

THE HODGE COHOMOLOGY AND CUBIC EQUIVALENCES

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In 1969, Mumford [8] proved that, for a complete non-singular algebraic surface F over the complex number field C , the dimension of the Chow group of zero-cycles on F is infinite if the geometric genus of F is positive. To this end, he defined a regular 2-form η_f on a non-singular variety S for a regular 2-form η on F and for a morphism $f: S \rightarrow S^n F$, where $S^n F$ is the n -th symmetric product of F , and he showed that η_f vanishes if all 0-cycles $f(s)$, $s \in S$, are rationally equivalent. Roitman [9] later generalized this to a higher dimensional smooth projective variety V . For $\omega \in H^0(V, \Omega_V^q)$, he has defined $\omega_n \in H^0(S^n V, \Omega^n)$ and proved that ω_n has the following property: if $f, g: S \rightarrow S^n V$ are morphisms such that the zero cycles $f(s)$ and $g(s)$ are rationally equivalent for every $s \in S$, then $f^*\omega_n = g^*\omega_n$. We may say this property, roughly, like this: $f^*\omega_n$ cannot distinguish the rational equivalence relation. The rational equivalence is the finest equivalence relation among the adequate equivalence relations (cf. [12]). We can therefore pose the problem: which equivalence relation can $f^*\omega_n$ distinguish and which one can $f^*\omega_n$ not?

On the other hand, Samuel has defined the cubic equivalences in [12]. Consider an algebraic family of cycles on a smooth projective variety V over an algebraically closed field k , parametrized by a smooth variety S . We can regard this family as a "function" on S with values in the set of cycles on V . A cycle algebraically equivalent to zero can be considered as the difference of values at two points for an appropriate "function" on a smooth projective curve. We shall assume that the parameter space S is a product of two curves $C_1 \times C_2$. Then we can define a difference of the second order: take two points $a_i^{(0)}$ and $a_i^{(1)}$ on C_i , $i = 1, 2$, respectively, and form a difference of values at $(a_1^{(0)}, a_2^{(0)})$ and $(a_1^{(1)}, a_2^{(0)})$. We also form a difference between $(a_1^{(0)}, a_2^{(1)})$ and $(a_1^{(1)}, a_2^{(1)})$. The difference of the second order is the difference of these two differences.

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Similarly we can consider the differences of higher order. The totality of differences of the ℓ -th order with values in the cycles on V defines an equivalence relation on the set of cycles on V , which we call the ℓ -cubic equivalence relation. This equivalence relation behaves nicely, i.e., if X and X' are cycles on V , ℓ -cube equivalent, then, for a morphism f , the direct images f_*X and f_*X' , or the pull-backs f^*X and f^*X' , if defined, are also ℓ -cube equivalent.

Now we assume, for simplicity, $k = C$ and let f be a family of effective r -cycles on V parametrized by S . To an element $\omega \in H^{r+\ell, r}(V) = H^r(V, \Omega^{r+\ell})$, we can attach a regular ℓ -form $f^*\omega$ on S , generalizing Roitman's $f^*\omega_n$. Then one of our theorems (5.7) replies partially to the problem raised above, by saying that $f^*\omega$ cannot distinguish the ℓ' -cubic equivalence relation for $\ell < \ell'$; but $f^*\omega$ happens to distinguish the ℓ -cubic equivalence, cf. (5.14). Our answer is, however, incomplete, as the following example indicates: if S is a point and $\ell = 0$, then $f^*\omega$ is the dual of the fundamental class of $\text{Im}(f)$, and it occurs that $f^*\omega = g^*\omega$ but $\text{Im}(f)$ and $\text{Im}(g)$ are not algebraically equivalent, i.e., not 1-cube equivalent by the example of Griffiths [3, 7].

Regarding the Hodge cohomology $H^{**}(V)$ as a functor from the category of smooth projective varieties to that of C -vector spaces, we shall denote by ${}^*H^{**}(V)$ the minimum subfunctor covariant and contravariant of $H^{**}(V)$, stable by the multiplication, and containing all the fundamental classes of algebraic cycles and $H^{0,1}(V)$. Then ${}^*H^{p,q}(V) = 0$ for $p > q$ and ${}^*H^{p-1,p}(V)$ can be interpreted as the tangent space for the algebraic part of the p -th intermediate Jacobian. If $\text{gr}^\ell CH^p(V)$ is the set of cycles on V of codimension p , ℓ -cube equivalent to zero modulo $(\ell + 1)$ -cubic equivalence, then the theorem (5.14) states that ${}^*H^{p-\ell,p}(V) \neq 0$ implies $\text{gr}^\ell CH^p(V) \otimes \mathbf{Q} \neq 0$. This shows that a part ${}^*H^{p-\ell,p}(V)$ of the Hodge cohomology $H^{p-\ell,p}(V)$ controls the structure of Chow group of codimension p .

In Chapter 1, we describe relations between cycles on $S \times V$ and the rational maps from S to the Chow schemes of V , which are reformulations of well-known facts about Chow schemes. In Chapter 2, we generalize a part of Roitman's theory to the case of cycles of intermediate dimension and see that $f^*\omega$ never distinguishes the rational equivalence. The statements of the propositions in this Chapter and their proofs, which we refer to [9] with slight modification, are the prototypes of

theorems in Chapter 5. In Chapter 3, we define cubic equivalences and describe the elementary properties of them. We also remark an interesting theorem essentially due to Swan: if the ground field is the algebraic closure of a finite field, then the theorem of square holds for arbitrary cycles on a smooth projective variety. In Chapter 4, we show that the set of rational maps from a projective scheme into another forms a scheme as $\text{Hom}_S(X, Y)$ does, for use in the following chapter. Chapter 5 is devoted mainly to the statements and the proofs of the theorems mentioned above.

§ 1.

(1.1) Let k be an arbitrary field, and V be a geometrically integral algebraic k -scheme, r an integer ≥ 0 . By an r -cycle on V , we understand a formal sum $\sum n_i X_i$ of r -dimensional integral subschemes X_i of V , where $n_i \in \mathbb{Z}$. If $n_i \geq 0$, X is said to be effective. We say also X a cycle on V of codimension $(m - r)$ where $m = \dim V$. If K/k is a field extension, we have a morphism $\varphi_{K/k}$ of schemes $V_K = V \times_k K \rightarrow V$. Given an integral subscheme X of V , we define $\varphi_{K/k, V}^*(X)$ by

$$\varphi_{K/k, V}^*(X) = \sum \text{length}(\mathcal{O}_{X_K, y}) \bar{y},$$

where y runs over the maximal points of X_K . By linearity, we can extend $\varphi_{K/k, V}^*$ to a map from the set of cycles on V to those on V_K . A cycle X' on V_K is said to be rational over k if there exists a cycle X on V such that $X' = \varphi_{K/k, V}^*(X)$. Note that such X is unique.

(1.2) Suppose further V projective over k and fix an embedding $V \hookrightarrow \mathbb{P}^n$ over k and take an algebraically closed field $\Omega \supset k$. Then we can speak of the degree of an r -cycle on V and the Chow scheme $C_r(V_\Omega)_d$ of effective r -cycles of degree d on V_Ω . $C_r(V_\Omega)_d$ is defined over k , i.e., there exists a scheme $C_r(V)_d$ over k such that $C_r(V_\Omega)_d = C_r(V)_d(\Omega)$. The Ω -valued points of $C_r(V_\Omega)_d$ correspond bijectively to the set of r -cycles on V_Ω of degree d . Let X be an effective r -cycle on V_Ω of degree d and \bar{x} the corresponding Ω -valued point of $C_r(V_\Omega)_d$. We say that \bar{x} is rational over k if $\kappa(x) = k$ for the image x of \bar{x} by $C_r(V_\Omega)_d = C_r(V)_d(\Omega) \rightarrow C_r(V)_d$. If X is rational over k , then \bar{x} is rational over k ; conversely if \bar{x} is rational over k , then there exists a purely inseparable finite extension K of k such that X is rational over K (cf. [11], p. 47). Therefore if $\text{char } k = 0$ and \bar{x} is rational over k , X is rational over k .

(1.3) If K/k is a field extension, and if T is a variety over k such that $k(T) = K$ then for r -cycle X' on V_K , we can define an r -cycle \bar{X}' on $T \times V$ by the following procedure: if X' is an integral subscheme of V_K , $\bar{X}' =$ the schematic image of X' by $V_K \rightarrow T \times V$. For general X' , we define \bar{X}' by linearity. If X is an integral subscheme of $T \times V$ such that $X \subset T \times V \rightarrow T$ is surjective, then X_K is an integral subscheme of V_K . For $X = \sum n_i X_i$, if $X_i \subset T \times V \rightarrow T$ are surjective for all i , $n_i \neq 0$, we say that X is non-degenerate on T . For a cycle X on $T \times V$ non-degenerate on T , we can define the cycle X_K by linearity. Then $X' \mapsto \bar{X}'$ and $X \mapsto X_K$ define a one-to-one correspondence between the set of r -cycles on V_K and the set of cycles on $T \times V$ of codimension $p = m - r$, non-degenerate on T .

For a cycle X on $T \times V$ nondegenerate on T , if d is the degree of the cycle X_K on V_K , X_K determines a K -rational point of $C_r(V)_d$, hence a k -rational map $f: T \dashrightarrow C_r(V)_d$. Conversely, given a k -rational map $f: T \dashrightarrow C_r(V)_d$, we denote by x the image of the generic point of T . The corresponding cycle is rational over a purely inseparable extension K' of K . If $K' = K$ (true when $\text{char } k = 0$), let X' be the corresponding cycle on V_K and \bar{X} the cycle on $T \times V$, the "closure" of X' ; then \bar{X} is (effective and) non-degenerate on T and $\bar{X}_K = X'$ is of degree d . If $f: T \dashrightarrow C_r(V)_d$ is obtained by a cycle on $T \times V$ as above, then we may assume $K' = K$ and $\bar{X} = X$. Starting from a rational map $\bar{f}: T \dashrightarrow C_r(V)_d$ and if $K' = K$, we obtain a cycle \bar{X} on $T \times V$. The rational map induced from the cycle \bar{X} is the rational map f . If $\text{char } k = 0$, it therefore follows that the correspondences $X \mapsto f$ and $f \mapsto \bar{X}$ define bijective maps between the set of rational maps of T to $C_r(V)_d$ and the set of cycles X on $T \times V$ of codimension p which is non-degenerate on T and such that $X(t)$ for general $t \in T$ is degree d .

Remark (1.3.1). Let $f: T \dashrightarrow C_r(V)_d$ be a rational map and K' a purely inseparable extension of $k(T)$ over which the corresponding cycle is rational. If there exists a smooth projective variety T' with the function field K' , we have a cycle \bar{X} on $T' \times V$ with the properties described above. Note that if C is a projective curve and if $K'/k(C)$ is a purely inseparable finite extension, there exists a smooth projective model C' of K' ; and if C is of genus zero, so is C' . Taking this fact into account, the reader will convince oneself that the propositions (2.2) and (2.3), and the theorems (5.5) and (5.6) below hold even in the positive characteristic

case (provided that k is uncountable if necessary).

(1.4) Let k be algebraically closed, V smooth projective, T smooth quasi-projective and X a cycle on $T \times V$ of codimension p , non-degenerate on T . There exists a normal projective variety \bar{T} containing T as an open set. Let \bar{X} be the closure of X in $\bar{T} \times V$, and d the degree of the cycle $\bar{X}_{k(\bar{T})}$ on $V_{k(\bar{T})}$. \bar{X} induces a k -rational map $f': \bar{T} \dashrightarrow C_r(V)_d$, or $f: T \dashrightarrow C_r(V)_d$. If $t \in \bar{T}$ is a closed point such that $\bar{X}(t)$ is defined, then $f'(t)$ is defined and corresponds to $\bar{X}(t)$ (cf. [11], p. 107, and Zariski Main Theorem). Conversely, if a closed point $t_0 \in T$ is in $\text{dom}(f)$, then $\bar{X}(t_0)$ is defined. In fact, let Y be the cycle on V corresponding to $f(t_0)$. Since $T \times V$ is smooth, if $\bar{X}(t_0)$ is not defined, $\text{Supp}(\bar{X}) \cap (t_0 \times V)$ is of dimension $> r$. Hence there exists an irreducible component X_i of $\text{Supp}(\bar{X})$ whose fibre over t_0 is not in $\text{Supp} Y$. Let C' be an irreducible curve in the component such that C' has a point $x \in V$ not in $\text{Supp} Y$ and $C'_{(t_0)} \neq \emptyset$, where T_0 is the open subset in T of points $t \in T$ such that $\bar{X}(t)$ is defined. We denote by C the image curve of C' on \bar{T} and by \bar{C} its normalization $\varphi: \bar{C} \rightarrow \bar{T}$. Putting $\varphi_0: \bar{C}_0 = \bar{C}_{(t_0)} \rightarrow T_0$, we have a cycle $X_0 = (\varphi_0 \times \text{id})^*(X_{T_0 \times V})$ on $C_0 \times V$ such that if \bar{X}_0 is its closure in $\bar{C} \times V$, we have $\bar{X}_0(s) = \bar{X}(\varphi_0(s))$ for all $s \in \bar{C}_0$. Moreover the rational map induced by the cycle \bar{X}_0 is the morphism $\bar{C} \xrightarrow{\varphi} \bar{T} \xrightarrow{f'} C_r(V)_d$, hence the point $f' \circ \varphi(s) \in C_r(V)_d$ corresponds to the cycle $\bar{X}_0(s)$ for $s \in \bar{C}$. But if s_0 is a point with $\varphi(s_0) = t_0$, $\text{Supp}(\bar{X}_0(t_0))$ will contain the point x , whence $\bar{X}_0(s_0) \neq Y$. This contradicts the fact $f \circ \varphi(s_0) = f(t_0)$.

(1.5) Let T be smooth projective over k and $\psi: T \dashrightarrow C_r(V)_d$ a rational map corresponding to a cycle X , and $t \in T$ a point. If C is a smooth projective curve and $f: C \rightarrow T$ is a morphism such that $f(C) \cap \text{dom} \psi \neq \emptyset$, then we have a rational map $g: C \xrightarrow{f} T \xrightarrow{\psi} C_r(V)_d$, which in fact is a morphism. If $a \in C$ is a point with $f(a) = t$, then as a cycle on V , $g(a) \in C_r(V)_d$ is uniquely determined up to rational equivalence, not depending on the choice of C, f and a , and is in the rational equivalence class $x(t)$, where $x \in CH^p(T \times V)$ is the class of X : If γ is a zero-cycle on T supported by $\text{dom} \psi$, by $[\psi_*(\gamma)]$ we shall understand the cycle $\sum n_i[\psi(t_i)]$, where $\gamma = \sum n_i(t_i)$ ($n_i \in \mathbb{Z}$) and $[\psi(t_i)]$ is the cycle on V corresponding to the Chow point $\psi(t_i)$. Note first that if γ and γ' are 0-cycles rationally equivalent on T , if $\dim T = 1$ (hence ψ is a morphism) and if ψ corresponds to a cycle on $T \times V$, then $\psi_*(\gamma)$ and $\psi_*(\gamma')$ are

rationally equivalent. Now we return to the general case. There exists a cycle γ on C supported by $f^{-1}(\text{dom } \psi)$ such that γ and the cycle (a) are rationally equivalent. Then $[g(a)] \underset{\text{rat}}{\sim} [g_*(\gamma)] = [\psi_*(f_*(\gamma))] = x(f_*(\gamma))$, $(t) = f_*(a) \underset{\text{rat}}{\sim} f_*(\gamma)$ on T , hence the class of $[g(a)] = x(f_*(\gamma)) = x(t)$.

§ 2.

(2.1) Let V be a smooth projective variety of dimension m over an algebraically closed field k of characteristic 0, with an embedding $V \subset P_N$, $C_r(V)_d$ its Chow scheme of effective r -cycles of degree d on V . We put

$$E_{\text{rat}}^p(V)_d = \{(X, Y) \in C_r(V)_d \times C_r(V)_d; X \underset{\text{rat}}{\sim} Y \text{ on } V\}.$$

PROPOSITION (2.2). $E_{\text{rat}}^p(V)$ is a countable union of closed subsets of $C_r(V)_d \times C_r(V)_d$. More precisely, for integers $\pi \geq 0, d', d'' > 0$, let

$$E_{\text{rat}}^p(V)_d^{\pi, d', d''} = \left\{ (X, Y); \exists f \in \text{Hom}^\pi(P_1, C_r(V)_{d'} \times C_r(V)_{d''}) \text{ such that } \right. \\ \left. X + \text{pr}_1 \circ f(0) + \text{pr}_2 \circ f(\infty) = Y + \text{pr}_1 \circ f(\infty) + \text{pr}_2 \circ f(0) \right\}.$$

Then $E_{\text{rat}}^p(V)_d$ is the union of $E_{\text{rat}}^p(V)_d^{\pi, d', d''}$ and the closures of the latter subsets in $C_r(V)_d \times C_r(V)_d$ are contained in $E_{\text{rat}}^p(V)_d$.

The proof is similar to that of [9], Theorem 1, replacing $S^n X$ by $C_r(V)_d$.

Note that if an integral subvariety F of P_N is a countable union of closed subsets F_n , then there exists an n such that $F = F_n$ provided that k is uncountable (This holds even if $\text{char } k \neq 0$).

PROPOSITION (2.3). If k is uncountable and $f, g: T \rightarrow C_r(V)$ are two morphisms from a smooth quasi-projective variety T to the Chow scheme such that for all $t \in T$, $f(t)$ and $g(t)$ are rationally equivalent, then there exists a smooth quasiprojective variety S , a dominant morphism $e: S \rightarrow T$ and a morphism $H: P_1 \times S \rightarrow C_r(V)_{d'} \times C_r(V)_{d''}$ such that

$$f \circ e + (\text{pr}_1 \circ H|_{0 \times S}) + (\text{pr}_2 \circ H|_{\infty \times S}) = g \circ e + (\text{pr}_1 \circ H|_{\infty \times S}) + (\text{pr}_2 \circ H|_{0 \times S}),$$

where “+” denotes the summation on the Chow scheme which are morphisms of the form $C_r(V)_d \times C_r(V)_{d'} \rightarrow C_r(V)_{d+d'}$.

Again the proof is similar to [9], Theorem 2.

(2.4) We shall recall the definitions and elementary properties of the

Hodge cohomology. For a smooth quasi-projective variety U , we set

$$H^{p,q}(U) = H^q(U, \Omega^p).$$

Then if $f: U \rightarrow V$ is a morphism of such varieties, we obtain a map

$$f^*: H^{p,q}(V) \rightarrow H^{p,q}(U)$$

with the functorial property. If f is proper, i.e., projective, we can define a map^{*)}

$$f_*: H^{p,q}(U) \rightarrow H^{p-d,q-d}(V)$$

where $d = \dim U - \dim V$. If U and V are projective, then f^* and f_* are dual provided that $H^{p,q}(U)$ and $H^{p-d,q-d}(V)$ are regarded as the duals of $H^{n-p,n-q}(U)$ ($n = \dim U$) and $H^{m-p,m-q}(V)$ ($m = \dim V = n - d$) respectively by Serre duality. Let V_0 be an open subscheme of V and consider the Cartesian diagram:

$$\begin{array}{ccc} U & \xrightarrow{f} & V \\ i \uparrow & & \uparrow j \\ U_0 & \xrightarrow{f_0} & V_0. \end{array}$$

Then the diagram

$$\begin{array}{ccc} H^{p,q}(U) & \xrightarrow{f_*} & H^{p-d,q-d}(V) \\ i^* \downarrow & & \downarrow j^* \\ H^{p,q}(U_0) & \xrightarrow{f_{0*}} & H^{p-d,q-d}(V_0) \end{array}$$

commutes.

The direct sum of $H^{p,q}(U)$ for $0 \leq p, q \leq \dim U$ has a bigraded (anticommutative) ring structure, and f^* is a ring homomorphism. For a proper morphism $f: U \rightarrow V$, the projection formula holds, i.e., $f_*(x.f^*(y)) = f_*(x).y$ for $x \in H^{**}(U)$ and $y \in H^{**}(V)$.

To a closed integral subscheme Z of U of codimension p , we can attach an element $\{Z\}$ of $H^{p,p}(U)$, called the fundamental class of Z .^{**)} By linearity we can attach an element $\{Z\}$ of $H^{p,p}(U)$ to a cycle Z of

^{*)} cf. [4]. The problem is to define f_* for a closed immersion f . Use the isomorphism $\mathcal{E}_{xV}^i(\mathcal{O}_U, \Omega_V^{p-d}) = \omega \otimes \Omega_V^{p-d}$ ($i = -d$), 0 ($i \neq -d$), where ω is \wedge^{-d} of the normal bundle of U in V [6].

^{**)} Define, for example, $\{Z\} = (-1)^{p-1} c_p(\mathcal{O}_Z) / (p-1)!$, where $c_p(?)$ is the p -th Chern class of $?$; note that $\text{ch}(\mathcal{O}_Z) = \{Z\} + \text{“higher terms”}$ so that the compatibility with pull-back, or with multiplication is obvious. The compatibility with direct image results from Riemann-Roch theorem, cf. [4], p. 151.

codimension p on U . If Z and Z' are cycles on U properly intersecting, then $\{Z, Z'\} = \{Z\}, \{Z\}$. If $f: U \rightarrow V$ and Y is a cycle on V such that f^*Y is defined, then $\{f^*Y\} = f^*\{Y\}$; if f is proper, and X is a cycle on U , $\{f_*X\} = f_*\{X\}$. Hence, if $Z \underset{\text{alg}}{\sim} 0$, then $\{Z\} = 0$.

Let V be a smooth projective variety and $f: T \rightarrow C_r(V)_d$ a morphism from a smooth quasi-projective variety into the Chow scheme of V , corresponding to a cycle X on $T \times V$ non-degenerate on T . For $\omega \in H^{r+\ell, r}(V)$ ($\ell \geq 0$), we define $f^\# \omega$ by

$$(2.4.1) \quad f^\# \omega = \text{pr}_{T^*}(\{X\}, \text{pr}_V^* \omega) =: \{X\}(\omega) = X(\omega).$$

Hence we obtain a map

$$f^\#: H^{r+\ell, r}(V) \longrightarrow H^{\ell, 0}(T).$$

(2.4.2) If $g: S \rightarrow T$ is a morphism, we have $(f \circ g)^\# = g^* f^\#$. In fact $(g \times \text{id}_V)^* X$ is defined, cf. (1.4), and corresponding to $f \circ g$, and we get

$$\begin{aligned} (f \circ g)^\# \omega &= \text{pr}'_{S^*}(\{(g \times \text{id}_V)^*(X)\}, \text{pr}'_V^* \omega) \\ &= \text{pr}'_{S^*}(\{(g \times \text{id}_V)^*\{X\}\}, (g \times \text{id}_V)^* \text{pr}_V^* \omega) \\ &= \text{pr}'_{S^*}(\{(g \times \text{id}_V)^*\{X\}, \text{pr}_V^* \omega\}) \quad (\text{K\"unnetth formula}) \\ &= g^*(\text{pr}_{T^*}(\{X\}, \text{pr}_V^* \omega)) = g^* f^\# \omega. \end{aligned}$$

(2.4.3) If $f: T \rightarrow C_r(V)_d$ and $g: T \rightarrow C_r(V)_d$, are morphisms, we have

$$f + g: T \xrightarrow{(f, g)} C_r(V)_d \times C_r(V)_d \xrightarrow{“+”} C_r(V)_{d+d'}.$$

Then $(f + g)^\# = f^\# + g^\#$. In fact if X and Y are corresponding to f and g , respectively, then $f + g$ corresponds to the cycle $X + Y$, from which the equality follows immediately.

PROPOSITION (2.5). *Under the same hypothesis as in (2.3), we have $f^\# = g^\#$.*

The proof is similar to [9], Theorem 4.

(2.6) Let $CH(V)$ be the Chow ring and $CH^p(V)$ the codimension p part of $CH(V)$. A map κ of a smooth quasi-projective variety T to $CH^p(V)$ ($p + r = m = \dim V$ as before) is said to be *regular* if there exist a smooth quasi-projective variety S , $z \in CH^p(S \times V)$ and a commutative diagram

$$(2.6.1) \quad \begin{array}{ccc} S & & \\ g \downarrow & \searrow z(?) & \\ T & \xrightarrow{\kappa} & CH^p(V), \end{array}$$

where g is proper surjective, and $z(s) = i_s^*(z)$, $i_s: V \simeq s \times V \subset S \times V$, the immersion. Cutting S by hyperplanes transversal to the generic fibre of g , and choosing an irreducible component dominating T and finally desingularizing it, we may assume that g is generically finite. Let Z be a representative of z and Z' the sum (with multiplicity) of components of Z non-degenerate on S , and $Z'' = Z - Z'$. Then each component of Z'' is degenerate on S . Since $z''(s) = 0$ for $s \in S$ where z'' is the class of Z'' in $CH(S \times V)$, we may assume that Z is non-degenerate on S . Z is a difference of two effective cycles Z^+ and Z^- . Let d_+ and d_- be the degrees of $Z^+(s)$ and $Z^-(s)$ for general $s \in S$. Then Z defines a rational map $f: S \dashrightarrow C_r(V)_{d_+} \times C_r(V)_{d_-}$ such that for $s \in S$ with $Z(s)$ defined, $Z(s) = \text{pr}_1 \circ f(s) - \text{pr}_2 \circ f(s)$. Resolving the indeterminacy of the rational map f , we can assume that f is a morphism, and we have a commutative diagram

$$\begin{array}{ccc}
 S & \xrightarrow{f} & C_r(V)_{d_+} \times C_r(V)_{d_-} \ni (X, Y) \\
 g \downarrow & & \downarrow \delta \\
 T & \xrightarrow{\kappa} & CH^p(V) \ni \text{the class of } X - Y.
 \end{array}$$

We suppose now that k is uncountable. For $\omega \in H^{r+l, r}(V)$, we put

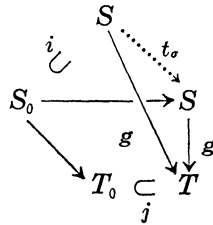
$$\kappa^\# \omega = g_* (f_1^* \omega - f_2^* \omega) / \deg g,$$

where $f_i = \text{pr}_i \circ f$ ($i = 1, 2$). For $\kappa^\# \omega$ to be well-defined, we must show that the second member is independent of S, f and g chosen. Let

$$\begin{array}{ccc}
 S' & \xrightarrow{f'} & C_r(V)_{d'_+} \times C_r(V)_{d'_-} \\
 g' \downarrow & & \downarrow \delta \\
 T & \xrightarrow{\kappa} & CH^p(V)
 \end{array}$$

be another one. We can find a smooth quasi-projective variety S'' , morphisms $h: S'' \rightarrow S$ and $h': S'' \rightarrow S'$ proper surjective and generically finite such that $g \circ h = g' \circ h'$. So we may assume $S = S'$ and $g = g'$ because of $h_* h^* = \deg h$ and $h'_* h'^* = \deg h'$. Since g is surjective, $f_1(s) - f_2(s) = f'_1(s) - f'_2(s)$ for $s \in S$, i.e., $f_1 + f'_2 = f_2 + f'_1$. The proposition (2.5) and (2.4.3) show that $\kappa^\# \omega$ is well-defined. We have further $g^* \kappa^\# \omega = f_1^* \omega - f_2^* \omega$. In fact, we have $k(T) \subset k(S)$. Let K be a finite Galois extension of $k(T)$ containing $k(S)$, and S' be a non-singular model of K such that there exists a proper morphism $S' \rightarrow S$ inducing the inclusion $k(S) \subset k(S') = K$. If g'

denotes the composite $S' \rightarrow S \xrightarrow{g} T$, it is sufficient to show that $g'^*\kappa^\#\omega = f_1^\#\omega - f_2^\#\omega$, where $f'_i: S' \rightarrow S \xrightarrow{f_i} C_r(V)_{a_\pm}$. We can therefore assume $k(S)/k(T)$ is a finite Galois extension. Let σ be a $k(T)$ -automorphism of $k(S)$. Then σ induces a rational map ${}^t\sigma: S \dashrightarrow S$ such that $g \circ {}^t\sigma = g$. Let S'_0 be the domain of definition of ${}^t\sigma$ and we put $T_0 = T \setminus g(S \setminus S'_0)$, an open subset of T and $S_0 = g^{-1}(T_0)$:



The restriction $g_0: S_0 \rightarrow T_0$ of g to S_0 is proper surjective. Setting $\kappa' = \kappa \circ j$ and ${}^i\sigma_0: S_0 \xrightarrow{i} S \dashrightarrow S$, we have two morphisms $f \circ i, f \circ {}^i\sigma_0: S_0 \rightarrow C_r(V)_{a_+} \times C_s(V)_{a_-}$ covering κ' . The argument above shows

$$(f_1 \circ i)^\#\omega - (f_2 \circ i)^\#\omega = (f_1 \circ {}^i\sigma_0)^\#\omega - (f_2 \circ {}^i\sigma_0)^\#\omega,$$

i.e.,

$$i^*(f_1^\#\omega - f_2^\#\omega) = {}^i\sigma_0^*(f_1^\#\omega - f_2^\#\omega),$$

which in turn implies $f_1^\#\omega - f_2^\#\omega$ viewed as a rational ℓ -form on S is invariant under the action of the Galois group. There is therefore a rational ℓ -form ω' on T such that $g^*\omega' = f_1^\#\omega - f_2^\#\omega$, from which

$$\omega' = g_*(f_1^\#\omega - f_2^\#\omega)/\deg g = \kappa^\#\omega, \quad \text{i.e.,} \quad g^*\kappa^\#\omega = f_1^\#\omega - f_2^\#\omega.$$

Note that if Z is a cycle on $S \times V$ of codimension p whose components are degenerate on S , then $\{Z\}(\?): H^{r+\ell, r}(V) \rightarrow H^{\ell, 0}(S)$ is a zero map, since if $S_0 \subset S$ is an open subset such that $Z_0 = Z|_{S_0 \times V}$ is zero, then

$$0 = \{Z_0\}(\?): H^{r+\ell, r}(V) \longrightarrow H^{\ell, 0}(S) \longrightarrow H^{\ell, 0}(S_0).$$

Thus $\kappa^\#\omega$ is characterized as follows: if in the diagram (2.6.1), g is generically finite, $\kappa^\#\omega \in H^{\ell, 0}(T)$ is the element $\omega' \in H^{\ell, 0}(T)$ such that $g^*\omega' = z(\omega)$.

(2.6.2) Suppose $\varphi: T' \rightarrow T$ is a morphism. Then $\kappa \circ \psi$ is also regular and

$$(\kappa \circ \psi)^\# = \psi^* \circ \kappa^\#.$$

(2.6.3) If U is a smooth projective variety and $u \in CH^{r+q}(V \times U)$, we have a map

$$u(?): CH^p(V) \longrightarrow CH^q(U), x \longmapsto u(x) = \text{pr}_{U^*}(u, \text{pr}_V^*(x)).$$

Then $u(?) \circ \kappa: T \rightarrow CH^q(U)$ is also regular and

$$(u(?) \circ \kappa)^\# = \kappa^\# \circ {}^t u: H^{n+l-q, n-q}(U) \longmapsto H^{l,0}(T).$$

These assertions follow readily from the above characterization of $\kappa^\# \omega$.

(2.6.4) If $\kappa_i: T \rightarrow CH^p(V)$ ($i = 1, 2$) are regular maps, then $\kappa_1 + \kappa_2: T \rightarrow CH^p(V)$ is also a regular map and $(\kappa_1 + \kappa_2)^\# = \kappa_1^\# + \kappa_2^\#$. In fact, let $z_i \in CH^p(S_i \times V)$ and $g_i: S_i \rightarrow T$ be proper surjective morphisms such that $\kappa_i \circ g_i = z_i(?)$ ($i = 1, 2$). We are immediately reduced to the case $S_1 = S_2$ and $g_1 = g_2$. Then the assertion is evident.

LEMMA (2.7). *If $\kappa: T \rightarrow CH^p(V)$ is regular, then for any $x \in CH^p(V)$, $\kappa^{-1}(x)$ in T is a countable union of closed subsets.*

In fact we have

$$\begin{array}{ccc} S & \xrightarrow{f} & C_r(V)_{d_+} \times C_r(V)_{d_-} \\ g \downarrow & & \downarrow \delta \\ T & \xrightarrow{\kappa} & CH^p(V). \end{array}$$

Then $g^{-1}(\kappa^{-1}(x)) = f^{-1}(\delta^{-1}(x))$. If x is the class of a difference of effective cycles $X_+ \in C_r(V)_{d'_+}$ and $X_- \in C_r(V)_{d'_-}$, then

$$\delta^{-1}(x) = \{(Y_+, Y_-) \in C_r(V)_{d_+} \times C_r(V)_{d_-}; Y_+ + X_- \underset{\text{rat}}{\sim} Y_- + X_+\}.$$

If $d_+ - d_- \neq d'_+ - d'_-$, $\kappa^{-1}(x) = \emptyset$; if $d_+ - d_- = d'_+ - d'_-$, we have a morphism

$$\begin{aligned} F: C_r(V)_{d_+} \times C_r(V)_{d_-} &\longrightarrow C_r(V)_{d''} \times C_r(V)_{d''}, \\ (Y_+, Y_-) &\longmapsto (X_+ + Y_-, X_- + Y_+), \end{aligned}$$

where $d'' = d_+ + d_-$. $\delta^{-1}(x)$ is a countable union of closed subsets in $C_r(V)_{d_+} \times C_r(V)_{d_-}$ by $\delta^{-1}(x) = F^{-1}(E_{\text{rat}}^p(V)_d)$ and by (2.2). This implies that $g^{-1}(\kappa^{-1}(x))$ is a countable union of closed subsets of S . Since g is proper surjective, $\kappa^{-1}(x) = g(g^{-1}(\kappa^{-1}(x)))$ is a countable union of closed subsets of T .

§ 3.

(3.1) We shall recall the definition of the cubic equivalences. Let k be an algebraically closed field of arbitrary characteristic, V a smooth projective variety over k .

DEFINITION (3.1.1) cf. [12]. Let ℓ be a positive integer. Two cycles X and Y on V of codimension p are said to be ℓ -cube equivalent and denoted by $X \underset{(\ell)}{\sim} Y$ if there exist ℓ curves (smooth projective over k) C_1, \dots, C_ℓ , a cycle Z on $C_1 \times \dots \times C_\ell \times V$ of codimension p , points $a_i^{(0)}$, $a_i^{(1)}$ of C_i ($i = 1, \dots, \ell$) such that

$$(3.1.2) \quad Z(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)}) \text{ are defined for all } i_1, \dots, i_\ell = 0, 1,$$

$$(3.1.3) \quad X - Y = \sum_{i_1, \dots, i_\ell = 0, 1} (-1)^{i_1 + \dots + i_\ell} Z(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)}).$$

For $\ell = 1$, ℓ -cubic equivalence is no other than the algebraic equivalence. 2-cubic equivalence is also called square equivalence and 3-cubic equivalence is called simply cubic equivalence. For convention we say that any two cycles on V are 0-cube equivalent.

Remark (3.1.4). Our definition is a priori different from Samuel's one because in our definition the objects are defined over k , while in Samuel's, they are considered over the universal domain.

PROPOSITION (3.2). 1) *The ℓ -cubic equivalence is an adequate equivalence relation (cf. [12]) so that the set of cycles on V modulo ℓ -cubic equivalence is naturally equipped with a structure of commutative rings and has functorial properties; in particular the ℓ -cubic equivalence is coarser than the rational equivalence.*

2) *If $\ell' < \ell$, the ℓ -cubic equivalence is finer than the ℓ' -cubic equivalence.*

3) *If X is a cycle on V , ℓ -cube equivalent to zero and if X' is a cycle on V' , ℓ' -cube equivalent to zero, then the cycle $X \times X'$ on $V \times V'$ is $(\ell + \ell')$ -cube equivalent to zero.*

The proofs, more or less formal, can be found in [12].

(3.3) We denote by $F^\ell CH^p(V)$ the cycles on V of codimension p , ℓ -cube equivalent to zero modulo rational equivalence and by $F^\ell CH^*(V)$ the direct sum of $F^\ell CH^p(V)$ for $0 \leq p \leq \dim V$. We also write $F^\ell CH^p(V) = F^\ell CH_r(V)$ if $p + r = \dim V$. By definition, $F^0 CH(V) = CH(V)$. We have thus a descending filtration

$$CH(V) = F^0 CH(V) \supset F^1 CH(V) \supset \dots \supset F^\ell CH(V) \supset F^{\ell+1} CH(V) \supset \dots$$

We put: $gr^\ell CH^p(V) = gr^\ell CH_r(V) = F^\ell CH^p(V) / F^{\ell+1} CH^p(V)$.

$$gr^* CH^*(V) = \bigoplus_{\ell, 0 \leq p \leq \dim V} gr^\ell CH^p(V).$$

By (3.2.3), $gr^*CH^*(V)$ has a structure of bigraded commutative rings, the multiplication being induced by that of $CH(V)$. For a morphism $f: V \rightarrow V'$, $f^*: CH(V') \rightarrow CH(V)$ induces a homomorphism of bigraded rings

$$f^*: gr^*CH^*(V') \longrightarrow gr^*CH^*(V);$$

$f_*: CH(V) \rightarrow CH(V')$ induces a group homomorphism

$$f_*: gr^*CH^*(V) \longrightarrow gr^*CH^*(V')$$

such that $f_*(gr^eCH_r(V)) \subset gr^eCH_r(V')$.

EXAMPLES (3.4) 1) $gr^0CH^p(V)$ is the so-called Néron-Severi group of codimension p .

2) $gr^1CH^1(V)$ is isomorphic to the Picard variety of V as groups.

3) We have $gr^1CH_0(V \times V') \simeq gr^1CH_0(V) \times gr^1CH_0(V')$ which results from the formula

$$\sum n_i((x_i, y_i)) \underset{(2)}{\sim} \{\sum n_i(x_i)\} \times (y_0) + (x_0) \times \{\sum n_i(y_i)\},$$

where $x_i, x_0 \in V$, and $y_i, y_0 \in V'$ and n_i are integers such that $\sum n_i = 0$. In particular, if V is a product of curves,

$$gr^1CH_0(V) \xrightarrow{\sim} \text{Alb } V.$$

LEMMA (3.5) (cf. [12, 13]). *Let X be a cycle on V of codimension p and ℓ a positive integer. Then the following conditions on X are equivalent:*

a) X is ℓ -cube equivalent to zero.

b) *There exist ℓ smooth projective varieties T_1, \dots, T_ℓ over k , a cycle Z on $T_1 \times \dots \times T_\ell \times V$ of codimension p and points $a_i^{(0)}$ and $a_i^{(1)}$ on T_i ($1 \leq i \leq \ell$) satisfying the conditions (3.1.2) and (3.1.3) with $Y = 0$.*

c) *The same condition as b), but we further require T'_i s to be jacobian varieties.*

Clearly c) implies b). a) follows from b) because any two points on a variety can be joined by a smooth projective curve not necessarily in the variety. It therefore suffices to show that a) implies c). Let C_i, Z and $a_j^{(i)}$ be as in (3.1.1) with $Y = 0$. We shall show that there exists a cycle X' on V satisfying the condition c) such that X and X' are rationally equivalent. We assume that the genera of C'_i s are positive, since if one of C_i is rational, we can take $X' = 0$. First we suppose Z is a prime cycle, i.e., Z is a variety. Let J_i be the jacobian of C_i and g_i the genus of C_i . For $1 \leq j_i \leq g_i$, let

$$\psi_{j_1 \dots j_\ell} : C_1^{g_1} \times \dots \times C_\ell^{g_\ell} \longrightarrow C_1 \times \dots \times C_\ell$$

be the product of j_i -th projections $C_i^{g_i} \rightarrow C_i$ for $i = 1, \dots, \ell$, and Y the reduced subscheme of $C_1^{g_1} \times \dots \times C_\ell^{g_\ell} \times V$ with underlying closed subset

$$\bigcup_{j_1, \dots, j_\ell} (\psi_{j_1, \dots, j_\ell} \times \text{id}_V)^{-1}(Z).$$

If S_{g_i} is the symmetric group of degree g_i , $G = S_{g_1} \times \dots \times S_{g_\ell}$ acts on $C_1^{g_1} \times \dots \times C_\ell^{g_\ell} \times V$ and Y is stable by the action of G . Moreover

$$(C_1^{g_1} \times \dots \times C_\ell^{g_\ell} \times V)/G = S^{g_1}C_1 \times \dots \times S^{g_\ell}C_\ell \times V.$$

LEMMA (3.5.1). *Let L/K be a finite Galois extension with the Galois group G . If X' is an algebraic K -scheme and Y is a G -stable reduced subscheme of $X = X' \times_K L$, then Y/G is a closed reduced subscheme of X and $(Y/G) \times_K L = Y$.*

Consider the diagram

$$\begin{array}{ccccc} L & \longleftarrow & X'_{(L)} & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ K & \longleftarrow & X' & \longleftarrow & Y' = Y/G. \end{array}$$

We have the natural morphism $k: Y \rightarrow Y'_{(L)}$. We shall show that $k: Y \rightarrow Y'_{(L)}$ is an isomorphism, which implies $Y' = Y/G \rightarrow X'$ is a closed immersion. Y' is reduced because of reducedness of Y . $Y'_{(L)}$ is also reduced since L/K is a finite separable extension. $Y \xrightarrow{k} Y'_{(L)} \rightarrow X$ is a closed immersion by hypothesis. Hence $k: Y \rightarrow Y'_{(L)}$ is also a closed immersion, so that it is sufficient to show that $k: Y \rightarrow Y'_{(L)}$ is bijective, or surjective. Let p be a point of $Y'_{(L)}$. There is a point q of Y such that the image of q in Y' is equal to that of p , by the surjectivity of $Y \rightarrow Y'$. G acts on $Y'_{(L)}$ and the actions of G on $Y'_{(L)}$ and on Y are compatible with the morphism k . Since the action of G on a fibre of $Y'_{(L)} \rightarrow Y'$ is transitive, there exists a $\sigma \in G$ such that $\sigma \cdot q = p$. As Y is G -stable, p is a point of Y .

We shall continue the proof of (3.5). Let η be the generic point of $C_1^{g_1} \times \dots \times C_\ell^{g_\ell}$. Then Y_η is a reduced closed subscheme of $V_{\kappa(\eta)}$. Hence if $K = \kappa(\eta)^G$, there exists a closed subscheme Y' of V_K such that $Y' \times_K \kappa(\eta) = Y_\eta$. If η' is the generic point of $J_1 \times \dots \times J_\ell$, then $\kappa(\eta') = K$; therefore $\bar{Y}'_{\eta'} = Y'$ for the schematic closure \bar{Y}' of Y' in $J_1 \times \dots \times J_\ell \times V$. Let $\pi_i: C_i^{g_i} \rightarrow J_i$ be the canonical morphism and

$$\pi = \pi_1 \times \cdots \times \pi_\ell: C_1^{g_1} \times \cdots \times C_\ell^{g_\ell} \longrightarrow J_1 \times \cdots \times J_\ell.$$

We have $\{(\pi \times \text{id}_V)^{-1}(\bar{Y}')\}_\eta = Y_\eta$. Let F be the union of irreducible components of $(\pi \times \text{id}_V)^{-1}(\bar{Y}')$ degenerate on $C_1^{g_1} \times \cdots \times C_\ell^{g_\ell}$; F is a closed subset of $C_1^{g_1} \times \cdots \times C_\ell^{g_\ell} \times V$ disjoint from $V_{\kappa(\eta')}$. If we denote by U the open subset

$$J_1 \times \cdots \times J_\ell \setminus (\text{Im}(F \longrightarrow C_1^{g_1} \times \cdots \times C_\ell^{g_\ell} \longrightarrow J_1 \times \cdots \times J_\ell)),$$

we have

$$(\pi_V \times \text{id}_V)^{-1}((\bar{Y}' \cap U) \times V) = (Y \cap \pi^{-1}(U)) \times V.$$

Shrinking U if necessary, we can assume that $\pi^{-1}(U) \times V \rightarrow U \times V$ is flat, hence denoting the associated cycles also by the same symbols, we have

$$(\pi_V \times \text{id}_V)^*((\bar{Y}'|U) \times V) = (Y|\pi^{-1}(U)) \times V.$$

By definition we have $Y = \sum_{j_1, \dots, j_\ell} (\psi_{j_1, \dots, j_\ell} \times \text{id}_V)^*(Z)$, hence for $x \in \pi^{-1}(U)$,

$$\bar{Y}'(\pi(x)) = \sum_{j_1, \dots, j_\ell} Z(\psi_{j_1, \dots, j_\ell}(x)).$$

By virtue of the moving lemma, for any $x \in C_1^{g_1} \times \cdots \times C_\ell^{g_\ell}$, we have

$$\bar{y}'(\pi(x)) = \sum_{j_1, \dots, j_\ell} z(\psi_{j_1, \dots, j_\ell}(x)),$$

where \bar{y}' and z are the rational equivalence classes of \bar{Y}' and Z , respectively. By linearity, we may suppose that this equality holds for an arbitrary cycle Z on $C_1 \times \cdots \times C_\ell \times V$. Denoting $\bar{a}_h^{(i)} = (a_h^{(i)}, a_h^{(0)}, \dots, a_h^{(0)}) \in C_h^{g_h}$ ($1 \leq h \leq \ell, i = 0, 1$), we have

$$\begin{aligned} & \sum_{i_1, \dots, i_\ell=0,1} (-1)^{i_1+\dots+i_\ell} \bar{y}'(\pi(\bar{a}_1^{(i_1)}, \dots, \bar{a}_\ell^{(i_\ell)})) \\ &= \sum_{i_1, \dots, i_\ell=0,1} (-1)^{i_1+\dots+i_\ell} \sum_{j_1, \dots, j_\ell} z(\psi_{j_1, \dots, j_\ell}(\bar{a}_1^{(i_1)}, \dots, \bar{a}_\ell^{(i_\ell)})) \\ &= \sum_{j_1, \dots, j_\ell} \sum_{i_1, \dots, i_\ell=0,1} (-1)^{i_1+\dots+i_\ell} z(\psi_{j_1, \dots, j_\ell}(\bar{a}_1^{(i_1)}, \dots, \bar{a}_\ell^{(i_\ell)})) \\ &= \sum_{i_1, \dots, i_\ell=0,1} (-1)^{i_1+\dots+i_\ell} z(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)}) = \text{the class of } X, \end{aligned}$$

since if one of j_1, \dots, j_ℓ is greater than 1, say $j_h > 1$, then

$$\psi_{j_1, \dots, j_\ell}(\bar{a}_1^{(i_1)}, \dots, \bar{a}_\ell^{(i_\ell)}) = (\dots, a_h^{(0)}, \dots).$$

From the moving lemma we deduce that there exists a cycle Z' on $J_1 \times \cdots \times J_\ell \times V$, rationally equivalent \bar{Y}' such that $Z'(\pi(\bar{a}_1^{(i_1)}, \dots, \bar{a}_\ell^{(i_\ell)}))$ are all defined. Putting $\pi_i(\bar{a}_i^{(j)}) = a_i^{(j)} \in J_i$ ($j = 0, 1$), and

$$X' = \sum_{i_1, \dots, i_\ell = 0, 1} Z'(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)}),$$

we have our assertion. If one of C_i is rational, let J_1, \dots, J_ℓ be arbitrary jacobians, $a_i^{(0)}$ and $a_i^{(1)}$ arbitrary points on J_i , and $Z' = 0$. Since X and X' are rationally equivalent, there is a cycle Z^* on $P_1 \times V$ non-degenerate on P_1 (hence the components are flat over P_1) such that $Z^*(0) = X$ and $Z^*(1) = X'$. Consider a rational function on $J_1 \times \dots \times J_\ell$ defined in a neighbourhood of the finite set $\{(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)}); i_1, \dots, i_\ell = 0, 1\}$, with its value 0 at $(a_1^{(0)}, \dots, a_\ell^{(0)})$, and with its value 1 at the other points of the finite set. Let Γ be the closure of its graph in $J_1 \times \dots \times J_\ell \times P_1$ and $Z'' = \Gamma \circ Z^*$, a cycle on $J_1 \times \dots \times J_\ell \times V$; since Z^* is “flat” over P_1 , the cycle is defined. The following first member is defined and

$$Z''(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)}) = \begin{cases} Z^*(0) = X & \text{if } i_1 = \dots = i_\ell = 0 \\ Z^*(1) = X' & \text{otherwise,} \end{cases}$$

hence,

$$\sum_{i_1, \dots, i_\ell = 0, 1} (-1)^{i_1 + \dots + i_\ell} (Z' + Z'')(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)}) = X' + X - X' = X, \tag{3.6}$$

q.e.d.

(3.6) Let A be an arbitrary abelian variety over k . We set

$$I_A = F^1 CH_0(A),$$

and let $*$ denote the Pontrjagin product.

LEMMA (3.6.1). *The elements of $F^\ell CH^p(V)$ are the elements of the form $z(\gamma)$, where $z \in CH^p(A \times V)$, $\gamma \in I_A^{*\ell}$ and A is an abelian variety.*

In fact since $I_A^{*\ell} \subset F^\ell CH_0(A)$, $z(\gamma) \in F^\ell CH^p(V)$. Conversely, let $x \in F^\ell CH^p(V)$ and X a cycle on V representing x . Then X is ℓ -cube equivalent to zero and by (3.5),

$$X = \sum (-1)^{i_1 + \dots + i_\ell} Z(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)}),$$

where Z is a cycle on $J_1 \times \dots \times J_\ell \times V$, and J_i are jacobian varieties. Let $A = J_1 \times \dots \times J_\ell$ and z the class of Z . Then

$$\begin{aligned} x &= \sum (-1)^{i_1 + \dots + i_\ell} z(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)}) \\ &= z(\{(a_1^{(0)} - a_1^{(1)})\} \times \dots \times \{(a_\ell^{(0)} - a_\ell^{(1)})\}). \end{aligned}$$

But we have

$$\begin{aligned} & \{(a_1^{(0)} - (a_1^{(1)})) \times \cdots \times (a_\ell^{(0)} - (a_\ell^{(1)}))\} \\ &= \{(a_1^{(0)}, 0, \dots, 0) - (a_1^{(1)}, 0, \dots, 0)\} \\ & \quad * \cdots * \{(0, \dots, 0, a_\ell^{(0)}) - (0, \dots, 0, a_\ell^{(1)})\}, \end{aligned}$$

which belongs to $I_A^{*\ell}$, so that x is of the form above.

COROLLARY (3.6.2) $F^\ell CH^p(V)$ is a divisible group for any $\ell > 1$

This follows from the fact that $I_A^{*\ell}$ is divisible.

THEOREM (3.7). *If k is the algebraic closure of a finite field, then $F^2 CH^p(V) = 0$ for any integer p and smooth projective variety. In other words, the theorem of square holds for cycles of arbitrary codimension on a smooth projective variety.*

By [1], $I_A^{*2} = 0$ for an abelian variety over k ; the theorem follows from the lemma (3.6.1).

LEMMA (3.8). 1) *Let X be an r -cycle on V . The following are equivalent:*

- i) X is ℓ -cube equivalent to zero.
- ii) There exist ℓ smooth projective curves C_1, \dots, C_ℓ , a rational map

$$\varphi: C_1 \times \cdots \times C_\ell \rightarrow C_r(V)_{a'} \times C_r(V)_{a''}$$

and points $a_i^{(0)}$ and $a_i^{(1)}$ on C_i such that $\varphi(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)})$ are defined for all $i_1, \dots, i_\ell = 0, 1$, and

$$X = \sum_{i_1, \dots, i_\ell = 0, 1} (-1)^{i_1 + \dots + i_\ell} \{ [\text{pr}_1 \circ \varphi(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)})] - [\text{pr}_2 \circ \varphi(a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)})] \}.$$

iii) *In ii), we require C_i 's to be jacobian varieties in stead of smooth projective curves.*

2) *Let C_i ($i = 1, \dots, \ell$) be projective curves (not necessarily smooth but irreducible), $\varphi: C_1 \times \cdots \times C_\ell \rightarrow C_r(V)_{a'}$ a rational map, $a_i^{(0)}$ and $a_i^{(1)}$ points of C_i ($i = 0, \dots, \ell$). For each (i_1, \dots, i_ℓ) , consider a smooth curve Γ and a morphism $\psi: \Gamma \rightarrow C_1 \times \cdots \times C_\ell$ such that $\psi^{-1}(\text{dom } \varphi \cap \text{Reg}(C_1 \times \cdots \times C_\ell)) \neq \emptyset$ and $\psi(x) = (a_1^{(i_1)}, \dots, a_\ell^{(i_\ell)})$ for some $x \in \Gamma$. Then the rational map $\varphi \circ \psi: \Gamma \rightarrow C_r(V)_{a'}$ is a morphism and let $X_{i_1, \dots, i_\ell} = [\varphi \circ \psi(X)]$. The cycle*

1). $i) \Rightarrow ii)$ follows from (1.4) and the fact that, in the definition (3.1.1), we may suppose Z non-degenerate on $C_1 \times \cdots \times C_\ell$. Assume $ii)$. Replacing C'_i s by their purely inseparable coverings and lifting the points $a_j^{(i)}$ to points on the covering if necessary, we may suppose that φ corresponds to a cycle on $C_1 \times \cdots \times C_\ell \times V$; then $i)$ results from (1.4). The equivalence of $i)$ and $iii)$ follows similarly, noticing (3.5).

2). Considering the normalization \bar{C}_i of C_i , replacing it by a purely inseparable covering (if necessary) and taking account of the fact that Γ lifts to a curve $\bar{\Gamma}$ such that the following diagram commutes:

$$\begin{array}{ccc} \bar{\Gamma} & \longrightarrow & \bar{C}_1 \times \cdots \times \bar{C}_\ell \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & C_1 \times \cdots \times C_\ell, \end{array}$$

we may assume that C'_i s are smooth projective and the rational map φ corresponds to a cycle on $C_1 \times \cdots \times C_\ell \times V$. Then 2) is a consequence of (1.5) and the fact that the ℓ -cubic equivalence is coarser than the rational equivalence.

Remark (3.9) Consider the condition on $F^1CH^p(V)$:

(B) There exist an abelian variety A and a group isomorphism $h: F^1CH^p(V) \simeq A$ such that for any $u \in CH^p(T \times V)$, where T is a smooth projective variety, the map

$$T \xrightarrow{u'} F^1CH^p(V) \xrightarrow{\sim} A$$

is a morphism; here u' is a map of the form $t \mapsto u((t) - (t_0))$, $t_0 \in T$.

Then it is immediate that if $F^2CH^p(V) \neq 0$, then $F^1CH^p(V)$ never satisfies the condition (B). Note that the condition (B) is a kind of boundedness condition on $CH^p(V)$.

§ 4.

(4.1) Suppose that k is an arbitrary field, that X and Y are algebraic k -schemes and that X is reduced. Let $f: X \dashrightarrow Y$ be a k -rational map, V the domain of definition of f , an open subset of X , and $f_0: V \rightarrow Y$ a representative of f . Consider the graph $\Gamma \subset V \times Y = V \times_k Y$ of f_0 . Let U be the schematic closure of Γ in $X \times Y$. Since $V \simeq \Gamma$ is reduced, (1) U is also reduced. We have a morphism $U \subset X \times Y \rightarrow X$. By base change $V \rightarrow X$, (2) $\Gamma = U_{(V)} \rightarrow V$ is an isomorphism and (3) $U_{(V)} = \Gamma$ is

dense in U . By the definition of rational map, (4) V is dense in X . Conversely, suppose given a closed subscheme U of $X \times Y$ and an open subset V of X with properties (1) ~ (4). Then by (2), we have a morphism $f_0: V \rightarrow U_{(V)} \rightarrow U \subset X \times Y \rightarrow Y$. The property (4) implies that f_0 defines a k -rational map $f: X \dashrightarrow Y$. It is readily checked that the correspondence $\Gamma \rightarrow U$ is one-to-one. We can therefore identify the set of k -rational maps from X to Y with the set of closed subschemes U of $X \times Y$ such that there exists an open subset V of X satisfying the properties (1) ~ (4). If a k -morphism from Y to X is given, the set of k -rational sections is identified with a set of closed subschemes of Y with some similar properties, cf. (4.2.1) ~ 4) below.

(4.2) Let S be a locally noetherian scheme, X and Y S -schemes of finite type, and $p: X \rightarrow Y$ an S -morphism:

$$\begin{array}{ccc} X & \xrightarrow{p} & Y \\ p_x \searrow & & \swarrow p_y \\ & S & \end{array}$$

Let $\Gamma_{S\text{-rat}}(X/Y)$ be the set of closed subschemes U of X flat over S such that there exists an open subscheme V of Y satisfying the following conditions:

- (4.2.1) $p'_V: U_{(V)} \rightarrow V$ induced by $p': U \subset X \rightarrow Y$ via base change $V \rightarrow Y$ is an isomorphism;
- (4.2.2) For each $s \in S$, $V_s \subset Y_s$ is dense,
- (4.2.3) $U_s \subset X_s$ is geometrically reduced for all $s \in S$;
- (4.2.4) $(U_{(V)})_s \subset U_s$ is dense for all $s \in S$.

Note that given U , there exists the maximum open subscheme V of Y satisfying the above conditions. Given $T \rightarrow S$, $U \mapsto U_{(T)}$ induces a map

$$\Gamma_{S\text{-rat}}(X/Y) \longrightarrow \Gamma_{T\text{-rat}}(X_T/Y_T).$$

We get therefore a contravariant functor

$$(4.2.5) \quad \Gamma_{S\text{-rat}}(X/Y): (\text{loc. noetherian schemes}/S) \rightarrow (\text{Sets}).$$

PROPOSITION (4.3). *Suppose X projective over S , and Y projective and flat over S . Then the functor $\Gamma_{S\text{-rat}}(X/Y)$ is represented by an open subscheme of the Hilbert scheme $\text{Hilb}_{X/S}$.*

It is sufficient to show the following: Suppose U a closed subscheme of X flat over S . For some $s \in S$, if there exists an open dense subset

V' of Y_s such that $U_{V'} \rightarrow V'$ is an isomorphism, that $U_{V'}$ is dense in U_s and that U_s is geometrically reduced, then there exist an open subset S_0 of S containing s such that there exists an open subset V of Y satisfying the conditions (4.2.1) ~ 4)) above when we replace S by S_0 . First of all, by [5], IV, 12.1.1, vii, and by the properness of U over S , the set of $s \in S$ satisfying the condition (4.2.3) is open. The set U' of $x \in U$ such that x is isolated in $p'^{-1}(p'(x))$ is open. Since p' is proper and $p'(U \setminus U') \cap V' = \phi$, $Y_0 = Y \setminus p'(U \setminus U')$ is open in Y and contains V' . Then $U_0 = p'^{-1}(Y_0)$ is proper over Y_0 and flat over S . The morphism $p'_{(Y_0)}: U_0 \rightarrow Y_0$ is finite. Let $U_0 = \text{Spec } A$ with A a coherent \mathcal{O}_{Y_0} -Algebra. Define the coherent \mathcal{O}_{Y_0} -Modules N and C by the exact sequence

$$0 \longrightarrow N \longrightarrow \mathcal{O}_{Y_0} \longrightarrow A \longrightarrow C \longrightarrow 0.$$

For $y \in V'$, the sequence

$$\mathcal{O}_{Y_0, y} / m_y \longrightarrow A_y / m_y A_y \longrightarrow C_y / m_y C_y \longrightarrow 0$$

is exact. $U_y \simeq y$ implies $C_y / m_y C_y = 0$, hence $C_y = 0$, i.e., $\text{Supp } C \cap V' = \phi$. Denote by Y_1 the open subset $Y_0 \setminus \text{Supp } C$ of Y_0 containing V' , and set $U_1 = p'^{-1}(Y_1)$. We have a diagram

$$\begin{array}{ccccccc} & & U_1 & \longrightarrow & U_0 & \longrightarrow & U \\ & & \downarrow & & \downarrow & & \downarrow \\ V' & \longrightarrow & Y_1 & \longrightarrow & Y_0 & \longrightarrow & Y \longrightarrow S. \end{array}$$

Since V' is open in Y_s , there is an open subset Y'_0 of Y such that $V' = Y'_0 \cap Y_s$. Replacing Y_1 by $Y_1 \cap Y'_0$, we may assume $V' = (Y_1)_s$. We denote the restriction of A to Y_1 also by A . Since U is flat over S , A is flat over S . We therefore get the exact sequence

$$0 \longrightarrow N / m_s N \longrightarrow \mathcal{O}_{Y_1} / m_s \mathcal{O}_{Y_1} \longrightarrow A / m_s A \longrightarrow 0.$$

$(U_1)_s \simeq (Y_1)_s = V'$ implies $N / m_s N = 0$, or $N = m_s N$, a fortiori, $N_y = m_y N_y$ for $y \in V'$. Hence $N_y = 0$ for $y \in V'$ or $\text{Supp } N \cap V' = \phi$. $V = Y_1 \setminus \text{Supp } N$ is open in Y and contains V' . It is clear that $U_{(V)} \rightarrow V$ is an isomorphism. The following lemma will complete the proof of the proposition.

LEMMA (4.3.1). *Suppose Y proper flat over S , V an open subset of Y , $s \in S$. If V_s is dense in Y_s , then there exists an open subset $S_0 \subset S$ such that for each $s' \in S_0$, $V_{s'}$ is dense in $Y_{s'}$.*

$F = Y \setminus V$ is a closed subset of Y . Denote the restriction of $p_Y: F \rightarrow S$ by p' . We define d° and d' by

$$\begin{aligned} d^\circ: Y &\longrightarrow Z, & y &\longmapsto \dim_y(p_Y^{-1}(p_Y(y))), \\ d': F &\longrightarrow Z, & y &\longmapsto \dim_y(p'^{-1}(p'(y))). \end{aligned}$$

Then d' is upper semi-continuous, and d° is continuous since Y is flat over S . The restriction d of d° to F is also continuous. Clearly we have $d(y) \geq d'(y)$ for $y \in F$. The denseness of V_s in Y_s is equivalent to $d(y) > d'(y)$ for all $y \in F_s$. By the continuity of d° and the upper semi-continuity of d' , for each $y \in F_s$, there exists an open neighbourhood $N(y)$ of y in F such that $d(y') > d'(y')$ if $y' \in N(y)$. If $O = \bigcup_{y \in F_s} N(y)$, O is open in F with $F_s \subset O \subset F$ and $d(y') > d'(y')$ for $y' \in O$. By the properness of p_Y , $p_Y(F \setminus O)$ is closed. We have on the other hand $s \notin p_Y(F \setminus O)$ by $F_s \subset O$. Let S_0 be the open subset $S \setminus p_Y(F \setminus O)$ in S . Then $s \in S_0$. If $s' \in S_0$ and if $y \in F_{s'}$, then $y \in O$ so that $d(y) > d'(y)$, hence $V_{s'}$ is dense in $Y_{s'}$.

(4.4) Given a locally noetherian scheme S , and S -schemes X and Y , we shall put

$$(4.4.1) \quad \text{Hom}_{S\text{-rat}}(X, Y) = \Gamma_{S\text{-rat}}(X \times_S Y/X),$$

i.e., the set of closed subschemes U of $X \times_S Y$, flat over S such that U_s is geometrically reduced for $s \in S$ and such that there exists an open subset V of X satisfying the conditions 1) $U_{(V)} \rightarrow V$ is an isomorphism, 2) V_s is dense in X_s for $s \in S$ and 3) $(U_{(V)})_s$ is dense in U_s for $s \in S$. We denote further $\Gamma_{S\text{-rat}}(X \times_S Y/X)$ by

$$(4.4.2) \quad \text{Hom}_{S\text{-rat}}(X, Y).$$

For a geometric point \bar{s} of S , if $X_{\bar{s}}$ is reduced,

$$(4.4.3) \quad \begin{aligned} \text{Hom}_{S\text{-rat}}(X, Y)(\bar{s}) &= \Gamma_{(\bar{s})\text{-rat}}(X_{\bar{s}} \times_{\kappa(\bar{s})} Y_{\bar{s}}/X_{\bar{s}}) \\ &= \text{the set of } \kappa(\bar{s})\text{-rational maps from } X_{\bar{s}} \text{ to } Y_{\bar{s}}. \end{aligned}$$

The proposition (4.3) immediately implies the

PROPOSITION (4.5). *Let S be a locally noetherian scheme, Y a projective scheme over S , and X a flat projective scheme over S . Then the functor $\text{Hom}_{S\text{-rat}}(X, Y)$ is representable by open subscheme $\text{Hom}_{S\text{-rat}}(X, Y)$ of $\text{Hilb}_{X \times_S Y/S}$.*

(4.6) Given an S -morphism $f: Z \rightarrow X$, we define

$$\text{Hom}_{S\text{-rat}}(Z, f; X, Y)$$

as the set of pairs (φ, U) of $\varphi \in \text{Hom}_S(S, Z)$ and a closed subscheme U of $X \times_S Y$ flat over S such that U_s is geometrically reduced for $s \in S$ and such that there exists an open subscheme V of X satisfying the conditions:

(4.6.1) the morphism $U_{(V)} \rightarrow V$ induced from $U \subset X \times_S Y \rightarrow X$ is an iso-

morphism;

(4.6.2) V_s is dense in X_s for $s \in S$;

(4.6.3) $(U_{(V)})_s$ is dense in U_s for $s \in S$;

(4.6.4) $f \circ \varphi(S) \subset V$.

Note that the condition (4.6.4) is the same as $(f \circ \varphi)^{-1}(V) = S$. The set $\text{Hom}_{S\text{-rat}}(Z, f; X, Y)$ is a subset of $\text{Hom}_S(S, Z) \times \text{Hom}_{S\text{-rat}}(X, Y)$. For a morphism $T \rightarrow S$, the correspondence $(\varphi, U) \mapsto (\varphi_T, U_T)$ defines a map

$$\text{Hom}_{S\text{-rat}}(Z, f; X, Y) \longrightarrow \text{Hom}_{T\text{-rat}}(Z_T, f_T; X_T, Y_T)$$

and a contravariant functor

$$(4.6.5) \quad \text{Hom}_{S\text{-rat}}(Z, f; X, Y): (\text{loc. noetherian schemes}/S) \longrightarrow (\text{Sets})$$

Clearly this is a subfunctor of $h_Z \times \text{Hom}_{S\text{-rat}}(X, Y)$. For a geometric point \bar{s} of S , if $X_{\bar{s}}$ is reduced, with the above identification,

$$(4.6.6) \quad \text{Hom}_{S\text{-rat}}(Z, f; X, Y)(\bar{s}) = \left\{ \begin{array}{l} z \in Z_{\bar{s}}(\bar{s}), \\ (z, \psi); \psi: X_{\bar{s}} \dashrightarrow Y_{\bar{s}} \text{ is a } \kappa(\bar{s})\text{-rational map,} \\ f(z) \text{ is in the domain of definition of } \psi \end{array} \right\}.$$

PROPOSITION (4.7). *Let S, X and Y be as in (4.5) and $f: Z \rightarrow X$ be an S -morphism. Then the functor $\text{Hom}_{S\text{-rat}}(Z, f; X, Y)$ is representable by an open subscheme $\text{Hom}_{S\text{-rat}}(Z, f; X, Y)$ of $Z \times_S \text{Hom}_{S\text{-rat}}(X, Y)$.*

It is sufficient to show that if $s \in S$ and if $(\varphi, U) \in \text{Hom}_S(S, Z) \times \text{Hom}_{S\text{-rat}}(X, Y)$ is such that $(\varphi_s, U_s) \in \text{Hom}_{s\text{-rat}}(Z_s, f_s; X_s, Y_s)$, then there exists an open neighbourhood S_0 of the point s in S such that, for $s' \in S_0$, $(\varphi_{s'}, U_{s'}) \in \text{Hom}_{s'\text{-rat}}(Z_{s'}, f_{s'}; X_{s'}, Y_{s'})$, where $s = \text{Spec}(\kappa(s))$ and $s' = \text{Spec}(\kappa(s'))$. If V' is an open subset of X_s , satisfying the conditions (4.6.1)~4) (where we replace S and V by s and V' , respectively), then by the proof of (4.3), there exist an open neighbourhood S_1 of s in S , and an open subscheme V_1 of X_{S_1} satisfying the conditions (4.6.1)~3)) such that $(V_1)_s = V'$. Then $S_0 = (f \circ \varphi)^{-1}(V_1)$ is the required open subset.

(4.8). Suppose $(\varphi, U) \in \text{Hom}_{S\text{-rat}}(Z, f; X, Y)$. With the above notations, $f \circ \varphi: S \rightarrow X$ is factorized as $S \rightarrow V \subset X$. If $p'': U_{(V)} \xrightarrow{\sim} V$ is induced from $U \subset X \times_S Y \rightarrow X$ by the base change $V \rightarrow X$, then we obtain a morphism

$$\psi: S \longrightarrow V \xrightarrow{p''^{-1}} U_{(V)} \subset X \times_S Y \longrightarrow Y.$$

Thus we get a map

$$\text{Hom}_{S\text{-rat}}(Z, f; X, Y) \longrightarrow \text{Hom}_S(S, Y), (\varphi, U) \longmapsto \psi,$$

and a collection of this kind of maps defines a morphism of functors

$$(4.8.1) \quad \text{ev}: \text{Hom}_{S\text{-rat}}(Z, f; X, Y) \longrightarrow h_Y.$$

In particular, if the hypothesis of (4.7) is satisfied, we obtain a morphism of schemes $\text{ev}: \text{Hom}_{S\text{-rat}}(Z, f; C, Y) \rightarrow Y$.

If $(z, \psi) \in \text{Hom}_{S\text{-rat}}(Z, f; X, Y)(\bar{s})$, where \bar{s} is a geometric point of S , and if $X_{\bar{s}}$ is reduced, then $z \in Z_{\bar{s}}(\bar{s})$ and $\psi: X_{\bar{s}} \dashrightarrow Y_{\bar{s}}$ is a rational map with $f(z) \in \text{dom } \psi$, and $\text{ev}((z, \psi)) = \psi(f(z))$.

§ 5.

(5.1) Let V be a smooth projective variety of dimension m over an algebraically closed field k of characteristic zero, and p an integer ≥ 0 . For integers $d, \ell > 0$, we put

$$(5.1.1) \quad E_{\ell}^p(V)_d = \{(X, Y) \in C_r(V)_d \times C_r(V)_d; X \underset{(\ell)}{\sim} Y \text{ on } V\},$$

where $r = m - p$, and we identify the cycles on V with their Chow points. We shall prove the analogues of Propositions (2.2), (2.3) and (2.5).

(5.2) There exists a countable number of families of smooth irreducible projective curve $\mathcal{F}_{\mu}/\mathcal{S}_{\mu}$ ($\mu \in M$) such that for any smooth irreducible projective curve C , there exist a $\mu \in M$ and a point $s \in \mathcal{S}_{\mu}$ such that $C \simeq (\mathcal{F}_{\mu})_s$; the indexing set M is countable and $\mathcal{F}_{\mu} \rightarrow \mathcal{S}_{\mu}$ is smooth projective and each fibre is irreducible of dimension 1. Note that the set of $(Q, \mu_1, \dots, \mu_{\ell}, d', d'')$ formed of $Q \in \mathcal{Q}[T]$, $\mu_1, \dots, \mu_{\ell} \in M$ and of positive integers d', d'' is countable.

(5.3) Fix for a moment $\mu_1, \dots, \mu_{\ell} \in M$ and $Q \in \mathcal{Q}[T]$, and denote the set of maps from $[1, \ell]$ to $\{0, 1\}$ by $2^{[1, \ell]}$, where $[1, \ell]$ is the set of integers between 1 and ℓ . To simplify the notations, we shall write:

$$\begin{aligned} \mathcal{F}_i &= \mathcal{F}_{\mu_i}, & \mathcal{S}_i &= \mathcal{S}_{\mu_i}, \\ \mathcal{F} &= \mathcal{F}_1 \times \cdots \times \mathcal{F}_\ell, \\ \mathcal{S} &= \mathcal{S}_1 \times \cdots \times \mathcal{S}_\ell; \end{aligned}$$

Then, \mathcal{F} is a scheme smooth projective over \mathcal{S} . For $\sigma \in 2^{[1, \ell]}$, we shall define the \mathcal{S} -morphism

$$(5.3.1) \quad \bar{\sigma}: \mathcal{F} \times_{\mathcal{S}} \mathcal{F} = (\mathcal{F}_1 \times_{\mathcal{S}_1} \mathcal{F}_1) \times \cdots \times (\mathcal{F}_\ell \times_{\mathcal{S}_\ell} \mathcal{F}_\ell) \longrightarrow \mathcal{F}_1 \times \cdots \times \mathcal{F}_\ell = \mathcal{F},$$

by

$$(x_1^{(0)}, x_1^{(1)}, \dots, x_\ell^{(0)}, x_\ell^{(1)}) \longmapsto (x_\ell^{(\sigma(1))}, \dots, x_1^{(\sigma(\ell))}).$$

Recall that $Hom_{\mathcal{S}\text{-rat}}^Q(\mathcal{F} \times_{\mathcal{S}} \mathcal{F}, \bar{\sigma}; \mathcal{F}, \mathcal{S} \times C_r(V)_{d'} \times C_r(V)_{d''})$ is an open subscheme of the quasi-projective scheme $(\mathcal{F} \times_{\mathcal{S}} \mathcal{F}) \times Hom_{\mathcal{S}\text{-rat}}(\mathcal{F}, \mathcal{S} \times C_r(V)_{d'} \times C_r(V)_{d''})$, where $Q \in \mathbb{Q}[T]$ is the corresponding Hilbert polynomial. Consider

$$(5.3.2) \quad W' = \bigcap_{\sigma \in 2^{[1, \ell]}} Hom_{\mathcal{S}\text{-rat}}^Q(\mathcal{F} \times_{\mathcal{S}} \mathcal{F}, \bar{\sigma}; \mathcal{F}, \mathcal{S} \times C_r(V)_{d'} \times C_r(V)_{d''}),$$

an open subscheme of $(\mathcal{F} \times_{\mathcal{S}} \mathcal{F}) \times Hom_{\mathcal{S}\text{-rat}}(\mathcal{F}, \mathcal{S} \times C_r(V)_{d'} \times C_r(V)_{d''})$. For $\sigma \in 2^{[1, \ell]}$, we have a morphism

$$(5.3.3) \quad \begin{aligned} \sigma_*: W' \subset Hom_{\mathcal{S}\text{-rat}}^Q(\mathcal{F} \times_{\mathcal{S}} \mathcal{F}, \bar{\sigma}; \mathcal{F}, \mathcal{S} \times C_r(V)_{d'} \times C_r(V)_{d''}) \\ \xrightarrow{\text{ev}} \mathcal{S} \times C_r(V)_{d'} \times C_r(V)_{d''} \xrightarrow{\text{proj}} C_r(V)_{d'} \times C_r(V)_{d''}. \end{aligned}$$

For $\sigma \in 2^{[1, \ell]}$, we put $|\sigma| = \sum_{1 \leq i \leq \ell} \sigma(i)$. We define the morphism

$$(5.3.4) \quad \Sigma': W' \longrightarrow C_r(V)_N \times C_r(V)_N, \quad N = 2^{\ell-1}(d' + d''),$$

by

$$\Sigma'(x) = \left(\sum_{|\sigma| \equiv 0} \text{pr}_1 \circ \sigma_*(x) + \sum_{|\sigma| \equiv 1} \text{pr}_2 \circ \sigma_*(x), \sum_{|\sigma| \equiv 1} \text{pr}_1 \circ \sigma_*(x) + \sum_{|\sigma| \equiv 0} \text{pr}_2 \circ \sigma_*(x) \right),$$

where \equiv means “modulo 2” and “+” are the morphisms of the form

$$C_r(V)_d \times C_r(V)_{d'} \longrightarrow C_r(V)_{d+d'}, \quad (X, Y) \longmapsto X + Y,$$

and pr_1, pr_2 are the projections on $C_r(V)_{d'}$ and $C_r(V)_{d''}$. Finally consider the morphism

$$(5.3.5) \quad \begin{aligned} \Sigma: C_r(V)_d \times C_r(V) \times W' \longrightarrow C_r(V)_{d+N} \times C_r(V)_{d+N}, \\ \downarrow \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \downarrow \\ (X, Y, x) \longmapsto \Phi((X, Y), \Sigma'(x)) \end{aligned}$$

where $\Phi: C_r(V)_d \times C_r(V)_d \times C_r(V)_N \times C_r(V)_N \rightarrow C_r(V)_{d+N} \times C_r(V)_{d+N}$ is such that $(X, Y, X', Y') \mapsto (X + X', Y + Y')$. We put $W = \sum^{-1}(\Delta)$, Δ being the diagonal of $C_r(V)_{d+N} \times C_r(V)_{d+N}$. We shall then set

$$(5.3.6) \quad E_i^q(V)_d^{(Q, \mu_1, \dots, \mu_\ell, d', d'')} = \text{pr } W,$$

where $\text{pr}: C_r(V)_d \times C_r(V)_d \times W' \rightarrow C_r(V)_d \times C_r(V)_d$ is the projection; hence $\text{pr } W$ is a constructible subset of $C_r(V)_d \times C_r(V)_d$.

(5.4) Let $\varphi: S \rightarrow W$, i.e., $\varphi \in W(S)$. Since $W \subset C_r(V)_d \times C_r(V)_d \times W'$, φ is of the form $\varphi = (f, g, h) \in C_r(V)_d(S) \times C_r(V)_d(S) \times W'(S)$. Note that

$$\mathcal{F}_S = \mathcal{F} \times_S S = (\mathcal{F}_1 \times \dots \times \mathcal{F}_\ell)_{\mathcal{S}_1 \times \dots \times \mathcal{S}_\ell} \times_S S = (\mathcal{F}_1)_S \times_S \dots \times_S (\mathcal{F}_\ell)_S,$$

where $(\mathcal{F}_i)_S = \mathcal{F}_i \times_S S$. By definition,

$$\begin{aligned} W'(S) &= \bigcap_{\sigma} \text{Hom}_{\mathcal{S}\text{-rat}}^Q(\mathcal{F} \times_S \mathcal{F}, \bar{\sigma}; \mathcal{F}, \mathcal{S} \times C_r(V)_{d'} \times C_r(V)_{d''})(S) \\ &= \bigcap_{\sigma} \text{Hom}_{\mathcal{S}\text{-rat}}^Q(\mathcal{F}_S \times_S \mathcal{F}_S, \bar{\sigma}_S; \mathcal{F}_S, S \times C_r(V)_{d'} \times C_r(V)_{d''}) \\ &\subset (\mathcal{F}_S \times_S \mathcal{F}_S)(S) \times \text{Hom}_{\mathcal{S}\text{-rat}}^Q(\mathcal{F}_S, S \times C_r(V)_{d'} \times C_r(V)_{d''}), \\ (\mathcal{F}_S \times_S \mathcal{F}_S)(S) &= \mathcal{F}_1(S) \times \mathcal{F}_1(S) \times \dots \times \mathcal{F}_\ell(S) \times \mathcal{F}_\ell(S). \end{aligned}$$

So $h = (s_1^{(0)}, s_1^{(1)}, \dots, s_\ell^{(0)}, s_\ell^{(1)}, \bar{\psi})$ for $s_i^{(j)} \in \mathcal{F}_i(S)$, $j = 0, 1$; $i = 0, \dots, \ell$, and for $\bar{\psi} \in \text{Hom}_{\mathcal{S}\text{-rat}}^Q(\mathcal{F}_S, S \times C_r(V)_{d'} \times C_r(V)_{d''})$. Then $\bar{\psi}$ defines a rational map

$$\psi: \mathcal{F}_S \dashrightarrow C_r(V)_{d'} \times C_r(V)_{d''}.$$

By definition, $\psi \circ s^\sigma$ are defined for any $\sigma \in 2^{[1, \ell]}$, where

$$s^\sigma = s^{(\sigma(1))} \times \dots \times s^{(\sigma(\ell))} \in ((\mathcal{F}_1)_S \times_S \dots \times_S (\mathcal{F}_\ell)_S)(S).$$

The definition of W implies that

$$(5.4.1) \quad \begin{aligned} f + \sum_{|\sigma|=0} \text{pr}_1 \circ \psi \circ s^\sigma + \sum_{|\sigma|=1} \text{pr}_2 \circ \psi \circ s^\sigma \\ = g + \sum_{|\sigma|=1} \text{pr}_1 \circ \psi \circ s^\sigma + \sum_{|\sigma|=0} \text{pr}_2 \circ \psi \circ s^\sigma. \end{aligned}$$

in $C_r(V)_{d+N}(S)$.

Conversely, suppose $S = \text{Spec } k$ and suppose that $\mu_1, \dots, \mu_\ell \in M$, $S \rightarrow \mathcal{S}_{\mu_i}$ are given. Hence \mathcal{F} can be defined as above. Let $f, g \in C_r(V)_d(S)$, and $\psi = \bar{\psi}: \mathcal{F}_S \dashrightarrow C_r(V)_{d'} \times C_r(V)_{d''} = S \times C_r(V)_{d'} \times C_r(V)_{d''}$ be a rational map. If $s_i^{(j)} \in \mathcal{F}_i(S)$ satisfy the conditions $\text{Im } s^\sigma \subset \text{dom } \psi$ for $\sigma \in 2^{[1, \ell]}$, and if the equality (5.4.1) holds, then $\varphi = (f, g, h) \in W(S)$, $h = (s_1^{(0)}, s_1^{(1)}, \dots, s_\ell^{(0)}, s_\ell^{(1)}, \bar{\psi})$, where W is defined as above for an appropriate $Q \in \mathbb{Q}[T]$.

THEOREM (5.5). *Under the hypotheses the notations as above, we have:*

$$(5.5.1) \quad E_\ell^p(V)_d = \bigcup E_\ell^p(V)_d^{(Q, \mu_1, \dots, \mu_\ell, d', d'')} \quad (\text{as closed points}),$$

where the union is taken over all $Q \in \mathbf{Q}[T]$, $\mu_1, \dots, \mu_\ell \in M$ and positive integers d' and d'' .

$$(5.5.2) \quad E_\ell^p(V)_d \supset \overline{E_\ell^p(V)_d^{(Q, \mu_1, \dots, \mu_\ell, d', d'')}}$$

for arbitrary Q, μ_i , and d' and d'' , where the bar denotes the closure in $C_r(V)_d \times C_r(V)_d$. In particular, $E_\ell^p(V)_d$ is a countable union of closed subsets of $C_r(V)_d \times C_r(V)_d$. Moreover, if we replace $\mathcal{F}_\mu/\mathcal{S}'_\mu$ s by their associated families of jacobians in (5.3), then the equality (5.5.1) also holds.

The equality (5.5.1) follows from (3.8, 1) and (5.4). To show (5.5.2), notice first that $E_\ell^p(V)_d^{(Q, \mu_1, \dots, \mu_\ell, d', d'')} = E_\ell^p(V)_d^{(Q, \dots, d'')}$ is constructible. Hence it is sufficient to verify that if Γ is a curve (maybe singular or non-complete, but irreducible) in $E_\ell^p(V)_d^{(Q, \dots, d'')}$, then the closure $\bar{\Gamma}$ of Γ in $C_r(V)_d \times C_r(V)_d$ is in $E_\ell^p(V)_d$. With the notations in (5.3), there exists a curve Γ' in W dominating Γ via the morphism pr . Let R' be the smooth projective model of Γ' , and R'_0 the open subvariety of R' such that R'_0 is the normalization of $\Gamma': R'_0 \rightarrow \Gamma'$. We get the morphisms

$$\begin{aligned} R'_0 &\longrightarrow \Gamma' \subset W \subset C_r(V)_d \times C_r(V)_d \times (\mathcal{F} \times \mathcal{F}) \\ &\quad \times \underset{\mathcal{S}}{\text{Hom}}_{\mathcal{S}\text{-rat}}(\mathcal{F}, \mathcal{S} \times C_r(V)_{d'} \times C_r(V)_{d''}) \\ &\longrightarrow \mathcal{S} = \mathcal{S}_1 \times \dots \times \mathcal{S}_\ell \longrightarrow \mathcal{S}_i, \quad i = 1, \dots, \ell, \end{aligned}$$

and thus the smooth families of curves $(\mathcal{F}_i)_{R'_0}/R'_0$, $i = 1, \dots, \ell$, by the base change $R' \rightarrow \mathcal{S}_i$. By virtue of [2], (2.7), if the genus of the generic fibre of $(\mathcal{F}_i)_{R'_0}/R'_0$ is ≥ 2 , there exist a smooth projective curve B_i and a stable curve F'_i/B_i such that B_i is a covering of R' and $(F'_i)_{B_i^0} \simeq (\mathcal{F}_i)_{B_i^0}$, where $B_i^0 = B_i \times_{R'} R'_0$. If the generic fibre of $(\mathcal{F}_i)_{R'_0}/R'_0$ has genus 1, by the argument in [2], § 2, there exist a smooth projective curve B_i which is a covering of R' , and a flat family of curves F'_i/B_i such that $(F'_i)_{B_i^0} \simeq (\mathcal{F}_i)_{B_i^0}$ and such that the (geometric) fibres of F'_i/B_i are reduced and have at worst ordinary double points. Finally, if the generic fibre is of genus 0, then there are a smooth projective curve B_i which is a covering of R' , and a flat family of curves F'_i/B_i with fibres P_1 such that $(F'_i)_{B_i^0} \simeq (\mathcal{F}_i)_{B_i^0}$. Let R be a smooth projective curve with $R \rightarrow B_i$, $i = 1, \dots, \ell$, such that the morphisms $R \rightarrow B_i \rightarrow R'$ are the same. The fibres of $F'_i \times_{B_i} R = (F'_i)_R/R$ are reduced

and have at worst ordinary double points and $(F'_i)_R \times_R R_0 \simeq (\mathcal{F}_i)_{R'_0} \times_{R'_0} R_0$, where $R_0 = R \times_{R'} R'_0$. Blowing up $(F'_i)_R$ at points over $R \setminus R_0$ if necessary, we obtain flat families of curves F_i/R such that F_i are smooth projective surfaces and $(F_i)_{R_0} \simeq (\mathcal{F}_i)_{R'_0} \times_{R'_0} R_0$ and such that the fibres of F_i/R are reduced and have at worst ordinary double points. We put $F = F_1 \times_{R'} \cdots \times_{R'} F_\ell$. Considering $R_0 \rightarrow R'_0 \rightarrow W$, we are in the situation remarked in (5.4): we have R_0 -sections $s_i^{(j)}: R_0 \rightarrow (F_i)_{R_0}$, $j = 0, 1$; $i = 1, \dots, \ell$, a rational map

$$\psi: F_{R_0} \dashrightarrow C_r(V)_{d'} \times C_r(V)_{d''},$$

and morphisms $f: R_0 \rightarrow C_r(V)_d$ and $g: R_0 \rightarrow C_r(V)_d$ satisfying the equality (5.4.1) in which

$$s^\sigma = s_1^{(\sigma(1))} \times \cdots \times s_\ell^{(\sigma(\ell))}: R_0 \longrightarrow F_{R_0} = (F_1)_{R_0} \times_{R_0} \cdots \times_{R_0} (F_\ell)_{R_0},$$

and $\text{Im } s^\sigma \subset \text{dom } \psi$. Since R is a smooth curve, and F_i/R is proper, the R_0 -sections $s_i^{(j)}$ have their extensions $\bar{s}_i^{(j)}: R \rightarrow F_i$. Moreover $f, g: R_0 \rightarrow C_r(V)_d$ are extended to $\bar{f}, \bar{g}: R \rightarrow C_r(V)_d$ by the completeness of $C_r(V)_d$. Note that F is integral because F/R is flat and the generic fibre of F/R is integral. Since F_{R_0} is an open subset of F , ψ defines a rational map $\bar{\psi}: F \dashrightarrow C_r(V)_{d'} \times C_r(V)_{d''}$. Defining \bar{s}^σ as s^σ , $\bar{\psi} \circ \bar{s}^\sigma$ are morphisms and the equality (5.4.1) holds replacing f, g, ψ and s^σ by $\bar{f}, \bar{g}, \bar{\psi}$ and \bar{s}^σ , respectively. To show $\bar{I} \in E_\ell^p(V)_d$, it suffices to prove that for any $r \in R$, we have $\bar{f}(r) \underset{(i)}{\sim} \bar{g}(r)$. But by the extended version of the equality (5.4.1), it is sufficient to verify

$$\sum_{|\sigma|=0} \text{pr}_i \circ \bar{\psi} \circ \bar{s}^\sigma(r) \underset{(i)}{\sim} \sum_{|\sigma|=1} \text{pr}_i \circ \bar{\psi} \circ \bar{s}(r), \quad i = 1, 2,$$

for any $r \in R$. To simplify the notations, replacing $\text{pr}_i \circ \bar{\psi}$ by $\bar{\psi}$, we can assume $\bar{\psi}: F \dashrightarrow C_r(V)_{d'}$. Let $r \in R$ be a point such that some of $(F_i)_r$ ($i = 1, \dots, \ell$) have singularities. We notice that the set of such r 's is a finite set. For $j, 1 \leq j \leq \ell$, let L_j be the set of points $x \in F_r = (F_1)_r \times \cdots \times (F_\ell)_r$ at least j of whose projections onto $(F_i)_r$'s fall to double points. Then $L_1 \supset L_2 \supset \cdots \supset L_\ell$ are closed subsets of F_r , hence of F . We endow the reduced subscheme structures to L_j 's. Blow up first F with center L_ℓ to obtain F' and let $L'_1 \supset L'_2 \supset \cdots \supset L'_{\ell-1}$ be the strict transforms of $L_1 \supset \cdots \supset L_{\ell-1}$; then blow up F' with center $L'_{\ell-1}$ to obtain F'' and so on until we blow up F virtually with center L_2 to obtain \tilde{F} and a morphism

$\tilde{F} \rightarrow F$. We assert:

I) \tilde{F} is non-singular at the points of the fibre over $r \in R$.

II) For each $x \in F$, any two points of the fibre \tilde{F}_x are joined by a finite numbers of rational curves in the fibre.

To show these assertions, we shall recall basic facts about the blowing-ups: Let X be a scheme, $Y \supset Z$ be its closed subschemes, X_1 the blowing-up of X with center Z and Y the strict transform of Y . If $X' \rightarrow X$ is a flat morphism, setting $Y' = Y_{(X')}$ and $Z' = Z_{(X')}$, $X \times_X X_1$ is the blowing-up of X' with center Z' , and $X' \times_X Y_1$ is the strict transform of Y' . Pick up a point $x \in F$ and put $\hat{\phi} = \text{Spec}(\hat{\mathcal{O}}_{F,x})$ where $\hat{\mathcal{O}}_{F,x}$ is the completion of $\mathcal{O}_{F,x}$ with respect to the maximal ideal. Then we have a flat morphism $\hat{\phi} \rightarrow F$. Denote by x_i the image of $x \in F$ by the projection $F \rightarrow F_i$. Rearranging the numbers of indices, we can assume that $x_1, \dots, x_{\ell'}$ are ordinary double points and $x_{\ell'+1}, \dots, x_{\ell}$ are smooth points. Hence we have $x \in L_{\ell'}$ but $x \notin L_{\ell'+1}$. Since F_i are smooth,

$$\begin{array}{ccc} \hat{\mathcal{O}}_{R,r} & \longrightarrow & \hat{\mathcal{O}}_{F_i,x_i} \\ \downarrow \wr & & \downarrow \wr \\ k[[t]] & \longrightarrow & k[[u_i, v_i, t]]/(u_i v_i - t) \end{array}$$

for $i \leq \ell'$ and

$$\begin{array}{ccc} \hat{\mathcal{O}}_{R,r} & \longrightarrow & \hat{\mathcal{O}}_{F_i,x_i} \\ \downarrow \wr & & \downarrow \wr \\ k[[t]] & \longrightarrow & k[[u_i, t]]. \end{array}$$

for $i > \ell'$. Also we have

$$\begin{aligned} \hat{\mathcal{O}}_{F,x} &= (\hat{\mathcal{O}}_{F_1,x_1} \otimes_{\hat{\mathcal{O}}_{R,r}} \cdots \otimes_{\hat{\mathcal{O}}_{R,r}} \hat{\mathcal{O}}_{F,x})^\wedge \\ &= k[[u_1, v_1, \dots, u_{\ell'}, v_{\ell'}, u_{\ell'+1}, \dots, u_{\ell}, t]]/(u_1 v_1 - t, \dots, u_{\ell'} v_{\ell'} - t). \end{aligned}$$

The ideal of $\hat{\mathcal{O}}_{F,x}$ defining $(L_i)_{\hat{\phi}} = \hat{L}_i$ is

$$(5.5.3) \quad \bigcap_{\lambda_1, \dots, \lambda_i} (u_{\lambda_1}, v_{\lambda_1}, \dots, u_{\lambda_i}, v_{\lambda_i}),$$

where the intersection is over those $(\lambda_1, \dots, \lambda_i)$ that satisfy $1 \leq \lambda_1 < \dots < \lambda_i \leq \ell'$. As we have remarked above, $\tilde{F}_{\hat{\phi}}$ is obtained by the same procedure as \tilde{F} replacing F by $F_{\hat{\phi}}$ and L_i 's by \hat{L}_i 's. We note that the following are equivalent:

- \tilde{F} is smooth at the points of the fibre over x ;
- $\tilde{F}_{\hat{\phi}}$ is regular at the closed points of the fibre over the unique closed point \bar{x} of $\hat{\phi}$.

Moreover, we have $\tilde{F}_x \simeq (\tilde{F}_{\hat{\phi}})_{\bar{x}}$. Starting with

$$\Phi = \text{Spec}(k[[u_1, v_1, \dots, u_{\ell'}, v_{\ell'}, u_{\ell'+1}, \dots, u_{\ell}, t]]/(u_1v_1 - t, \dots, u_{\ell'}v_{\ell'} - t))$$

and A_i defined by the ideal of the form (5.5.3) in the affine ring of Φ , we define $\tilde{\Phi}$ by the same process as \tilde{F} , replacing F by Φ and L_i 's by A_i 's. We have an obvious flat morphism $\hat{\phi} \rightarrow \Phi$ and $\hat{L}_i = (A_i)_{\hat{\phi}}$. To verify the assertions I) and II) above, it suffices to show that $\tilde{\Phi}$ is smooth at the points of the fibre over the point x_0 corresponding to the ideal $(u_1, v_1, \dots, u_{\ell'}, v_{\ell'}, u_{\ell'+1}, \dots, u_{\ell}, t)$ and that $\tilde{\Phi}_{x_0}$ has the property described in II), because of $\tilde{\Phi} \times_{\hat{\phi}} \hat{\phi} \simeq \tilde{F}_{\hat{\phi}}$. We shall prove our assertions by induction on ℓ , so that we assume our assertions for

$$\text{Spec}(k[[u_1, v_1, \dots, u_{n'}, v_{n'}, u_{n'+1}, \dots, u_n, t]]/(u_1v_1 - t, \dots, u_{n'}v_{n'} - t))$$

for $n' \leq n \leq \ell$. Considering the smooth base change

$$\begin{aligned} &\text{Spec}(k[[u_1, v_1, \dots, u_{\ell'}, v_{\ell'}, u_{\ell'+1}, \dots, u_{\ell}, t]]/(u_1v_1 - t, \dots, u_{\ell'}v_{\ell'} - t)) \\ &\longrightarrow \text{Spec}(k[[u_1, v_1, \dots, u_{\ell'}, v_{\ell'}, t]]/(u_1v_1 - t, \dots, u_{\ell'}v_{\ell'} - t)) \end{aligned}$$

we may assume $\ell' = \ell$.

In $\text{Spec}(k[[u_1, v_1, \dots, u_{\ell}, v_{\ell}]])$, Φ is defined by

$$u_1v_1 = \dots = u_{\ell}v_{\ell} = t.$$

Then $A_{\ell} = x_0$ is defined by $u_1 = v_1 = \dots = u_{\ell} = v_{\ell} = t = 0$. The blowing-up Φ' of Φ at A_{ℓ} is gluing of affine pieces of the following type: in the coordinate system $u_1/u_{\ell}, v_1/u_{\ell}, \dots, u_{\ell-1}/u_{\ell}, v_{\ell-1}/u_{\ell}, u_{\ell}, v_{\ell}/u_{\ell}, t$,

$$(u_1/u_{\ell})(v_1/u_{\ell}) = \dots = (u_{\ell-1}/u_{\ell})(v_{\ell-1}/u_{\ell}) = v_{\ell}/u_{\ell},$$

Φ'° :

$$t = u_{\ell}^2(v_{\ell}/u_{\ell}).$$

On the other hand, $A_{\ell-1}$ is the union of

$$u_1 = v_1 = \dots = u_{i-1} = v_{i-1} = u_{i+1} = v_{i+1} = \dots = u_{\ell} = v_{\ell} = 0,$$

for $i = 1, \dots, \ell$; hence its strict transform $A'_{\ell-1}$ in this piece is

$$u_1/u_{\ell} = v_1/u_{\ell} = \dots = u_{\ell-1}/u_{\ell} = v_{\ell-1}/u_{\ell} = 0.$$

Moreover the strict transform $A'_{\ell-2}$ in this affine piece is the union of

$$\begin{aligned}
 u_1/u_\ell &= v_1/u_\ell = \cdots = u_{i-1}/u_\ell = v_{i-1}/u_\ell = u_{i+1}/u_\ell = v_{i+1}/u_\ell \\
 &= \cdots = u_{\ell-1}/u_\ell = v_{\ell-1}/u_\ell = 0,
 \end{aligned}$$

for $i = 1, \dots, \ell; \dots$; and the strict transform A'_2 of A_2 is, in the piece, given by the union of

$$u_i/u_\ell = v_i/u_\ell = u_j/u_\ell = v_j/u_\ell = 0,$$

for $1 \leq i < j < \ell$. Denoting $u'_i = u_i/u$, $v'_i = v_i/u$ ($i = 1, \dots, \ell - 1$) and $t' = v_\ell/u_\ell$, we have

$$\Phi'^\circ = \text{Spec}(k[u'_1, v'_1, \dots, u'_{\ell-1}, v'_{\ell-1}, t']/(u'_1 v'_1 - t', \dots, u'_{\ell-1} v'_{\ell-1} - t')),$$

and the ideal of A'_i is of the form (5.5.3), replacing u_i and v_i by u'_i and v'_i ; hence by inducting hypothesis, we see the first assertion.

We shall verify the second assertion; let \tilde{x} and \tilde{x}' be the points of $\tilde{\Phi}$ whose images x and x' in Φ are identical, and let x_1 and x'_1 be the images of \tilde{x} and \tilde{x}' in Φ' . If $x_1 = x'_1$, then by induction hypothesis, we are done. Suppose $x_1 \neq x'_1$. Since the morphism $\Phi' \rightarrow \Phi$ is an isomorphism outside A_ℓ , we have $x = x' \in A_\ell$, so that $x_1, x'_1 \in \Phi'_{A_\ell}$. We claim that there exists a finite number of rational curves Γ_i ($i = 1, \dots, n$) in Φ'_{A_ℓ} such that $\Gamma_i \not\subset A'_2$ for all i , $x_1 \in \Gamma_1$, $x'_1 \in \Gamma_n$ and $\Gamma_i \cap \Gamma_{i+1} \neq \emptyset$ ($1 \leq i < n$); This claim implies our second assertion: the curves Γ_i 's lift to the rational curves $\tilde{\Gamma}_i$ in $\tilde{\Phi}$. If $y_i \in \Gamma_i \cap \Gamma_{i+1}$ ($1 \leq i < n$), if $\tilde{y}_i \in \tilde{\Gamma}_i$ and $\tilde{y}_{i+1} \in \tilde{\Gamma}_{i+1}$ are points over y_i and if \tilde{x} and \tilde{x}' are points on $\tilde{\Gamma}_1$ and on $\tilde{\Gamma}_n$ over x_1 and x'_1 respectively, by induction hypothesis, \tilde{y}_i and \tilde{y}_{i+1} can be joined by a finite number of rational curves and the same is true for \tilde{x} and \tilde{y}_1 , and also for \tilde{x}' and \tilde{y}_n . Moreover \tilde{y}_i and \tilde{y}_i are on the curve $\tilde{\Gamma}_i$, whence the assertion II). We show the claim for arbitrary $x_1, x'_1 \in \Phi'$ over A_ℓ , hence we may assume that x_1 and x'_1 are contained in the same piece of Φ' above, say, the above piece. We shall denote the varieties intersected with the above affine piece by the same symbols as the varieties. The variety Φ'_{A_ℓ} is defined by

$$\begin{aligned}
 (u_1/u_\ell)(v_1/u_\ell) &= \cdots = (u_{\ell-1}/u_\ell)(v_{\ell-1}/u_\ell) = v_\ell/u_\ell. \\
 u_\ell &= t = 0.
 \end{aligned}$$

We can therefore take $(u_1/u_\ell, v_1/u_\ell, \dots, v_{\ell-1}/u_\ell, v_\ell/u_\ell)$ as the coordinate system for Φ'_{A_ℓ} . Let $(a_1, b_1, \dots, a_{\ell-1}, b_{\ell-1}, c)$ and $(a'_1, b'_1, \dots, a'_{\ell-1}, b'_{\ell-1}, c')$ be the coordinates of the points x_1 and x'_1 . To show x_1 and x'_1 joined by a rational curve in Φ'_{A_ℓ} , it is sufficient to find polynomial functions $f_i(w)$,

$g_i(w), \dots, f_{\ell-1}(w)$ and $g_{\ell-1}(w)$ with $f_i(0) = a_i, g_i(0) = b_i, f_i(1) = a'_i$ and $g_i(1) = b'_i$ such that

$$f_i(w)g_i(w) = f_j(w)g_j(w)$$

for all $1 \leq i < j < \ell$. Note that a point with $v_i/u_i \neq 0$ is not in A'_2 . First we shall assume that c and c' are non-zero. Considering the functions $f_i(w) = a_i w$ and $g_i(w) = b_i w$, x_1 and the point $(0, \dots, 0)$ are joined. Similarly x'_1 and the point $(0, \dots, 0)$ are joined. Hence in this case we are done. We then assume $c = 0$. We shall show that x_1 and a point with $v_i/u_i \neq 0$ can be joined by a rational curve. Since $a_i b_i = c = 0$ for $i < \ell$, a_i or b_i is zero. For simplicity, we suppose (and we can suppose) $a_1 \cdots a_\nu \neq 0, b_1 = \dots = b_\nu = a_{\nu+1} = b_{\nu+1} = \dots = b_{\ell-1} = 0$. We put

$$f_i(w) = a_i + (1 - a_i)w,$$

$$g_i(w) = \begin{cases} w^2 \prod_{\substack{1 \leq j \leq \nu \\ j \neq i}} (a_j + (1 - a_j)w) & \text{if } i \leq \nu \\ w \prod_{1 \leq j \leq \nu} (a_j + (1 - a_j)w) & \text{if } \nu < i < \ell. \end{cases}$$

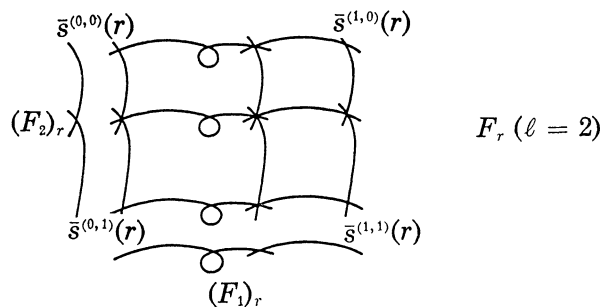
Thus x_1 and the point with coordinates $(1, \dots, 1)$ are joined by a rational curve. Since any two points with $v_i/u_i \neq 0$ can be joined by rational curves, this completes the proof of our assertions.

Iterating the above procedure at the fibres containing a singularity of F , we get a smooth projective variety, again denoted by \tilde{F} and a morphism $\tilde{F} \rightarrow F$ such that for any $x \in F$, the fibre of \tilde{F} over x has the property II), or that the points of the fibre \tilde{F}_x are rationally equivalent on \tilde{F} . If z' denotes the rational equivalence class of the cycle on $\tilde{F} \times V$ corresponding to the rational map $\tilde{\psi}: \tilde{F} \dashrightarrow C_r(V)_a$, then $z'(a')$ ($a' \in \tilde{F}$) depends only on the image a of a' in F . Hence we denote the rational equivalence class $z'(a')$ by $z(a)$. We notice that the set of singular points of F is of codimension ≥ 3 in F and hence the complement of the domain of definition of the rational map $\bar{\psi}: F \dashrightarrow C_r(V)_a$ is of codimension ≥ 2 . Let $r \in R$ be an arbitrary point and C'_i an irreducible component of $(F_i)_r$ for each $1 \leq i \leq \ell$. Then $(C'_1 \times \dots \times C'_\ell) \cap \text{dom } \bar{\psi} \neq \emptyset$, so that $\bar{\psi}$ determines a rational map $\bar{\psi}': C'_1 \times \dots \times C'_\ell \dashrightarrow C_r(V)_a$. Note that

$$T = \text{dom } \bar{\psi} \cap \text{Reg } F \cap \text{Reg } (C'_1 \times \dots \times C'_\ell)$$

is a non-empty open subset of $C'_1 \times \dots \times C'_\ell$. Let $a \in C'_1 \times \dots \times C'_\ell, \Gamma$ be a smooth complete curve, $b \in \Gamma$ a point and $\varphi: \Gamma \rightarrow C'_1 \times \dots \times C'_\ell$ a mor-

phism such that $\varphi^{-1}(T) \neq \emptyset$ and $\varphi(b) = a$. Then $\bar{\psi}' \circ \varphi$ is a morphism from Γ to $C_r(V)_{a'}$; we have therefore a cycle $\bar{\psi}' \circ \varphi(b) \in C_r(V)_{a'}$. Since $\varphi: \Gamma \rightarrow C'_1 \times \cdots \times C'_\ell \subset F$ lifts to $\tilde{\varphi}: \Gamma \rightarrow \tilde{F}$, and $\tilde{\psi}' \circ \tilde{\varphi} = \bar{\psi}' \circ \varphi$, the cycle $\bar{\psi}' \circ \varphi(b)$ is in the class $z(a)$ by (1.5). Since $(F_i)_r$ is connected for $i = 1, \dots, \ell$, there are points $\bar{s}_i^{(0)} = c_i(0), \dots, c_i(n_i) = \bar{s}_i^{(1)}$ on $(F_i)_r$ such that $c_i(n)$ and $c_i(n + 1)$ are on the same irreducible component of $(F_i)_r$ for $n = 0, \dots, n_i - 1$.



For (j_1, \dots, j_ℓ) ($0 \leq j_i \leq n_i$), let $x_{j_1, \dots, j_\ell} = z((c_i(j_1), \dots, c_i(j_\ell)))$. In virtue of (3.8, 2)), we have

$$(5.5.4) \quad \sum_{\sigma} (-1)^{|\sigma|} x_{j_1 + \sigma(1), \dots, j_\ell + \sigma(\ell)} \in F^\ell CH^p(V).$$

On the other hand, $x_{n_1 \sigma(1), \dots, n_\ell \sigma(\ell)}$ is the class of $\bar{\psi} \circ \bar{s}^\sigma(r)$ for any $\sigma \in 2^{[1, \ell]}$ by (1.5). Summing (5.5.4) over $0 \leq j_i < n_i$ ($i = 1, \dots, \ell$), we see that the rational equivalence class of $\sum_{\sigma} (-1)^{|\sigma|} \bar{\psi} \circ \bar{s}^\sigma(r)$ is in $F^\ell CH^p(V)$, i.e.,

$$\sum_{|\sigma| \equiv 0} \bar{\psi} \circ \bar{s}^\sigma(r) \sim_{(\ell)} \sum_{|\sigma| \equiv 1} \bar{\psi} \circ \bar{s}^\sigma(r), \quad \text{q.e.d.}$$

COROLLARY (5.5.5). *If $\kappa: T \rightarrow CH^p(V)$ is regular, then for any $\ell > 0$, $\kappa^{-1}(F^\ell CH^p(V))$ is a countable union of closed subsets of T .*

The proof is similar to that of (2.7).

From now on, we suppose that k is uncountable.

THEOREM (5.6). *Let V be a smooth projective variety, T a smooth quasi-projective variety, f and $g: T \rightarrow C_r(V)_a$ morphisms such that for $t \in T$, $f(t) \sim_{(a)}$ $g(t)$. Then there exist a smooth quasi-projective variety S , families $\Gamma_i|_S$ of curves (resp. jacobian varieties) over S (each fibre is a smooth projective curve (resp. jacobian variety)), sections $s_i^{(0)}, s_i^{(1)}: S \rightarrow \Gamma_i$ ($1 \leq i \leq \ell$) of $\Gamma_i|_S$, a dominant morphism $e: S \rightarrow T$, and a rational map*

$$H: \Gamma_1 \times_S \cdots \times_S \Gamma_\ell \cdots \rightarrow C_r(V)_{d'} \times C_r(V)_{d''}$$

($d', d'' > 0$) such that

$$(5.6.1) \quad H \text{ is defined on } s^\sigma(S) \subset \Gamma_1 \times_S \cdots \times_S \Gamma_\ell,$$

$$(5.6.2) \quad \begin{aligned} f \circ e + \sum_{|\sigma| \equiv 0} \text{pr}_1 \circ H \circ s^\sigma + \sum_{|\sigma| \equiv 1} \text{pr}_2 \circ H \circ s^\sigma \\ = g \circ e + \sum_{|\sigma| \equiv 1} \text{pr}_1 \circ H \circ s^\sigma + \sum_{|\sigma| \equiv 0} \text{pr}_2 \circ H \circ s^\sigma, \end{aligned}$$

where $s^\sigma = s_1^{(\sigma(1))} \times \cdots \times s_\ell^{(\sigma(\ell))}: S \rightarrow \Gamma_1 \times_S \cdots \times_S \Gamma_\ell$ for $\sigma \in 2^{[1, \ell]}$.

Consider $h = (f, g): T \rightarrow C_r(V)_d \times C_r(V)_d$. The hypothesis implies $\text{Im } h \subset E_\ell^p(V)_d$. Since $\text{Im } h$ is constructible and T is irreducible, there exists an irreducible open dense subset T_0 of $\text{Im } h$. Since $E_\ell^p(V)_d^{(Q, \mu_1, \dots, \mu_\ell, d', d'')}$ are constructible in $C_r(V)_d \times C_r(V)_d$ and since h is uncountable, by virtue of (5.5.1), there exist $Q, \mu_1, \dots, \mu_\ell, d',$ and d'' such that $T_0 \cap E_\ell^p(V)_d^{(Q, \mu_1, \dots, \mu_\ell, d', d'')}$ contains a non-void open subset in T_0 . Employing the above notations, consider

$$\begin{array}{ccc} W \times_{C_r(V)_d^2} T & \longrightarrow & W \\ \text{pr}' \downarrow & & \downarrow \text{pr} \\ T & \longrightarrow & C_r(V)_d \times C_r(V)_d, \end{array}$$

where W is that in (5.3) for $Q, \mu_1, \dots, \mu_\ell, d', d''$. Then pr' is a dominant morphism; picking up some irreducible component S' of the fibre product dominating T and replacing S' by some open subscheme of S'_{red} , we may suppose that in the diagram

$$(5.6.3) \quad \begin{array}{ccccccc} S & \longrightarrow & W \times_{C_r(V)_d^2} T & \longrightarrow & W \subset C_r(V)_d \times W' & \longrightarrow & W' \longrightarrow \mathcal{S} \\ & \searrow e & \downarrow & & \downarrow & & \\ & & T & \longrightarrow & C_r(V)_d \times C_r(V)_d, & & \end{array}$$

S is smooth and $e: S \rightarrow T$ is dominant. By the base change $S \rightarrow \mathcal{S}$, we have families of curves (resp. jacobians) $\Gamma_i = (\mathcal{F}_i)_S$. Then the theorem follows from (5.4) and the commutativity of the diagram (5.6.3).

THEOREM (5.7). *Let $\kappa: T \rightarrow CH^p(V)$ be a regular map, where T is a smooth quasiprojective variety. If $\kappa(T) \subset F'CH^p(V)$, then, for $0 \leq \ell' < \ell$, we have*

$$\kappa^\#(H^{r+\ell', r}(V)) = 0,$$

where $r + p = \dim V$.

By the construction of $\kappa^\#$, it is sufficient to show that if $f, g: T \rightarrow C_r(V)_a$ satisfy the hypothesis of (5.6), then $f^\#\omega = g^\#\omega$ for $\omega \in H^{r+\ell',r}(V)$. By virtue of (5.6), there exist a smooth quasi-projective variety S , a dominant morphism $e: S \rightarrow T$, families Γ_i/S of jacobians, their sections $s_i^{(j)}: S \rightarrow \Gamma_i$ ($1 \leq i \leq \ell; j = 0, 1$) and a rational map $H: \Gamma_1 \times_S \cdots \times_S \Gamma_\ell \dashrightarrow C_r(V)_{a'} \times C_r(V)_{a''}$ satisfying the conditions (5.6.1) and (5.6.2). Note that $e^*: H^0(T, \Omega^\ell) \rightarrow H^0(S, \Omega^\ell)$ is injective since e is dominant; it therefore suffices to verify $(f \circ e)^\#\omega = (g \circ e)^\#\omega$. The theorem will result from the next two lemmas, taking account of the property (2.4.3).

LEMMA (5.7.1). *Suppose S and T be smooth quasi-projective varieties and $\omega \in H^{r+\ell',r}(V)$. If $\varphi: T \dashrightarrow C_r(V)_a$ is a rational map, there exists an element $\eta \in H^{\ell',0}(T)$ such that, for any morphism $f: S \rightarrow T$ with $f(S) \subset \text{dom } \varphi$ (hence $\varphi \circ f$ is defined and a morphism), we have*

$$(\varphi \circ f)^\#\omega = f^*\eta.$$

Let $T_0 = \text{dom } \varphi$ and $i: T_0 \rightarrow T$ be the inclusion, $\varphi_0: T_0 \rightarrow C_r(V)_a$ a representative of φ , X the cycle on $T \times V$ corresponding to φ and X_0 the cycle on $T_0 \times V$ corresponding to φ_0 . Then $(i \times \text{id}_V)^*X$ is defined and equal to X_0 . We set $\eta = X(\omega)$. Then, as shown in the proof of (2.4.2), we have $i^*\eta = X_0(\omega)$. The morphism f as above is factorized as $f: S \xrightarrow{f_0} T_0 \xrightarrow{i} T$. Then by definition, $\varphi \circ f = \varphi_0 \circ f_0$, hence we have $(\varphi \circ f)^\#\omega = (\varphi_0 \circ f_0)^\#\omega = f_0^*(\varphi_0^\#\omega) = f_0^*(X_0(\omega)) = f_0^*(i^*(\eta)) = f^*\eta$.

LEMMA (5.7.2). *Let S be a smooth quasi-projective variety, Γ_i/S families of jacobian varieties ($1 \leq i \leq \ell$), and $s_i^{(j)}: S \rightarrow \Gamma_i$ their sections ($1 \leq i \leq \ell; j = 0, 1$). For any $\eta \in H^{\ell',0}(\Gamma_1 \times_S \cdots \times_S \Gamma_\ell)$, if $\ell' < \ell$, we have*

$$\sum_{\sigma} (-1)^{|\sigma|} s^{\sigma*}(\eta) = 0 \quad \text{in } H^{\ell',0}(S).$$

Since we are dealing with a finite number of varieties and morphisms, and since $H^{\ell',0}(?)$ commutes with the extension of the ground fields, we may assume $k = \mathbb{C}$. Let S^h, Γ_i^h , etc. denote the complex analytic manifolds associated to S, Γ_i , etc. Since the diagram

$$\begin{CD} H^{\ell',0}(\Gamma_1 \times_S \cdots \times_S \Gamma_\ell) = H^0(\Gamma_1 \times_S \cdots \times_S \Gamma_\ell, \Omega^{\ell'}) @>>> H^0(\Gamma_1^h \times_{S^h} \cdots \times_{S^h} \Gamma_\ell^h, \Omega^{\ell'}) \\ @V s^\sigma VV @VV (s^\sigma)^h V \\ H^{\ell',0}(S) = H^0(S, \Omega^{\ell'}) @>>> H^0(S^h, \Omega^{\ell'}) \end{CD}$$

is commutative, where $(s^\sigma)^h = (s^h)^\sigma = (s_1^{(\sigma(1))})^h \times \cdots \times (s_\ell^{(\sigma(\ell))})^h$ and the horizontal maps are obvious ones, and since they are injective, it suffices to prove the lemma for the complex analytic version; we therefore omit the superscript h . The problem being local on S , we may suppose S an open ball in \mathbb{C}^m , $m = \dim S$. Since Γ_i/S are families of jacobian varieties, $S \times \mathbb{C}^{n_i}$ are the universal coverings of Γ_i , where $n_i = \dim \Gamma_i - m$. We get S -morphisms $S \times \mathbb{C}^{n_i} \rightarrow \Gamma_i$ and

$$p: S \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_\ell} = (S \times \mathbb{C}^{n_1}) \times_S \cdots \times_S (S \times \mathbb{C}^{n_\ell}) \longrightarrow \Gamma_1 \times_S \cdots \times_S \Gamma_\ell.$$

Let $\tilde{s}_i^{(j)}: S \rightarrow S \times \mathbb{C}^{n_i}$ be liftings of $s_i^{(j)}: S \rightarrow \Gamma_i$, and

$$\tilde{s}^\sigma = \tilde{s}_1^{(\sigma(1))} \times_S \cdots \times_S \tilde{s}_\ell^{(\sigma(\ell))}.$$

We have $(s^\sigma)^*\eta = (\tilde{s}^\sigma)^* \circ p^*\eta$. Let $z_i^1, \dots, z_i^{n_i}$ be a coordinate system for \mathbb{C}^{n_i} , and x^1, \dots, x^m for $S \subset \mathbb{C}^m$. We set

$$\begin{aligned} dx^I &= dx^{i_1} \wedge \cdots \wedge dx^{i_\lambda}, \\ dz_i^{J_i} &= dz_i^{j_1} \wedge \cdots \wedge dz_i^{j_\nu} \end{aligned}$$

for $I = \{i_1 < \cdots < i_\lambda\}$ and $J_i = \{j_1 < \cdots < j_\nu\}$. Then $p^*\eta$ is of the form

$$p^*\eta = \sum f_{I, J_1, \dots, J_\ell} dx^I \wedge dz_1^{J_1} \wedge \cdots \wedge dz_\ell^{J_\ell}$$

where the summation is over (I, J_1, \dots, J_ℓ) such that $I \subset \{1, \dots, m\}$, $J_i \subset \{1, \dots, n_i\}$ with $|I| + |J_1| + \cdots + |J_\ell| = \ell'$, and where $f_{I, J_1, \dots, J_\ell} \in \Gamma(S \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_\ell}, \mathcal{O})$. For $s \in S$ fixed, since $(\Gamma_1 \times \cdots \times \Gamma_\ell)_s$ is compact, $f_{I, J_1, \dots, J_\ell}(s, ?)$ are constants. So we may regard $f_{I, J_1, \dots, J_\ell} \in \Gamma(S, \mathcal{O})$. Denoting

$$\tilde{s}_i^{(j)}: S \xrightarrow{\tilde{s}_i^{(j)}} S \times \mathbb{C}^{n_i} \longrightarrow \mathbb{C}^{n_i},$$

and

$$\zeta_i(\sigma, J_i) = (\tilde{s}_i^{(\sigma(i))})^*(dz_i^{J_i}),$$

we have

$$(\tilde{s}^\sigma)^* \circ p^*\eta = \sum f_{I, J_1, \dots, J_\ell} dx^I \wedge \zeta_1(\sigma, J_1) \wedge \cdots \wedge \zeta_\ell(\sigma, J_\ell).$$

Hence,

$$\sum_\sigma (-1)^{|\sigma|} (s^\sigma)^*\eta = \sum f_{I, J_1, \dots, J_\ell} dx^I \wedge [\sum_\sigma (-1)^{|\sigma|} \zeta_1(\sigma, J_1) \wedge \cdots \wedge \zeta_\ell(\sigma, J_\ell)],$$

where by the first \sum on the right side, we understand the summation over (I, J_1, \dots, J_ℓ) as above. For a fixed (I, J_1, \dots, J_ℓ) , since $|J_1| + \cdots + |J_\ell|$

$= \ell' - |I| < \ell$, there is a number i such that $J_i = \phi$. Then the expression in the bracket is equal to zero because

$$\begin{aligned} & \sum_{\sigma} (-1)^{|\sigma|} \zeta_1(\sigma, J_1) \wedge \cdots \wedge \zeta_{\ell}(\sigma, J_{\ell}) \\ &= \sum_{\sigma(i)=0} (-1)^{|\sigma|} \zeta_1(\sigma, J_1) \wedge \cdots \wedge \widehat{\zeta_i(\sigma, J_i)} \wedge \cdots \wedge \zeta_{\ell}(\sigma, J_{\ell}) \\ & \quad + \sum_{\sigma(i)=1} (-1)^{|\sigma|} \zeta_1(\sigma, J_1) \wedge \cdots \wedge \widehat{\zeta_i(\sigma, J_i)} \wedge \cdots \wedge \zeta_{\ell}(\sigma, J_{\ell}) \\ &= 0, \end{aligned}$$

where $\widehat{?}$ means the absence of $?$. The lemma is thus proven.

Remark (5.8). We have proven the ℓ -cubic equivalence versions of [9], Theorems 1, 2 and 4. The results in [9, 10] deduced formally from them are, of course, valid in the ℓ -cubic equivalence case. Here, we shall indicate two of such results (without proofs).

For any effective 0-cycle X on V of degree d let

$$V_x = \{x \in V; (x) + Y \underset{(\ell+1)}{\sim} X \quad \text{for some } Y \in S^{d-1}V\}.$$

Then V_x is a countable union of closed subsets of V . By $\dim V_x$, we understand the maximum of the dimensions of “irreducible components” of V_x and let $\text{codim } V_x = \dim V - \dim V_x$. Then the set $\{X \in S^d V; \text{codim } V_x < a\}$ is a countable union of closed subsets of $S^d V$ for any integer a , so that there is a countable intersection U of open subsets in $S^d V$ that the maximum δ_a of $\text{codim } V_x$ for $X \in S^d V$ is attained on U . Then $\delta_a \leq \delta_{a+1}$ and we set

$$d_i(V) = \lim_{d \rightarrow \infty} \delta_d = \delta_a \quad \text{for } d \gg 0.$$

Then $0 \leq d_i(V) \leq \dim V$ and $d_i(V) \neq 1$. Clearly, $d_0(V) = 0$. By (3.4, 3), $d_i(V) = 0$ if V is a product of curves.

COROLLARY (5.8.1). If $H^0(V, \Omega^q) \neq 0$, $q \geq 2$, and $\ell \geq q$, then $d_i(V) \geq q$ (cf. [10], Theorem 3).

A canonical morphism $V \rightarrow \text{Alb } V$ induces the natural map $F^1 CH_0(V) \rightarrow \text{Alb } V$, hence $gr^1 CH_0(V) \rightarrow \text{Alb } V$.

COROLLARY (5.8.2). If $d_i(V) = 0$, then $gr^1(V) \rightarrow \text{Alb } V$ is an isomorphism of groups (cf. [10], Theorem 4), and vice versa.

One can ask whether $d_i(V) \leq \ell$ or not for each ℓ . If this is true for $\ell = 1$, $gr^1 CH_0(V)$ is always isomorphic to $\text{Alb } V$. For an abelian variety

V , $d_1(V) = 0$ and $gr^1CH_0(V) \simeq \text{Alb } V$ (cf. [12]).

(5.9) To state the next theorem, we shall define ${}^{\#}H^{p,q}(V)$.

Let $\mathfrak{A}^p(V)$ be the k -vector subspace of $H^{p,p}(V)$ generated by the fundamental classes of algebraic cycles on V of codimension p . $\mathfrak{A}^*(V) = \bigoplus_p \mathfrak{A}^p(V)$ is a subring of $H^{**}(V)$ and stable by pull-backs or direct images. For a positive integer ℓ , let $(H^{0,1}(V))^\ell$ denote the image of

$$\overbrace{H^{0,1}(V) \otimes \dots \otimes H^{0,1}(V)}^{\ell \text{ times}} \longrightarrow H^{0,\ell}(V)$$

induced by the multiplication of $H^{**}(V)$.

DEFINITION (5.9.1). Let $0 \leq p \leq q$ be integers. ${}^{\#}H^{p,q}(V)$ is by definition the k -vector subspace of $H^{p,q}(V)$ generated by the elements of the form $u(x)$, where $u \in \mathfrak{A}^{p+\dim T}(T \times V)$, $x \in (H^{0,1}(T))^{(q-p)}$ and T is a smooth projective variety. Similarly, ${}^bH^{p,q}(V)$ is the k -vector subspace generated by $u(x)$, where $u \in \mathfrak{A}^{p+\dim T}(T \times V)$ and $x \in H^{0,q-p}(T)$.

Clearly, ${}^{\#}H^{p,q}(V) \subset {}^bH^{p,q}(V)$ and ${}^{\#}H^{p,p}(V) = {}^bH^{p,p}(V) = \mathfrak{A}^p(V)$.

LEMMA (5.10). 1) For any morphism $f: U \rightarrow V$, $f^*({}^{\#}H^{p,q}(V)) \subset {}^{\#}H^{p,q}(U)$.

2) For any morphism $f: U \rightarrow V$, $f_*({}^{\#}H^{p,q}(U)) \subset {}^{\#}H^{p-d,q-d}(V)$, where $d = \dim U - \dim V$.

3) If $x \in {}^{\#}H^{p,q}(U)$ and $y \in {}^{\#}H^{p',q'}(V)$, then $x \times y \in {}^{\#}H^{p+p',q+q'}(U \times V)$.

The similar properties hold for ${}^bH^{p,q}(V)$.

1) Indeed, for $v \in \mathfrak{A}^*(T \times V)$, $y \in (H^{0,1}(T))^{(q-p)}$, we have

$$f^*(v(y)) = ((\text{id}_T \times f)^*(v))(y).$$

2) For $u \in \mathfrak{A}^*(T \times U)$, $x \in (H^{0,1}(T))^{(q-p)}$, we have

$$f_*(u(x)) = ((\text{id}_T \times f)_*(u))(x).$$

3) If $u \in \mathfrak{A}^*(T \times U)$, $v \in \mathfrak{A}^*(S \times V)$, $x \in (H^{0,1}(T))^{(q-p)}$ and

$$y \in (H^{0,1}(S))^{(q'-p')},$$

then $u(x) \times v(y) = (u \times v)(x \times y)$, $u \times v \in \mathfrak{A}^*(T \times S \times U \times V)$ and

$$x \times y \in (H^{0,1}(T \times S))^{(q+q'-p-p')}.$$

COROLLARY (5.10.4). ${}^{\#}H^{**}(V) = \bigoplus_{0 \leq p \leq q} {}^{\#}H^{p,q}(V)$ is a subring of $H^{**}(V)$, and similarly for ${}^bH^{**}(V)$.

(5.11) We have clearly inclusions $\mathfrak{A}^*(V) \subset {}^{\#}H^{**}(V) \subset {}^bH^{**}(V) \subset$

$H^{**}(V)$. We shall show that in the definition of ${}^*H^{p,q}(V)$, we may restrict T to varieties of the form

$$T = C_1 \times \cdots \times C_{q-p}, \quad C_i: \text{curves},$$

or to abelian varieties. Suppose, in fact, given subspaces $\bar{H}^{p,q}(V)$ for all $p \leq q$, with the properties 1), 2), and 3) above and assume that $\bar{H}^{p,p}(V) \supset \mathfrak{A}^p(V)$ and $\bar{H}^{0,1}(V) = H^{0,1}(V)$ for all V and p . Then we have $H^{p,q}(V) \supset {}^*H^{p,q}(V)$ for all $p \leq q$ by the properties 1), 2), and 3). Denote the k -vector subspace of $H^{p,q}(V)$ which, in the definition of ${}^*H^{p,q}(V)$, is obtained by restricting T to abelian varieties (resp. products of curves as above) by $\bar{H}^{p,q}(V)$. Since $\bar{H}^{p,q}(V)$ has the properties 1), 2) and 3) above, and $\bar{H}^{p,p}(V) = \mathfrak{A}^p(V)$, we have only to check $\bar{H}^{0,1}(V) = H^{0,1}(V)$. In the case of T being abelian varieties, if $i: V \rightarrow \text{Alb } V = A$ is a canonical morphism, we have $i^*: H^{0,1}(A) \simeq H^{0,1}(V)$; but since A is an abelian varieties, $H^{0,1}(A) = \bar{H}^{0,1}(A)$, so that $H^{0,1}(V) = i^*H^{0,1}(A) = i^*\bar{H}^{0,1}(A) \subset \bar{H}^{0,1}(V) \subset H^{0,1}(V)$. In the case of T being products of curves, it therefore suffices to show that $\bar{H}^{0,1}(V) = H^{0,1}(V)$ for an arbitrary abelian variety V . We can find a jacobian variety J and a homomorphism $h: V \rightarrow J$ with finite kernel. Then since $h^*H^{0,1}(J) = H^{0,1}(V)$, we are reduced to the case where V is the jacobian variety of a curve C of genus g . We have a natural generically finite morphism $f: C^g \rightarrow V$. By projection formula,

$$(\deg f) \text{id}: H^{0,1}(V) \xrightarrow{f^*} H^{0,1}(C^g) \xrightarrow{f_*} H^{0,1}(V),$$

hence f_* is surjective. So if $\bar{H}^{0,1}(C^g) = H^{0,1}(C^g)$, we are done. The map

$$K: H^{0,1}(C)^{\oplus g} \longrightarrow H^{0,1}(C^g)$$

defined by

$$K(x_1, \dots, x_g) = \sum_{i=1}^g \text{pr}_i^* x_i$$

is an isomorphism, where $\text{pr}_i: C^g \rightarrow C$ is the i -th projection. Since $H^{0,1}(C) = \bar{H}^{0,1}(C)$, $H^{0,1}(C^g) \supset \bar{H}^{0,1}(C^g) \supset \text{Im}(K: \bar{H}^{0,1}(C)^{\oplus g} \rightarrow H^{0,1}(C^g)) = \text{Im } K = H^{0,1}(C^g)$. By a similar method, we can show that, in the definition of ${}^bH^{p,q}(V)$, we can restrict T to varieties of dimension $q - p$.

COROLLARY (5.11.1). *For any V and p , ${}^*H^{p-1,p}(V) = {}^bH^{p-1,p}(V) \subset H^{p-1,p}(V)$.*

It is well-known that this subspace is equal to the tangent space of the algebraic part of the p -th intermediate jacobian of V if $k = \mathbb{C}$.

PROPOSITION (5.12). $\dim {}^{\#}H^{0,q}(V)$ is a birational invariant of V for each q .

If $V' \rightarrow V$ is the blowing-up of V with a non-singular center W , then $f^*: {}^{\#}H^{0,q}(V) \rightarrow {}^{\#}H^{0,q}(V')$ is an isomorphism. In fact we have an isomorphism

$$H^{p,q}(V) \oplus \bigoplus_{0 < i < r} H^{p-i, q-1}(W) \longrightarrow H^{p,q}(V'),$$

where r is the codimension of W in V , and this isomorphism and its inverse are composed of pull-backs, direct images and multiplications by fundamental classes of algebraic cycles. Hence the isomorphism remains valid when we replace H by ${}^{\#}H$. Since ${}^{\#}H^{0-i, q-i}(W) = 0$ for $i > 0$, $f^*: {}^{\#}H^{0,q}(V) \simeq {}^{\#}H^{0,q}(V')$. Now if V and V' are birationally equivalent, we have a diagram

$$V' \longleftarrow V_s \longrightarrow \dots \longrightarrow V_1 \longrightarrow V_0 = V,$$

where V_{i+1} is a blowing up of V_i with non-singular center, and $V_s \rightarrow V'$ is a birational morphism. We have $\dim {}^{\#}H^{0,q}(V') \leq \dim {}^{\#}H^{0,q}(V_s) = \dim {}^{\#}H^{0,q}(V)$. Switching the roles of V and V' , we get the reversed inequality, hence $\dim {}^{\#}H^{0,q}(V) = \dim {}^{\#}H^{0,q}(V')$.

Remark (5.13). It is well-known that $\dim {}^bH^{0,q}(V) = \dim H^{0,q}(V)$ is a birational invariant of V . We may ask whether $H^{0,q}(V) = {}^{\#}H^{0,q}(V)$ for any V and p , which is equivalent to

$${}^bH^{p,q}(V) = {}^{\#}H^{p,q}(V)$$

for any V and $p \leq q$.

THEOREM (5.14). If ${}^{\#}H^{p-\ell, p}(V) \neq 0$, then we have $gr^{\ell}CH^p(V) \otimes \mathbf{Q} \neq 0$.

By hypothesis, we can find ℓ curves C_1, \dots, C_{ℓ} , $u \in \mathfrak{A}^{p+\ell}(C_1 \times \dots \times C_{\ell} \times V)$ and $x \in (H^{0,1}(C_1 \times \dots \times C_{\ell}))^{\ell} = H^{0,\ell}(C_1 \times \dots \times C_{\ell})$ such that $u(x) \neq 0$. We may further assume that u is the fundamental class of a cycle U on $C_1 \times \dots \times C_{\ell} \times V$. By abuse of notation, we shall denote the rational equivalence class of a cycle by the same symbol as the cycle itself. Fix points a_i of C_i ($1 \leq i < \ell$), and consider the regular maps

$$\kappa': C_1 \times \dots \times C_{\ell} \longrightarrow CH_0(C_1 \times \dots \times C_{\ell})$$

defined by

$$\kappa'(x_1, \dots, x_{\ell}) = ((x_1) - (a_1)) \times \dots \times ((x_{\ell}) - (a_{\ell})).$$

and

$$\kappa: C_1 \times \cdots \times C_\ell \xrightarrow{\kappa'} CH_0(C_1 \times \cdots \times C_\ell) \xrightarrow{U(?)} CH^p(V).$$

$\text{Im } \kappa' \subset F^\ell CH_0(C_1 \times \cdots \times C_\ell)$ implies $\text{Im } \kappa \subset F^\ell CH^p(V)$. On the other hand, by (2.6.3), we have $\kappa^\# = \kappa'^\# \circ {}^t u(?)$, where

$${}^t u(?): H^{r+\ell, r}(V) \longrightarrow H^{\ell, 0}(C_1 \times \cdots \times C_\ell);$$

but ${}^t u(?)$ is the transpose of

$$u(?): H^{0, \ell}(C_1 \times \cdots \times C_\ell) \longrightarrow H^{p-\ell, p}(V),$$

hence we see ${}^t u(?) \neq 0$. If we write $\kappa'(x_1, \dots, x_\ell) = (x_1) \times \cdots \times (x_\ell) + R$, then R is a sum of elements of the form $\pm (y_1) \times \cdots \times (y_\ell)$, where at least one of y_i 's is a_i and the rests are x_i 's. In other words,

$$\kappa' = \kappa_0 + \sum \kappa_j,$$

where

$$\kappa_0: C_1 \times \cdots \times C_\ell \longrightarrow CH_0(C_1 \times \cdots \times C_\ell)$$

is the regular map

$$\kappa_0(x_1, \dots, x_\ell) = (x_1) \times \cdots \times (x_\ell)$$

and κ_j are regular maps such that

$$\kappa_j: C_1 \times \cdots \times C_\ell \xrightarrow{f_j} T_j \xrightarrow{\bar{\kappa}_j} CH_0(C_1 \times \cdots \times C_\ell)$$

with $\bar{\kappa}_j$ regular, a morphism f_j , and $\dim T_j < \ell$. For $\eta \in H^{\ell, 0}(C_1 \times \cdots \times C_\ell)$, we have $\kappa'^\# \eta = \kappa_0^\# \eta + \sum \kappa_j^\# \eta$. But $\kappa_0^\# \eta = \eta$, and $\kappa_j^\# \eta = 0$, because

$$\kappa_j^\#: H^{\ell, 0}(C_1 \times \cdots \times C_\ell) \xrightarrow{\bar{\kappa}_j^\#} H^{\ell, 0}(T_j) \xrightarrow{f_j^*} H^{\ell, 0}(C_1 \times \cdots \times C_\ell)$$

and $\dim T_j < \ell$; thus $\kappa'^\# = \text{id}$ on $H^{\ell, 0}(C_1 \times \cdots \times C_\ell)$ so that $\kappa^\# = {}^t u(?) \neq 0$ on $H^{r+\ell, r}(V)$. For each integer $n > 0$, $(n\kappa)^\#(H^{r+\ell, r}(V)) \neq 0$, which implies $\text{Im } (n\kappa) \not\subset F^{\ell+1} CH^p(V)$ in virtue of (5.7). Since

$$\{c \in C_1 \times \cdots \times C_\ell; n\kappa(c) \in F^{\ell+1} CH^p(V)\}$$

is a countable union of closed subsets of $C_1 \times \cdots \times C_\ell$ distinct from $C_1 \times \cdots \times C_\ell$ for each n by (5.5.5),

$$\bigcup_n \{c \in C_1 \times \cdots \times C_\ell; n\kappa(c) \in F^{\ell+1} CH^p(V)\} \neq C_1 \times \cdots \times C_\ell.$$

If $c \in C_1 \times \cdots \times C_\ell$ does not belong to the first member, then for any

$n > 0$, $n\kappa(c) \notin F^{\ell+1}CH^p(V)$, i.e., $\kappa(c) \in F^\ell CH^p(V)$ gives a non-torsion element of $gr^\ell CH^p(V)$, hence $gr^\ell CH^p(V) \otimes \mathbf{Q} \neq 0$.

Remark (5.15). From ${}^bH^{p-\ell,p}(V) \neq 0$, we deduce only

$$(F^1 CH^p(V)/F^{\ell+1} CH^p(V)) \otimes \mathbf{Q} \neq 0$$

EXAMPLE (5.16). If V is an abelian variety, we have $H^{0,p}(V) = \wedge^p H^{0,1}(V)$, hence ${}^*H^{p,q}(V) \neq 0$ for $0 \leq q \leq \dim V$, which in turn implies

$${}^*H^{p,q}(V) \neq 0 \quad \text{for all } 0 \leq p \leq q \leq \dim V,$$

by the Hard Lefschetz Theorem. Hence we find $gr^\ell CH^p(V) \otimes \mathbf{Q} \neq 0$ if $0 \leq \ell \leq p \leq \dim V$. If W is the associated Kummer variety, we have ${}^*H^{p,q}(W) \neq 0$ for $p+q \equiv 0 \pmod{2}$, so that $gr^\ell CH^p(W) \otimes \mathbf{Q} \neq 0$ if ℓ is even and $0 \leq \ell \leq q \leq \dim W$. Notice that ${}^*H^{p,q}(W) = 0$ for $p+q \equiv 0 \pmod{2}$; it would be interesting to know if $gr^\ell CH^p(W) = 0$ for odd ℓ .

REFERENCES

- [1] Bloch, S., Some elementary theorems about algebraic cycles on abelian varieties, *Invent. Math.*, **37** (1975), 215–228.
- [2] Deligne, P. and Mumford, D., The irreducibility of the space of curves of given genus. *IHES, Publ. Math.* N° 36, 75–109.
- [3] Griffiths, P. A., On the periods of certain rational integrals: I, II, *Ann. of Math.*, **90** (1969), 460–495 and 496–541.
- [4] Grothendieck, A., La théorie des classes de Chern, *Bull. Soc. Math. France*, **86** (1958), 137–154.
- [5] — and Dieudonné, J., *Eléments de géométrie algébrique*, IV (Troisième Partie). *IHES, Publ. Math.* N° 28.
- [6] Hartshorne, R., Residue and duality. *Lecture Notes in Math.* **20**, Springer 1966.
- [7] Katz, N., Le Théorème de Griffiths. *Lecture Notes in Math.* **340**, p. 341. Springer 1973.
- [8] Mumford, D., Rational equivalence of 0-cycles on surfaces, *J. Math. Kyoto Univ.*, **9** (1969), 195–204.
- [9] Roitman, A. A., Γ -equivalence of zero-dimensional cycles, *Mat. Sb.*, Tom **86** (128) (1971) 557–570 = *Math. USSR Sb.*, **15** (1971), 555–567.
- [10] —, Rational equivalence of zero-cycles, *Mat. Sb.* Tom **89** (131) (1972), 567–585 = *Math. USSR Sb.*, **18** (1972), 571–588.
- [11] Samuel, P., *Methodes d'algèbre abstraite en géométrie algébrique*. *Ergebniss*, No. 4. Berlin-Göttingen-Heidelberg, Springer 1955.
- [12] —, *Relations d'équivalence en géométrie algébrique*. *Proc. Int. Congress Math.* 1958, New York, Cambridge Univ. Press 1960.
- [13] Weil, A., Sur les critères d'équivalence en géométrie algébrique, *Math. Ann.*, **128** (1954), 95–127.

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