A RELATION BETWEEN ULTRASPHERICAL AND JACOBI POLYNOMIAL SETS

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1. Introduction. The Jacobi polynomials may be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_2F_1\left[\begin{matrix} -n,\ 1+\alpha+\beta+n; \\ 1+\alpha; \end{matrix}\ \frac{1-x}{2}\right], \tag{1}$$

where $(a)_n = a\,(a+1)\ldots(a+n-1)$. Putting $\beta = \alpha$ gives the ultraspherical polynomials $P_n^{(\alpha,\alpha)}(x)$ which have as a special case the Legendre polynomials $P_n(x) = P_n^{(0,0)}(x)$.

Now the Jacobi polynomials can be expressed in terms of the simpler ultraspherical polynomials of course by a relation such as

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} c_k\, P_k^{(\alpha,\alpha)}(x),$$

where the c_k may be found by the orthogonality properties of the $P_k^{(\alpha,\alpha)}(x)$. In general, the c_k will be very complicated.

This paper will show that a Jacobi polynomial may also be formed by summing across a set of sets of ultraspherical polynomials.

Let

$$f_{j,0} = P_j^{(\frac{1}{2}(\alpha+\beta),\,\frac{1}{2}(\alpha+\beta))}(x), \qquad j = 0, 1, 2, \ldots,$$

$$f_{j,2k} = \frac{(-1)^k\left(\frac{\beta-\alpha}{2}\right)_k\left(\frac{\alpha-\beta}{2}\right)_k}{2^{2k}k!\left(\frac{1}{2}\right)_k}\, P_j^{(\frac{1}{2}(\alpha+\beta)+k,\,\frac{1}{2}(\alpha+\beta)+k)}(x), \quad k = 1, 2, 3, \ldots, \tag{2}$$

$$f_{j,2k+1} = \left(\frac{\alpha-\beta}{2}\right)\frac{(-1)^k\left(\frac{\beta-\alpha+1}{2}\right)_k\left(\frac{\alpha-\beta+1}{2}\right)_k}{k!\left(\frac{3}{2}\right)_k\, 2^{2k}}\, P_j^{(\frac{1}{2}(\alpha+\beta+1)+k,\,\frac{1}{2}(\alpha+\beta+1)+k)}(x),$$

$$k = 0, 1, 2, \ldots.$$

Then consider the matrix

$$||f_{j,k}||, \tag{3}$$

where j is the column index and k is the row index, and where $j, k = 0, 1, 2, \ldots$. Note that each row, given by fixed k with $j = 0, 1, 2, \ldots$, represents a true ultraspherical polynomial set multiplied through by a common constant factor.

The result of this paper is that Jacobi polynomials may be formed from the diagonals of array (3), viz.,

$$P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} f_{j,n-j} = \sum_{k=0}^{n} f_{n-k,k}. \tag{4}$$

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Note that for the special case, $\beta = \alpha$, of the ultraspherical polynomials,

$$(5) \qquad f_{j,k} = 0 \qquad k = 1, 2, 3 \ldots,$$
$$f_{j,0} = P_j^{(\alpha,\alpha)}(x).$$

The arrays corresponding to other special Jacobi polynomials have certain simplifications. For example, in the array for $P_n^{(\alpha,\beta)}(x)$ when $\beta = \alpha + 2m$ for m a positive integer, it happens that $f_{j,2k} = 0$ for $m + 1 \leqslant k$, that is, all even-numbered rows after a finite number vanish. Similar results hold for $\beta = \alpha - 2m$, $\beta = \alpha \pm (2m + 1)$.

2. Proof of results (4). First note

$$(6) \qquad f_{0,0} = 1 = P_0^{(\alpha,\beta)}(x).$$

For $n \geqslant 1$, the result (4) will be proved in the form

$$(7) \quad P_n^{(\alpha,\beta)}(x) = P_n^{(\frac{1}{2}(\alpha+\beta),\frac{1}{2}(\alpha+\beta))}(x) + \left(\frac{\alpha-\beta}{2}\right)\sum_{k=1}^{n} \frac{\left(\frac{\alpha-\beta+2-k}{2}\right)_{k-1} P_{n-k}^{(\frac{1}{2}(\alpha+\beta+k),\frac{1}{2}(\alpha+\beta+k))}(x)}{k!}.$$

Let the right side of (7) be denoted by $g_n(x)$. The remainder of this section will be devoted to showing that $g_n(x) = P_n^{(\alpha,\beta)}(x)$, for $n \geqslant 1$. By splitting the summation in (7) into even and odd indices and using definition (2), the reader may show easily that

$$g_n(x) = \sum_{j=0}^{n} f_{j,n-j}.$$

Then (4) will be proven once it is demonstrated that $g_n(x) = P_n^{(\alpha,\beta)}(x)$.

By using definition (1), it follows that

$$(8) \quad g_n(x) = \frac{\left(\frac{2+\alpha+\beta}{2}\right)_n}{n!} \sum_{j=0}^{n} \frac{(-n)_j(1+\alpha+\beta+n)_j\left(\frac{1-x}{2}\right)^j}{\left(\frac{2+\alpha+\beta}{2}\right)_j \; j\,!}$$
$$+ \left(\frac{\alpha-\beta}{2}\right)\sum_{k=1}^{n}\sum_{j=0}^{n-k} \frac{\left(\frac{\alpha-\beta+2-k}{2}\right)_{k-1}\left(\frac{2+\alpha+\beta+k}{2}\right)_{n-k}(-n+k)_j(1+\alpha+\beta+n)_j\left(\frac{1-x}{2}\right)^j}{k!\,(n-k)!\,\left(\frac{2+\alpha+\beta+k}{2}\right)_j\, j!}$$
$$= \frac{(-1)^n(1+\alpha+\beta+n)_n}{n!}\left(\frac{1-x}{2}\right)^n + \sum_{j=0}^{n-1}\left[\frac{\left(\frac{2+\alpha+\beta}{2}\right)_n(1+\alpha+\beta+n)_j\left(\frac{x-1}{2}\right)^j}{(n-j)!\,\left(\frac{2+\alpha+\beta}{2}\right)_j\, j!}\right.$$
$$\left. + \left(\frac{\alpha-\beta}{2}\right)\sum_{k=1}^{n-j}\frac{\left(\frac{\alpha-\beta+2-k}{2}\right)_{k-1}\left(\frac{2+\alpha+\beta+k}{2}\right)_{n-k}(1+\alpha+\beta+n)_j\left(\frac{x-1}{2}\right)^j}{k!\,j!\,(n-k-j)!\,\left(\frac{2+\alpha+\beta+k}{2}\right)_j}\right].$$

Thus

$$(9) \quad g_n(x) = \frac{(-1)^n(1+\alpha+\beta+n)_n}{n!}\left(\frac{1-x}{2}\right)^n + \sum_{j=0}^{n-1}\frac{(1+\alpha+\beta+n)_j\left(\frac{x-1}{2}\right)^j}{j!} Z_{n-j},$$

where

$$(10)\qquad Z_p = \frac{(\frac{2+\alpha+\beta}{2}+j)_p}{p!} + \left(\frac{\alpha-\beta}{2}\right)\sum_{k=1}^{p}\frac{(\frac{\alpha-\beta+2-k}{2})_{k-1}(\frac{2+\alpha+\beta+k}{2}+j)_{p-k}}{k!\,(p-k)!},\quad p = 1,2,3,\ldots.$$

The functions Z_p will be simplified by means of a generating function. From (10), form

$$(11)\qquad \sum_{p=1}^{\infty} Z_p t^p = \sum_{p=1}^{\infty}\frac{(\frac{2+\alpha+\beta}{2}+j)_p t^p}{p!} + \left(\frac{\alpha-\beta}{2}\right)\sum_{p=1}^{\infty}\sum_{k=1}^{p}\frac{(\frac{\alpha-\beta+2-k}{2})_{k-1}(\frac{2+\alpha+\beta+k}{2}+j)_{p-k}t^p}{k!\,(p-k)!}$$

$$= -1 + (1-t)^{-j-\frac{1}{2}(2+\alpha+\beta)} + \left(\frac{\alpha-\beta}{2}\right)\sum_{k=1}^{\infty}\sum_{p=0}^{\infty}\frac{(\frac{\alpha-\beta+2-k}{2})_{k-1}(\frac{2+\alpha+\beta+k}{2}+j)_p t^{p+k}}{k!\,p!},$$

or

$$(12)\qquad \sum_{p=1}^{\infty} Z_p t^p = -1 + (1-t)^{-j-\frac{1}{2}(2+\alpha+\beta)} + \left(\frac{\alpha-\beta}{2}\right)(1-t)^{-j-\frac{1}{2}(2+\alpha+\beta)}\sum_{k=1}^{\infty}\frac{(\frac{\alpha-\beta+2-k}{2})_{k-1}t^k(1-t)^{-\frac{1}{2}k}}{k!}.$$

In (11), (12), and subsequently, it is intended that the branch of the power of $(1 - t)$ is the one which approaches $+1$ as t approaches zero.

Next, by splitting into summations over even and odd indices, it follows that

$$(13)\qquad \sum_{k=1}^{\infty}\frac{(\frac{\alpha-\beta+2-k}{2})_{k-1}u^k}{k!}$$

$$= u\sum_{k=0}^{\infty}\frac{(\frac{\alpha-\beta+1}{2})_k(\frac{1+\beta-\alpha}{2})_k}{(\frac{3}{2})_k\,k!}\left(-\frac{u^2}{4}\right)^k - \sum_{k=1}^{\infty}\frac{(\frac{\alpha-\beta}{2})_k(\frac{2-\alpha+\beta}{2})_{k-1}}{(\frac{1}{2})_k\,k!}\left(-\frac{u^2}{4}\right)^k$$

$$= u\,{}_2F_1\left[\begin{matrix}\frac{\alpha-\beta+1}{2}, \frac{1+\beta-\alpha}{2}; \\ \frac{3}{2}\quad ;\end{matrix} -\frac{u^2}{4}\right] - \left(\frac{2}{\beta-\alpha}\right)\left\{{}_2F_1\left[\begin{matrix}\frac{\alpha-\beta}{2}, \frac{\beta-\alpha}{2}; \\ \frac{1}{2}\quad ;\end{matrix} -\frac{u^2}{4}\right] - 1\right\},$$

where, for $\beta = \alpha$, the indeterminate form vanishes. Substituting the results of (13) into (12) gives

$$(14)\qquad \sum_{p=1}^{\infty} Z_p t^p = -1 + (1-t)^{-j-\frac{1}{2}(2+\alpha+\beta)}\cdot Q,$$

with

$$Q = {}_2F_1\left[\begin{matrix}\frac{\alpha-\beta}{2}, \frac{\beta-\alpha}{2}; \\ \frac{1}{2}\quad ;\end{matrix}\frac{-t^2}{4(1-t)}\right] + \frac{\alpha-\beta}{2}\frac{t}{(1-t)^{\frac{1}{2}}}\,{}_2F_1\left[\begin{matrix}\frac{\alpha-\beta+1}{2}, \frac{1+\beta-\alpha}{2}; \\ \frac{3}{2}\quad ;\end{matrix}\frac{-t^2}{4(1-t)}\right].$$

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Using the well-known relation

(15) $${}_2F_1(a, b; c; z) = (1 - z)^{-a}\, {}_2F_1(a, c - b; c; \tfrac{z}{z-1}),$$

it follows that

(16) $$Q = \left[\frac{(2-t)^2}{4(1-t)}\right]^{\frac{1}{2}(\beta-\alpha)} {}_2F_1\left[\begin{matrix}\frac{\alpha-\beta}{2}, \frac{1+\alpha-\beta}{2}; \\ \frac{1}{2}\;\;;\end{matrix}\left(\frac{t}{(2-t)}\right)^2\right]$$
$$+ \left(\frac{\alpha-\beta}{2}\right)\frac{t}{(1-t)^{\frac{1}{2}}}\left[\frac{(2-t)^2}{4(1-t)}\right]^{\frac{1}{2}(\beta-\alpha-1)} {}_2F_1\left[\begin{matrix}\frac{\alpha-\beta+1}{2}, \frac{2+\alpha-\beta}{2}; \\ \frac{3}{2}\;\;;\end{matrix}\left(\frac{t}{2-t}\right)^2\right]$$

or

(17) $$Q = \frac{(2-t)^{\beta-\alpha}}{[4(1-t)]^{\frac{1}{2}(\beta-\alpha)}}\left\{ {}_2F_1\left[\begin{matrix}\frac{\alpha-\beta}{2}, \frac{1+\alpha-\beta}{2}; \\ \frac{1}{2}\;;\end{matrix}\left(\frac{t}{2-t}\right)^2\right]\right.$$
$$\left. + \tfrac{1}{2}(\alpha-\beta)\left(\frac{t}{2-t}\right) {}_2F_1\left[\begin{matrix}\frac{\alpha-\beta+1}{2}, \frac{2+\alpha-\beta}{2}; \\ \frac{3}{2}\;;\end{matrix}\left(\frac{t}{2-t}\right)^2\right]\right\}.$$

On expansion,

(18) $$Q = \frac{(2-t)^{\beta-\alpha}}{[4(1-t)]^{\frac{1}{2}(\beta-\alpha)}}\left[\sum_{n=0}^{\infty}\frac{(\alpha-\beta)_{2n}}{(2n)!}\left(\frac{t}{2-t}\right)^{2n}\right.$$
$$\left. + (\alpha-\beta)\sum_{n=0}^{\infty}\frac{(\alpha-\beta+1)_{2n}}{(2)_{2n}}\left(\frac{t}{2-t}\right)^{2n+1}\right].$$

Thus

(19) $$Q = \frac{(2-t)^{\beta-\alpha}}{[4(1-t)]^{\frac{1}{2}(\beta-\alpha)}}\left[\sum_{n=0}^{\infty}\frac{(\alpha-\beta)_{2n}}{(2n)!}\left(\frac{t}{2-t}\right)^{2n}\right.$$
$$\left. + \sum_{n=0}^{\infty}\frac{(\alpha-\beta)_{2n+1}}{(2n+1)!}\left(\frac{t}{2-t}\right)^{2n+1}\right],$$

or

(20) $$Q = \frac{\left(\frac{2-t}{2}\right)^{\beta-\alpha}}{(1-t)^{\frac{1}{2}(\beta-\alpha)}}\sum_{n=0}^{\infty}\frac{(\alpha-\beta)_n}{n!}\left(\frac{t}{2-t}\right)^n.$$

The series in (20) represents a binomial expansion, and (20) becomes

(21) $$Q = (1-t)^{\frac{1}{2}(\beta-\alpha)}.$$

The result of (15)–(21) changes (14) into

(22) $$\sum_{p=1}^{\infty} Z_p\, t^p = -1 + (1-t)^{-j-1-\alpha}.$$

Expanding the right side of (22) gives

(23) $$\sum_{p=1}^{\infty} Z_p\, t^p = \sum_{p=1}^{\infty}\frac{(1+\alpha+j)_p\, t^p}{p!}$$

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and thus

$$Z_p = \frac{(1 + \alpha + j)_p}{p!}. \tag{24}$$

Substitute this in (9) to find

$$\begin{aligned} g_n(x) &= \frac{(-1)^n(1 + \alpha + \beta + n)_n}{n!} \left(\frac{1-x}{2}\right)^n \\ &\qquad + \sum_{j=0}^{n-1} \frac{(1 + \alpha + \beta + n)_j \ (\frac{x-1}{2})^j \ (1 + \alpha + j)_{n-j}}{j! \ (n - j)!} \\ &= \frac{(-1)^n \ (1 + \alpha + \beta + n)_n \ (\frac{1-x}{2})^n}{n!} \\ &\qquad + \frac{(1 + \alpha)_n}{n!} \sum_{j=0}^{n-1} \frac{(1 + \alpha + \beta + n)_j \ (\frac{1-x}{2})^j \ (-n)_j}{j! \ (1 + \alpha)_j} \\ &= P_n^{(\alpha,\beta)}(x), \end{aligned} \tag{25}$$

as desired. This completes the proof of (7).

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