Bull. Aust. Math. Soc. **83** (2011), 231–240
doi:10.1017/S0004972710001735

ON λ-STRICT IDEALS IN BANACH SPACES

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(Received 10 May 2010)

Abstract

We define and study λ-strict ideals in Banach spaces, which for $\lambda = 1$ means strict ideals. Strict u-ideals in their biduals are known to have the unique ideal property; we prove that so also do λ-strict u-ideals in their biduals, at least for $\lambda > 1/2$. An open question, posed by Godefroy *et al.* ['Unconditional ideals in Banach spaces', *Studia Math.* **104** (1993), 13–59] is whether the Banach space X is a u-ideal in $\mathrm{Ba}(X)$, the Baire-one functions in X^{**}, exactly when $\kappa_u(X) = 1$; we prove that if $\kappa_u(X) = 1$ then X is a strict u-ideal in $\mathrm{Ba}(X)$, and we establish the converse in the separable case.

2000 *Mathematics subject classification*: primary 46B20.
Keywords and phrases: strict ideal, u-ideal, *VN*-subspace.

1. Introduction

In this paper we restrict ourself to working with real Banach spaces, although many of the results will also hold in the complex case. Let Y be a Banach space. Recall that a (not necessarily) closed subspace X of Y is called an ideal if there is a norm-one projection P on Y^* with kernel $X^\perp$ (such a P is called an ideal projection on Y^*). When X is an ideal in Y we have $Y^* = X^\perp \oplus P(Y^*)$, where the range of P is isometrically isomorphic to X^*. The concept of an ideal was introduced by Godefroy *et al.* in their seminal paper [5].

Let X be a closed subspace of a Banach space Y. Given a projection P on Y^* with kernel $X^\perp$, then we can define an operator $T : Y \to X^{**}$ by

$$\langle Ty, (i_X)^* y^* \rangle = \langle P y^*, y \rangle$$

where $y \in Y$, $y^* \in Y^*$, and where $i_X : X \to Y$ is the natural embedding operator. The fact that T is well defined follows since the kernel of P is $X^\perp$. Since P is linear, T also is, and $\|T\| \leq \|P\|$ so T is bounded. Note also that $Tx = x$ for every $x \in X$. Hence T is an extension of k_X, the canonical embedding of X into its bidual. Note that T is one-to-one if and only if $P(Y^*)$ is weak* dense in Y^*.

Let $0 \leq \lambda \leq 1$ and X be an ideal in Y with an ideal projection P on Y^*. If $P(Y^*)$ is weak* dense in Y^*, we will say that X is a λ-strict ideal in Y if $P(Y^*)$ is λ-norming for Y, that is,

$$\sup_{y^*\in Y^*, \|Py^*\|=1} |Py^*(y)| \geq \lambda\|y\| \quad \text{for all } y \in Y.$$

When X is a 1-strict ideal in Y we simply call X a strict ideal in Y, as in [5] and later papers. In [5, 10] it is observed that X is a strict ideal in Y if and only if T is isometric. Thus k_X extends to an isometry on Y exactly when X is a strict ideal in Y.

The paper is organized as follows. In Section 2 we study λ-strict ideals in general. We will show that when X is a λ-strict ideal in Y for some $\lambda > 0$, k_X extends to an isomorphism on Y and $P(Y^*)$ automatically gets a slightly stronger property than being λ-norming (namely weak* thickness) (Proposition 2.1). An application of this result (Corollary 2.2) is also given. Then we let $Y = X^{**}$ and show, in Theorem 2.5, that if every norm-preserving extension T of k_X to X^{**} is injective, then the only possible norm-preserving extension of k_X to X^{**} is the identity on X^{**}.

In Section 3 we turn our attention to λ-strict u-ideals. Recall that a space X is called a u-ideal in Y if X is an ideal in Y with an ideal projection P on Y^* with $\|I - 2P\| = 1$. If in addition the range of this P is λ-norming for Y, X is called a λ-strict u-ideal in Y. First we extend the known result that proper strict u-ideals contain a copy of c_0 to λ-strict u-ideals, for $\lambda \geq 0$. Then we turn to the special case when $Y = X^{**}$ and show that three known results valid for strict u-ideals are valid for λ-strict u-ideals as well, as soon as $\lambda > 1/2$ (Proposition 3.3, Theorem 3.5 and Corollary 3.6).

In many cases (typically when ℓ_1 is involved) it is interesting to let $Y = \mathrm{Ba}(X)$, the Baire-one functions in X^{**}. In Section 4 we concentrate on studying when X is a strict u-ideal in $\mathrm{Ba}(X)$. Building on arguments of Godefroy *et al.* and combining with one of the results that we extended in Section 3, we examine [5, Question 9] (Theorem 4.2). As a consequence of this we obtain, in Corollary 4.4, a sufficient condition for a separable X to be a strict u-ideal in $\mathrm{Ba}(X)$ when X is a u-ideal in $\mathrm{Ba}(X)$.

Our notation is mostly standard. When some notation or term is used which we do not think is standard or self-explanatory, we explain its meaning there and then.

The reader will surely observe that some of our proofs could have been simplified and some results could have been strengthened by using [2, Proposition 2.26(b)]. However, it seems that there is a gap in the proof of that result, and thus we have not used it.

2. λ-strict ideals

Let X be a closed subspace of a Banach space Y with $P(Y^*)$ weak* dense in Y^*. Recall that we call X a λ-strict ideal in Y if $P(Y^*)$ is λ-norming for Y, $0 \leq \lambda \leq 1$.

Recall (see, for example, the survey paper [12]) that a set A in Y^* is weak* thick if it has the following boundedness deciding property: whenever a sequence $(y_n) \subset Y$ is pointwise bounded on A, it is bounded in norm in Y.

PROPOSITION 2.1. *Let X be a closed subspace of Y and P a projection on Y^* with kernel $X^\perp$. Then the following conditions are equivalent.*

(a) *$P(Y^*)$ is λ-norming for some $0 < \lambda \leq 1$.*
(b) *$P(Y^*)$ is weak* thick.*
(c) *T is an isomorphism.*

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PROOF. (a) $\Leftrightarrow$ (c). Statement (a) holds if and only if there exists $0 < \lambda \leq 1$ such that, for every $y \in S_Y$,

$$\lambda < \sup_{\|Py^*\|=1} |\langle Py^*, y\rangle| = \sup_{\|(i_X)^*y^*\|=1} |\langle (i_X)^*y^*, Ty\rangle|.$$

This again is equivalent to T being an isomorphism.

(b) $\Rightarrow$ (a). This is clear from the definition of weak* thickness.

(c) $\Rightarrow$ (b). Suppose that $(y_n) \subset Y$ is pointwise bounded on $P(Y^*) \cap B_{Y^*}$, that is,

$$\infty > \sup_n |\langle Py^*, y_n\rangle| = \sup_n |\langle (i_X)^*y^*, Ty_n\rangle|$$

for every $y^* \in Y^*$ with $\|Py^*\| = 1$. Then (Ty_n) is pointwise bounded on B_{X^*} and from the uniform boundedness principle (Ty_n) has to be bounded in X^{**}. Since T is an isomorphism, (y_n) must also be bounded. □

Let us give an application. Recall first that a Banach space X is co-reflexive if the quotient X^{**}/X is reflexive.

PROPOSITION 2.2. *Let X be a λ-strict ideal in Y for some $\lambda > 0$. If X is co-reflexive, then Y/X is reflexive.*

PROOF. From Proposition 2.1 the operator $T : Y \to X^{**}$ corresponding to the ideal projection on Y^* in this case is an isomorphism. Now introduce a mapping $S : Y/X \to X^{**}/X$ by $S[y] = [Ty]$ for all cosets $[y]$ in Y/X. S is well defined and linear. It is also straightforward to show that S is an isomorphism. Thus Y/X is reflexive since $S(Y/X)$ is a subspace of X^{**}/X, which is reflexive by the co-reflexivity of X. □

REMARK 2.3. In the proof of Proposition 2.2 we used the fact that the operator S is an isomorphism. Actually, it is straightforward to show that S is an isomorphism if and only if T is.

Let X be a closed subspace of Y and $\mathcal{L}(Y, X^{**})$ the space of bounded linear operators from Y to X^{**}. Denote by $\mathcal{E}(Y, X^{**})$ the set

$$\{T \in \mathcal{L}(Y, X^{**}) : Tx = x \ \forall x \in X, \|T\| = 1\}$$

of norm-preserving extensions to Y of the canonical embedding k_X of X into its bidual. Note that the connection

$$\langle Ty, (i_X)^*y^*\rangle = \langle Py^*, y\rangle$$

for all $y \in Y$, $y^* \in Y^*$, where P is an ideal projection on Y^*, puts $\mathcal{E}(Y, X^{**})$ in a one-to-one correspondence with the set of all ideal projections on Y^*.

From now on, and throughout the section, we study the particular case when $Y = X^{**}$. Define an order relation $\leq$ on $\mathcal{E}(X^{**}, X^{**})$ by $U \leq V$ if $\|Ux^{**}\| \leq \|Vx^{**}\|$ for every $x^{**} \in X^{**}$. Elements of minimal order in $(\mathcal{E}(X^{**}, X^{**}), \leq)$ are denoted by $\mathcal{M}(X^{**}, X^{**})$.

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The following result and argument are implicit in [4, Theorem III.1].

PROPOSITION 2.4. *The set $\mathcal{M}(X^{**}, X^{**})$ is nonempty and consists of projections.*

PROOF. By using Zorn's lemma one can verify that $(\mathcal{E}(X^{**}, X^{**}), \leq)$ contains a minimal element P, so $\mathcal{M}(X^{**}, X^{**})$ is nonempty. Since P is minimal and $\|U\| = 1$ for all $U \in \mathcal{E}(X^{**}, X^{**})$ we have $\|UPx^{**}\| = \|Px^{**}\|$ for all $U \in \mathcal{E}(X^{**}, X^{**})$ and all $x^{**} \in X^{**}$. Applying this observation to

$$U_n = \frac{1}{n}\left(\sum_{i=1}^{n} P^i\right),$$

which by convexity is in $\mathcal{E}(X^{**}, X^{**})$, gives

$$\begin{aligned}\|(U_n P^2 - U_n P)x^{**}\| &= \|U_n P(Px^{**} - x^{**})\| \\ &= \|P(Px^{**} - x^{**})\| \\ &= \|P^2 x^{**} - Px^{**}\|.\end{aligned}$$

Since

$$U_n P^2 - U_n P = \frac{1}{n}(P^{n+2} - P^2),$$

we get that $\|P^2 x^{**} - Px^{**}\| \leq 2/n$ for all $n \geq 1$. It follows that P is a projection. □

Using Proposition 2.4, we now easily obtain the following result.

THEOREM 2.5. *If every $T \in \mathcal{E}(X^{**}, X^{**})$ is one-to-one, then $\mathcal{E}(X^{**}, X^{**}) = \{I_{X^{**}}\}$, where $I_{X^{**}}$ is the identity on X^{**}.*

PROOF. Since a projection is one-to-one only if it is the identity, $\mathcal{M}(X^{**}, X^{**}) = \{I_{X^{**}}\}$. Thus we are done if we can show that $\mathcal{E}(X^{**}, X^{**}) = \mathcal{M}(X^{**}, X^{**})$. To this end let $S, T \in \mathcal{E}(X^{**}, X^{**})$ and suppose that $S \leq T$. Now, since $\|SI_{X^{**}}x^{**}\| \leq \|I_{X^{**}}x^{**}\|$ for all $x^{**} \in X^{**}$, we have $\|Sx^{**}\| = \|x^{**}\|$ by minimality of $I_{X^{**}}$. Thus $\|Tx^{**}\| \leq \|x^{**}\| = \|Sx^{**}\|$ for all $x^{**} \in X^{**}$ so $S \geq T$, and we are done. □

In other words, if whenever X is placed in X^{**} as an ideal and it sits there as a λ-strict ideal, then the natural way (using the Dixmier projection) is the only way. A similar result will be obtained in Corollary 3.6.

3. λ-strict u-ideals

Recall that X is a u-summand in Y if X is the range of a norm-one projection P on Y with $\|I - 2P\| = 1$. If the range of an ideal projection P on Y^* is a u-summand in Y^*, then X is a u-ideal in Y. We call such an ideal projection P on Y^* a u-projection and the corresponding $T \in \mathcal{E}(Y, X^{**})$ an unconditional extension operator. Note that a u-projection is always unique [5, Lemma 3.1] (see also [1, Proposition 4.2]). An equivalent formulation of an operator P being a u-projection is that, whenever $z^* \in X^{\perp}$ and $y^* \in Y^*$, then $\|z^* + Py^*\| = \|z^* - Py^*\|$.

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We start with a result on λ-strict u-ideals which is known for strict u-ideals (see [10, Theorem 2.7]).

PROPOSITION 3.1. *Let* $0 \leq \lambda \leq 1$. *Proper* λ*-strict u-ideals must contain an isomorphic copy of* c_0.

PROOF. Let X be a proper λ-strict u-ideal in Y with u-projection P and Suppose that X does not contain a copy of c_0. Since X is a u-ideal in Y, by [5, Theorem 3.5], X has to be a u-summand in Y. Thus P is weak* continuous, hence onto Y^* since X is λ-strict. This contradicts the assumption that X is a proper subspace of Y. □

In the remaining part of this section we will make use of another equivalent formulation of an ideal: let X be an ideal in Y with corresponding projection $P : Y^* \to Y^*$. Then Py^* is a norm-preserving extension of the restriction $y^*|_X \in X^*$. This induces a linear extension operator (a Hahn–Banach extension operator) $\varphi : X^* \to Y^*$, depending on P. Conversely, if $\varphi : X^* \to Y^*$ is a Hahn–Banach extension operator, then φ induces an ideal projection P_φ. The correspondence $\varphi \leftrightarrow P_\varphi$ is given by $P_\varphi = \varphi(i_X)^*$. It is helpful to observe that $(i_X)^*$ is simply the linear operator from Y^* to X^* that restricts $y^* \in Y^*$ to X.

Let $\mathrm{HB}(X, Y)$ (as usual) denote the set of norm-preserving linear extension operators from X^* into Y^*. Of course, X is an ideal in Y if and only if $\mathrm{HB}(X, Y) \neq \emptyset$. If $\varphi \in \mathrm{HB}(X, Y)$ corresponds to a projection P_φ which makes X a λ-strict ideal in Y, we call φ λ-strict. Moreover, if the ideal projection P_φ in addition makes X a u-ideal in Y, φ is called unconditional λ-strict.

Here is another result known for strict u-ideals that extends to λ-strict u-ideals.

PROPOSITION 3.2. *Let X be a u-ideal in Y and* $0 \leq \lambda \leq 1$. *Then X is a* λ*-strict u-ideal in Y if and only if X is a* λ*-strict u-ideal in* $Z = \mathrm{span}(X, \{y\})$ *for every* $y \in Y$.

PROOF. Since X is a u-ideal in Y, X is a u-ideal in Z by local characterization of u-ideals [5, Proposition 3.6]. Denote by P_Z and P_Y respectively the u-projections on Z^* and Y^*, and by $T_Z \in \mathcal{E}(Z, X^{**})$ and $T_Y \in \mathcal{E}(Y, X^{**})$ the corresponding unconditional extension operators. Now, from [9, Lemmas 2.2 and 3.1], $T_Y|_Z = T_Z$. By Proposition 2.1, the result follows for $\lambda > 0$. For $\lambda = 0$ one only needs to recall from the introduction that the range of an ideal projection is weak* dense if and only if the corresponding extension operator is one-to-one. □

When X is a u-ideal in Y there is a unique ideal projection making it a u-ideal, but there may very well be other ideal projections for which X is an ideal. The next proposition shows that in some cases, the possible other ideal projections at least do not make X a 1-complemented subspace in Y. This result is similar to [10, Proposition 2.5]; we state it in the more general setting of λ-strict u-ideals.

PROPOSITION 3.3. *If X is a proper* λ*-strict u-ideal in Y for some* $\frac{1}{2} < \lambda \leq 1$, *then every* $T \in \mathcal{E}(Y, X^{**})$ *is one-to-one. In particular, if P is a projection of Y onto X, then* $\|P\| > 1$.

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PROOF. Let $\varphi \in \mathrm{HB}(X, Y)$ be unconditional with corresponding unconditional $T_\varphi \in \mathcal{E}(Y, X^{**})$. Choose $\psi \in \mathrm{HB}(X, Y)$ with corresponding $T_\psi \in \mathcal{E}(Y, X^{**})$. Then, by [1, Proposition 2.2], φ is the center of symmetry in $\mathrm{HB}(X, Y)$, so $2\varphi - \psi \in \mathrm{HB}(X, Y)$. Thus $\|2T_\varphi - T_\psi\| \leq 1$. Let $0 \neq y \in Y$. Then

$$\|T_\psi(y)\| \geq 2\|T_\varphi(y)\| - \|(2T_\varphi - T_\psi)(y)\| \geq (2\lambda - 1)\|y\| > 0.$$

Hence $T_\psi \in \mathcal{E}(Y, X^{**})$ is one-to-one and thus not onto X. The last part follows since left composition of every norm-one projection P on Y onto X with k_X is in $\mathcal{E}(Y, X^{**})$. □

Using Proposition 3.3 we can observe that the known result (see [10, Proposition 2.5] and the remark thereafter) that dual spaces never can be strict u-ideals in their biduals can be pushed further to conclude that they never can be λ-strict u-ideals for $\lambda > 1/2$. Similar to [10, Corollary 2.6], we actually get that a dual space never can be a λ-strict u-ideal for $1/2 < \lambda \leq 1$ in any superspace.

COROLLARY 3.4. *If X is a u-ideal in Y and X is 1-complemented in its bidual, then X is not a λ-strict u-ideal in Y for any $1/2 < \lambda \leq 1$.*

PROOF. The argument is similar to the proof of [10, Corollary 2.6] except that Propositions 3.3 and 3.2 are used instead of [10, Propositions 2.5 and 2.1]. □

An ideal X in Y has the *unique ideal property* in Y if $\mathrm{HB}(X, Y)$ consists of a singleton, that is, there is only one ideal projection for which X is an ideal in Y. A subspace X of a Banach space Y is said to be a *very nonconstrained subspace* (*VN*-subspace) in Y if, for all $y \in Y$,

$$\bigcap_{x \in X} B_Y(x, \|y - x\|) = \{y\}.$$

The notion of a *VN*-subspace was introduced in [2] where it is shown (see [2, Theorem 2.12]) that the above definition is equivalent to the condition that, for all $y \in Y \setminus X$,

$$\bigcap_{x \in X} B_X(x, \|y - x\|) = \emptyset.$$

It is known that strict u-ideals in their biduals have the unique ideal property [10, Remark 2.1]. A consequence of Proposition 3.3, Theorem 2.5, and the following result is that this is also true for λ-strict u-ideals in their biduals whenever $\lambda > 1/2$ (see Corollary 3.6).

THEOREM 3.5. *Let $\frac{1}{2} < \lambda \leq 1$. Then λ-strict u-ideals are VN-subspaces.*

PROOF. Let X be a closed subspace of Y and $y \in Y \setminus X$. Now, $X \cap \bigcap_{x \in X} B_{X^{**}}(x, \|y - x\|) = \emptyset$. Otherwise this would define a norm-one projection P on $\operatorname{span}(X, \{y\})$ onto X by $P(ay + x) = aPy + x$, where Py is some element in $X \cap \bigcap_{x \in X} B_{X^{**}}(x, \|y - x\|)$. But this contradicts Proposition 3.3. It follows that $\bigcap_{x \in X} B_X(x, \|y - x\|) = \emptyset$, and thus X is a *VN*-subspace of Y. □

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COROLLARY 3.6. *Let X be a Banach space and let the Dixmier projection on X^{***} be denoted by π.*

(a) *If, whenever $P : X^{***} \to X^{***}$ is an ideal projection on X^{***} with* $\ker P = X^{\perp}$, *PX^{***} is weak* dense in X^{***}, then X has the unique ideal property in X^{**}.*

(b) *Let $\frac{1}{2} < \lambda \leq 1$. If X is a λ-strict u-ideal in X^{**}, then X has the unique ideal property in X^{**}; moreover, $\|I_{X^{***}} - 2\pi\| = 1$.*

Note that (a) can be used in combination with Proposition 3.3 to obtain (b).

4. When X is a u-ideal in $\mathrm{Ba}(X)$

Let $\mathrm{Ba}(X)$ denote, as usual, the Banach space of elements in X^{**} of the first Baire class, that is, the set of $x^{**} \in X^{**}$ which are weak* limits of sequences from X.

The number $\kappa_u(X)$ is defined on [5, pp. 22–23]. We repeat the definition here for convenience: for each $x^{**} \in X^{**}$ define $\kappa_u(x^{**})$ to be the infimum over all a such that $x^{**} = \sum_n x_n$ in the weak* topology of X^{**}, with $x_n \in X$ and such that for any $n \in \mathbb{N}$ and $\theta_k = \pm 1$ for $1 \leq k \leq n$, we have $\|\sum_{k=1}^{n} \theta_k x_k\| \leq a$. Put $\kappa_u(x^{**}) = \infty$ if no such a exists. Recall that X has property (u) if every $x^{**} \in \mathrm{Ba}(X)$ has $\kappa_u(x^{**}) < \infty$. In this case it follows from the closed graph theorem that there exists a constant C such that $\kappa_u(x^{**}) \leq C\|x^{**}\|$ for all $x^{**} \in \mathrm{Ba}(X)$. The smallest such constant is $\kappa_u(X)$.

The following proposition will be used to prove Theorem 4.2.

PROPOSITION 4.1. *Let X be a separable u-ideal in* $\mathrm{Ba}(X)$ *with corresponding unconditional $T \in \mathcal{E}(\mathrm{Ba}(X), X^{**})$. Assume that X is also a VN-subspace in* $\mathrm{Ba}(X)$. *Then $T(\mathrm{Ba}(X)) \subset \mathrm{Ba}(X)$. In fact, $T = \mathrm{id}_{X^{**}}|_{\mathrm{Ba}(X)}$.*

PROOF. Since X is separable there is a sequence $(x_i^*)_{i=1}^{\infty} \subset S_{X^*}$ such that $M = \overline{\mathrm{span}}\{x_i^*\}$ is 1-norming for X. Let $x^{**} \in \mathrm{Ba}(X)$ with $\|x^{**}\| = 1$ and put

$$A_n = \left\{ x \in X : |Tx^{**}(x_i^*) - x(x_i^*)| < \frac{1}{n}, i = 1, 2, \ldots, n \right\}.$$

Note that A_n is convex and nonempty and that $Tx^{**} \in H_n$, the weak* closure of A_n in X^{**}, for each n. Since X is a u-ideal in $\mathrm{Ba}(X)$, by [5, Lemma 3.4], for every $\varepsilon > 0$ there exists $\chi \in \bigcap_n H_n$ such that $\kappa_u(\chi) \leq \|x^{**}\| + \varepsilon$. In particular, $\chi \in \mathrm{Ba}(X)$. Since $\chi \in \bigcap_n H_n$, $\chi(f) = Tx^{**}(f)$ for all $f \in M$.

Now take an arbitrary $x^* \in X^*$ and put $N = \mathrm{span}\{M, \{x^*\}\}$. The same argument as above produces a Baire-one function $\chi_1 \in \bigcap_n H_n$ with $\chi_1(f) = Tx^{**}(f)$ for all $f \in M$ and $\chi_1(x^*) = Tx^{**}(x^*)$.

We now use the fact that X is a VN-subspace of $\mathrm{Ba}(X)$. By [2, Theorem 2.12, Lemma 2.10] $\chi_1 = \chi$ since $\ker(\chi - \chi_1)|_X \subset X^*$ contains the norming subspace M. Since $\chi_1 = \chi$ for $x^* \in X^*$ we obtain $Tx^{**} = \chi \in \mathrm{Ba}(X)$.

The final part of the proposition follows by [2, Proposition 2.26(a)]. □

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Godefroy *et al.* [5, Question 9, p. 56] ask whether $\kappa_u(X) = 1$ if and only if X is a u-ideal in $\mathrm{Ba}(X)$. Note that if this is true, then it follows from the argument of the following theorem that X is a strict u-ideal in $\mathrm{Ba}(X)$ whenever it is a u-ideal in $\mathrm{Ba}(X)$.

THEOREM 4.2. *Let X be a Banach space. If $\kappa_u(X) = 1$, then X is a strict u-ideal in* $\mathrm{Ba}(X)$. *If X is separable and X is a λ-strict u-ideal in* $\mathrm{Ba}(X)$ *for some $\frac{1}{2} < \lambda \leq 1$, then $\kappa_u(X) = 1$.*

PROOF. Suppose that $\kappa_u(X) = 1$ and let $x^{**} \in \mathrm{Ba}(X)$. Now choose a sequence (x_n) in X such that $s_n := \sum_{k=1}^n x_k \to x^{**}$ is weak* in X^{**} and $\|\sum_{k=1}^n \theta_k x_k\| < \|x^{**}\| + \varepsilon$ for all n and $\theta_k = \pm 1$. Then

$$\|x^{**} - 2s_n\| \leq \liminf_m \Big\| \sum_{k=1}^m x_k - 2s_n \Big\| = \liminf_m \Big\| \sum_{k=1}^m x_k - 2\sum_{k=1}^n x_k \Big\|$$
$$\leq \liminf_m \Big\| \sum_{k=1}^m \theta_k x_k \Big\| \leq \|x^{**}\| + \varepsilon.$$

Since the above inequality holds for every n, we get $\limsup_n \|x^{**} - 2s_n\| < \|x^{**}\| + 2\varepsilon$. Now, since the natural embedding $i_{\mathrm{Ba}(X)}$ of $\mathrm{Ba}(X)$ into X^{**} is in $\mathcal{E}(\mathrm{Ba}(X), X^{**})$, it follows from the above inequality in combination with [5, Lemma 2.2], that X is a u-ideal in $\mathrm{Ba}(X)$. Moreover, $i_{\mathrm{Ba}(X)}$ is isometric, so X is indeed a strict u-ideal in $\mathrm{Ba}(X)$.

Assume that $\frac{1}{2} < \lambda \leq 1$ and that X is a separable λ-strict u-ideal in $\mathrm{Ba}(X)$. Then, by Theorem 3.5, X is a *VN*-subspace in $\mathrm{Ba}(X)$. From the proof of 4.1 it follows directly that $\kappa_u(X) = 1$. □

Our next result gives a condition for X to be a *VN*-subspace in $\mathrm{Ba}(X)$.

THEOREM 4.3. *Let X be a Banach space. If $\kappa_u(X) < 2$, then X is a VN subspace in* $\mathrm{Ba}(X)$.

PROOF. From [5, Lemma 6.3] it follows that $\ker x^{**}$ cannot be a 1-norming subspace of X^* for any $0 \neq x^{**} \in \mathrm{Ba}(X)$. Using [2, Lemma 2.10, Theorem 2.12] it follows that the ortho-complement $\mathcal{O}(X, \mathrm{Ba}(X))$, of X in $\mathrm{Ba}(X)$, is $\{0\}$. Thus X is a *VN* subspace of $\mathrm{Ba}(X)$. □

The following result should be compared with [5, Theorem 7.5].

COROLLARY 4.4. *Let X be a separable u-ideal in* $\mathrm{Ba}(X)$ *such that $\kappa_u(X) < 2$. Then $\kappa_u(X) = 1$ and X is a strict u-ideal in* $\mathrm{Ba}(X)$.

PROOF. This follows from Theorem 4.3 and by an argument similar to the proof of Proposition 4.1. □

REMARK 4.5. From [5, Theorem 7.5] (see also [10, Corollary 2.10]) we know that strict u-ideals in their biduals do not contain copies of ℓ_1. However, this is not the case for strict u-ideals in general. Indeed, let $X = \ell_1 \oplus_\infty c_0$. Since ℓ_1 has the Schur

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property, $\mathrm{Ba}(\ell_1) = \ell_1$, and therefore $\mathrm{Ba}(X) = \ell_1 \oplus_\infty \ell_\infty$. Thus $\kappa_u(X) = \kappa_u(c_0) = 1$, and X is therefore a strict u-ideal in $\mathrm{Ba}(X)$ by Theorem 4.2. Note that X is a u-ideal, but not a strict u-ideal, in its bidual.

Note that if, in addition to the assumptions in Theorem 4.3, we assume that X does not contain a copy of ℓ_1 we get that $I_{X^{**}}$ is the unique extension of k_X to X^{**}. Indeed, by combining [5, Proposition 2.7, Lemma 5.3] with [6, Proposition 2.5] we arrive at the following result.

THEOREM 4.6. *Let X be a Banach space which contains no copies of ℓ_1 and with $\kappa_u(X) < 2$. Then X is a VN-subspace in X^{**}. In particular, $\mathcal{E}(X^{**}, X^{**}) = \{I_{X^{**}}\}$.*

REMARK 4.7. Both the James space J and the James tree space JT are separable dual spaces which contain no copies of ℓ_1 [7, 8]. Thus, by a result of Belobrov [3, Corollary 1], both $\mathcal{E}(J^{**}, J^{**})$ and $\mathcal{E}(JT^{**}, JT^{**})$ contain more than one element and from Theorem 4.6 it follows that both $\kappa_u(J) \geq 2$ and $\kappa_u(JT) \geq 2$.

In [5, Proposition 7.1] it is proven that if X contains no copies of ℓ_1 and is a u-ideal in its bidual with u-projection P on X^{***}, then $V = P(X^{***})$ is weak* dense in X^{***}. In Corollary 4.10 we will see that for a separable u-ideal in its bidual this happens exactly when ℓ_1 is not present.

PROPOSITION 4.8. *Let X be a u-ideal in Y with u-projection P on Y^*. If X is a u-summand in $Z = \mathrm{span}(X, \{y\})$ for some $y \in Y \setminus X$, then the unconditional $T \in \mathcal{E}(Y, X^{**})$ corresponding to P is not one-to-one.*

PROOF. By [9, Lemmas 2.2 and 3.1], the unconditional $T_Z \in \mathcal{E}(Z, X^{**})$ is given by $T|_Z$. Since X is a u-summand in Z and the u-projection is unique, T_Z must be a projection. Thus T_Y cannot be one-to-one. □

COROLLARY 4.9. *Let X be separable u-ideal in its bidual containing a copy of ℓ_1. Then $T \in \mathcal{E}(X^{**}, X^{**})$ corresponding to the u-projection on X^{**} is not one-to-one.*

PROOF. By a result of Maurey [11], there is $x^{**} \in X^{**}$ such that $\|x^{**} - x\| = \|x^{**} + x\|$ for all $x \in X$. Thus X is a u-summand in $Z = \mathrm{span}(X, \{x^{**}\})$. Indeed, let P be the natural projection from Z onto X. Then

$$\|(I - 2P)(rx^{**} + x)\| = \|rx^{**} - x\| = \|rx^{**} + x\|,$$

so P is a u-projection. Using Proposition 4.8, T cannot be one-to-one. □

COROLLARY 4.10. *Let X be a separable u-ideal in its bidual with u-projection P. Then the following statements are equivalent.*

(a) *$V = P(X^{***})$ is weak* dense in X^{***}.*
(b) *X does not contain a copy of ℓ_1.*

PROOF. (a) ⇔ (b) follows from Corollary 4.9 and [5, Proposition 7.1]. □

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Note that u-ideals in their biduals always contain a copy of c_0 or ℓ_1. Indeed, suppose that X is a u-ideal in X^{**} and does not contain a copy of c_0. Then, by [5, Theorem 3.5], X is a u-summand in X^{**}. Thus, for every $x^{**} \in X^{**}$, we have $\|x^{**} - x\| = \|x^{**} + x\|$. By a result of Maurey, X then contains a copy of ℓ_1 (see [11]).

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https://doi.org/10.1017/S0004972710001735 Published online by Cambridge University Press