DISTRIBUTION OF THE SUM OF TRUNCATED BINOMIAL VARIATES

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1. Introduction. Let X be a random variable defined by a Bernoullian probability function

$$(1) \quad f(x) = \binom{N}{x} p^x q^{N-x} \qquad x = 0, 1, 2, \ldots, N$$
$$p + q = 1, \quad p > 0, \quad q > 0 .$$

The probability function of the restricted random variable which is truncated away from zero is then

$$(2) \quad f^*(x) = \frac{\binom{N}{x} p^x q^{N-x}}{1 - q^N} \qquad x = 1, 2, \ldots, N .$$

The divisor $1 - q^N$ arises from the condition excluding zero.

Finney (1949) has treated the truncated binomial distribution and has mentioned several practical problems in which a truncated binomial distribution might be met. In this note the distribution is obtained for the sum of n independent identically distributed truncated binomial random variables.

2. Distribution of the sum. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a population with probability function (2). We shall try to find a distribution of

$$Y = \sum_{j=1}^{n} X_j .$$

Now the characteristic function of a random variable X having probability function $\overset{*}{f}(x)$ is given by

$$(3) \quad \overset{*}{\phi}(t;x) = \sum_{x=1}^{N} \binom{N}{x} (p\, e^{it})^x \, q^{N-x} \Big/ 1 - q^N$$
$$= [(q + p\, e^{it})^N - q^N] \Big/ 1 - q^N$$

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Since the X_j's are assumed independent, the characteristic function $\overset{*}{\phi}(t;y)$ of the sum $Y = X_1 + X_2 + \dots X_n$ is given

$$\overset{*}{\phi}(t;y) = \left[(q + p e^{it})^N - q^N \right]^n \Big/ (1 - q^N)^n$$

(4) $$= \sum_{r=0}^{n} \binom{n}{r} (-1)^r (q + p e^{it})^{N(n-r)} q^{Nr} \Big/ (1 - q^N)^n$$

or

(5) $$\overset{*}{\phi}(t;y) = \sum_{r=0}^{n} b_r (q + p e^{it})^{N(n-r)} q^{Nr}$$

where

$$b_r = \binom{n}{r} (-1)^r \Big/ (1 - q^N)^n$$

Since $(q + p e^{it})^N$ is the characteristic function of $\binom{N}{x} p^x q^{N-x}$, the rth term of (5), excluding b_r, is the characteristic function of

$$\binom{N(n-r)}{y} p^y q^{N(n-r) - y} .$$

Thus, we have the exact distribution of the sum $Y = \sum_{j=1}^{n} X_j$.

(6) $$f(y) = \sum_{r=0}^{n-1} b_r \binom{N(n-r)}{y} p^y q^{N(n-r) - y}$$

$$y = n, n+1, \dots, N(n-r)$$

where $\binom{N(n-r)}{y}$ is defined to be zero when $y > N(n-r)$.

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