($\Gamma_{\max}$), $\mathbb{R}$) *has a density* $p(g)$ *with respect to the Haar measure such that* $p(g)$ *is positive on some open subset of* SL (m, $\mathbb{R}$). *Then* (A_2) *is also true.*

Proof. Set $S_{\max}=\{z\in\Gamma_{\max}: \|z\|=1\}$ and define the Markov chain Z_n on $S_{\max}$ in the same way as in (3.12) i.e.

$$Z_n=X_nZ_{n-1}\|X_nZ_{n-1}\|^{-1} \quad \text{and} \quad Z_0\in\Gamma_{\max}. \tag{5.11}$$

From the assumption on $\Gamma_{\max}$ one can see that there exist some functions $q(n, z, y)$ such that for any $z\in S_{\max}$ and Borel set $Q\subset S_{\max}$

$$P_z\{Z_n\in Q\}=P\{X_n\cdots X_1z\|X_n\cdots X_1z\|^{-1}\in Q\}=\int_Q q(n, z, y)\,dy, \tag{5.12}$$

and there exist $\tilde{\beta}>0$ and a positive integer N such that

$$q(N, z, y)>\tilde{\beta}>0 \tag{5.13}$$

for all $z, y\in S_{\max}$, where dy is an element of the volume on $S_{\max}$.

It is known (see [**4**, chapter 5, § 5]) that (5.12) and (5.13) imply that there exist a positive function $q(y)$ and positive numbers C and α such that

$$|q(n, z, y)-q(y)|\le C\cdot\exp(-\alpha n/N) \tag{5.14}$$

where $q(y)$ is the density of the invariant measure of Z_n on $S_{\max}$ i.e.

$$\int_{S_{\max}} q(z)P_z\{Z_n\in Q\}\,dz=\int_Q q(y)\,dy. \tag{5.15}$$

From (3.24), (5.12), (5.14) and (5.15) one obtains easily that for any $j<n$,

$$\sup_{z\in S_{\max}} P\{n^{-1}\log\|X_n\cdots X_1z\|<\Lambda(\mu)-\delta\}$$

$$\le \sup_{z\in S_{\max}} P_z\left\{n^{-1}\log\|X_n\cdots X_{j+1}Z_j\|<\Lambda(\mu)-\delta+\frac{j}{n}M\right\}$$

$$=\sup_{z\in S_{\max}}\int_{S_{\max}} P\left\{n^{-1}\log\|X_n\cdots X_{j+1}y\|<\Lambda(\mu)-\delta+\frac{j}{n}M\right\}q(j, z, y)\,dy$$

$$\le\int_{S_{\max}} P\left\{n^{-1}\log\|X_n\cdots X_{j+1}y\|<\Lambda(\mu)-\delta+\frac{j}{n}M\right\}q(y)\,dy+Ce^{-\alpha j/N}. \tag{5.16}$$

Letting $n\to\infty$ we see by [**7**] that the last integral in (5.16) tends to zero and since j is arbitrarily large one obtains (A_2), completing the proof of proposition 2.

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