Can. J. Math., Vol. XXVIII, No. 4, 1976, pp. 753–759

FOURIER-YOUNG COEFFICIENTS OF A FUNCTION OF WIENER'S CLASS V_p

RAFAT N. SIDDIQI

1. Introduction. N. Wiener [**12**] introduced the idea of the class V_p. A 2π-periodic function f is said to have *bounded p-variation* $V_p(f)(1 \leqq p < \infty)$, or to *belong to the class* V_p, if

$$(1) \qquad V_p(f) = \lim_{\epsilon \to 0} \sup_{\mu(P) \leqq \epsilon} \left\{ \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p \right\}^{1/p} < \infty,$$

where $P : 0 = t_0 < t_1 < t_2 < \ldots < t_n = 2\pi$ is an arbitrary partition of $[0, 2\pi]$ with $\mu(P) = \max_{1 \leqq i \leqq n} |t_i - t_{i-1}|$. We write simply V_p for the class of functions of bounded p-variation on $[0, 2\pi]$. When $p = 1$, V_1 is an ordinary class of functions of bounded variation. We have $V_{p_1} \subset V_{p_2}$ $(1 \leqq p_1 < p_2 < \infty)$, (see [**11**]), a strict inclusion. Hence for $(1 < p < \infty)$, Wiener's class V_p is strictly larger class than the class V_1. In connection with the existence of Riemann-Stieltjes integral of functions of V_p, Young [**13**] proved the following theorem.

THEOREM A. *If an $f \in V_p$ and a $g \in V_q$ where $p, q > 0$, $1/p + 1/q > 1$, have no common points of discontinuity, their Stieltjes integral $\int_0^{2\pi} fdg$ exists in the Riemann sense.*

From Theorem A, $\hat{f}(n)$ defined by

$$\hat{f}(n) = (2\pi)^{-1} \int_0^{2\pi} e^{int} df(t) \quad (n = 0, \pm 1, \pm 2, \ldots)$$

exists for every $f \in V_p(1 \leqq p < \infty)$. We shall call the series $\sum_{-\infty}^{\infty} \hat{f}(k) e^{ikx}$ the *Fourier-Young series* of f and $\hat{f}(n)$ will be called a *sequence of Fourier-Young coefficients* of $f \in V_p(1 < p < \infty)$.

2. An infinite matrix $\Lambda = (\lambda_{n,k})(n, k = 0, 1, 2, \ldots)$ of real or complex numbers is called *admissible* if $\sup_{n \geqq 0} \sum_{k=0}^{\infty} |\lambda_{n,k}| < \infty$. A sequence $\{s_k\}$ is said to be *summable* Λ if $\lim_{n \to \infty} \sum_{k=0}^{\infty} \lambda_{n,k} s_k$ exists; it is said to be *summable* F_Λ if $\lim_{n \to \infty} \sum_{k=0}^{\infty} \lambda_{n,k} s_{k+\nu}$ exists uniformly in $\nu = 0, 1, 2, \ldots$. The summability method F_Λ corresponding to the arithmetic mean is called *almost convergence*

Received June 25, 1975 and in revised form March 4, 1976. This research work was supported in part by NRC of Canada grant given to the Department of Physics-Mathematics, Université de Moncton, Moncton, N.B., and in part by the grant of a fellowship of SRI of the Canadian Mathematical Congress held at Dalhousie University, Halifax, N.S.

[7]. Recently Siddiqi [**10**] generalized a classical theorem of Fejér [**3**] on determination of jump of a function of the class V_1.

THEOREM B. *If* $\Lambda = (\lambda_{n,k})$ *is an admissible matrix, then for every* $f \in V_1$ *and for every* $x \in [0, 2\pi]$, *the sequence*

(2) $$\{\hat{f}(k)e^{ikx} + \hat{f}(-k)e^{-ikx} - \pi^{-1}D(x)\}$$

is summable Λ (*or* F_Λ) *to zero if and only if* $\{\cos kt\}$ *is summable* Λ (*respectively,* F_Λ) *to zero for all* $t \not\equiv 0 \pmod{2\pi}$, *where* $D(x) = f(x + 0) - f(x - 0)$.

In this paper we study the problem of summability of the sequence (2) and of allied sequences in the strictly larger class $V_p (1 < p < \infty)$. This enables us to obtain extensions of various theorems of Fejér [**3**], Wiener [**12**], Lozinskiĭ [**8**], Matveyev (cf. Bari [**1**, p. 256]), Keogh and Petersen [**6**], Siddiqi [**10**] and DeLeeuw and Katznelson [**2**]. We give a simple proof of Theorem 1 which depends only on the application of Fatou's lemma and on the properties of limit superior. More precisely we prove the following theorem.

THEOREM 1. *If* $\Lambda = (\lambda_{n,k})$ *is an admissible matrix such that* $\{\cos kt\}$ *is summable* Λ (*or* F_Λ) *to zero for all* $t \not\equiv 0 \pmod{2\pi}$, *then for every* $f \in V_p (1 < p < \infty)$ *and for every* $x \in [0, 2\pi]$, *the sequence* (2) *is summable* Λ (*respectively,* F_Λ) *to zero. Conversely, if*

$$\underline{\lim}_{n\to\infty} \left(\sum_{k=0}^{\infty} \lambda_{n,k}(\hat{f}(k)e^{ikx} + \hat{f}(-k)e^{-ikx} - \pi^{-1}D(x) \right) = 0$$

for every $f \in V_1$ *and for every* $x \in [0, 2\pi]$, *then*

$$\underline{\lim} \sum_{k=0}^{\infty} \lambda_{n,k} \cos kt = 0$$

for all $t \not\equiv 0 \pmod{2\pi}$.

3. *Proof.* We shall give the proof of summability Λ only. The proof of summability F_Λ is similar. Suppose that $\{\cos kt\}$ is summable Λ to zero for all $t \not\equiv 0 \pmod{2\pi}$. We can write

$$\hat{f}(k)e^{ikx} + \hat{f}(-k)e^{-ikx} = \pi^{-1} \int_0^{2\pi} \cos k(x - t) df(t).$$

If $d(x_j)$ denotes the jump of f at $x_j \in [0, 2\pi]$, then $\sum_{j=0}^{\infty} [d(x_j)]^p \leqq V_p(f)$ which is finite (cf. Wiener [**12**, p. 76]). Hence we can define

(3) $$h(t) = f(t) - \pi^{-1} \sum_{j=0}^{\infty} d(x_j)\phi(t - x_j)$$

where $\phi(t) = (\pi - t)/2 \quad (0 < t < 2\pi)$, $\phi(0) = \phi(2\pi) = 0$, and outside of $[0, 2\pi]$, ϕ is defined by periodicity. It is clear that $h \in V_p \ (1 \leqq p < \infty)$ and is continuous everywhere, and hence we can define

$$\{A_k(x)\} = \left\{\hat{f}(k)e^{ikx} + \hat{f}(-k)e^{-ikx} - \pi^{-1}\sum_{j=0}^{\infty} d(x_j)\cos k(x - x_j)\right\}$$
$$= \pi^{-1}\int_0^{2\pi}\cos k(x-t)dh(t),$$

so that

$$\sum_{k=0}^{\infty}\lambda_{n,k}A_k(x) = \pi^{-1}\int_0^{2\pi}K_n(x-t)dh(t)$$

where

$$K_n(t) = \sum_{k=0}^{\infty}\lambda_{n,k}\cos kt.$$

Breaking $K_n(t)$ into its positive and negative parts, we can write

$$K_n(t) = K_n^+(t) - K_n^-(t)$$

where $K_n^+(t) = \max\ (K_n(t), 0)$ and $K_n^-(t) = \max\ (0, -K_n(t))$. We also denote

$$\phi_n(x) = \int_0^{2\pi}K_n(x-t)dh(t)$$
$$(4) \qquad = \int_0^{2\pi}K_n^+(x-t)dh(t) - \int_0^{2\pi}K_n^-(x-t)dh(t)$$
$$= \phi_n^+(x) - \phi_n^-(x).$$

Using the properties of limit superior (cf. Royden [**9**, p. 36]), we obtain

$$(5) \qquad \overline{\lim}\ \phi_n(x) \leqq \overline{\lim}\ \phi_n^+(x) - \underline{\lim}\ \phi_n^-(x).$$

But by Fatou's lemma (cf. Hildebrandth [**4**, p. 25]), we have

$$(6) \qquad \begin{aligned} \overline{\lim}\ \phi_n^+(x) &\leqq \int_0^{2\pi}\overline{\lim}\,K_n^+(x-t)dh(t) \quad \text{and} \\ -\underline{\lim}\ \phi_n^-(x) &\leqq -\int_0^{2\pi}\underline{\lim}\,K_n^-(x-t)dh(t) \end{aligned}$$

Adding (6) together and using (5), we obtain

$$(7) \qquad \overline{\lim}\ \phi_n(x) \leqq \int_0^{2\pi}(\overline{\lim}\,K_n^+(x-t) - \underline{\lim}\,K_n^-(x-t))dh(t).$$

But $\overline{\lim}\,K_n^+(x-t) = \underline{\lim}\,K_n^-(x-t)$ by hypothesis, hence

$$(8) \qquad \overline{\lim}\ \phi_n(x) \leqq 0.$$

Similarly using Fatou's lemma [**4**] again and by the properties of limit inferior, we obtain

$$(9)\quad \overline{\lim}\ \phi_n(x) \geqq \underline{\lim}\ \phi_n(x) \geqq \underline{\lim}\ \phi_n^+(x) - \overline{\lim}\ \phi_n^-(x)$$

$$\geqq \int_0^{2\pi} (\underline{\lim}\, K_n^+(x-t) - \overline{\lim}\, K_n^-(x-t))dh(t).$$

which is equal to zero by hypothesis. Hence

$$(10)\quad \overline{\lim}\ \phi_n(x) \geqq 0.$$

From (8) and (10), we conclude that $\overline{\lim}\ \phi_n(x) = 0$. Also it follows from (9) that $\underline{\lim}\ \phi_n(x) \geqq 0$ and from (8), we have $\underline{\lim}\ \phi_n(x) \leqq 0$. Hence $\underline{\lim}\ \phi_n(x) = 0$, and hence we obtain finally

$$(11)\quad \lim_{n\to\infty} \phi_n(x) = 0.$$

This implies that if $K_n(t) \to 0$ as $n \to \infty$ for all $t \not\equiv 0 \pmod{2\pi}$, then the sequence $\{A_k(x)\}$ and hence the sequence

$$\{\hat{f}(k)e^{ikx} + \hat{f}(-k)e^{-ikx} - \pi^{-1}D(x)\}$$

is summable Λ to zero for every $f \in V_p$ and for every $x \in [0, 2\pi]$. Conversely if

$$\underline{\lim}\left(\sum_{k=0}^{\infty} \lambda_{n,k}(\hat{f}(k)e^{ikx} + \hat{f}(-k)e^{-ikx} - \pi^{-1}D(x)\right) = 0$$

for every $f \in V_p$ and for every $x \in [0, 2\pi]$, then we choose $f(t) = 2\phi(t)$ where $\phi(t)$ has already been defined in (3). It can easily be verified that $f \in V_p(1 \leqq p < \infty)$ and $\hat{f}(k) = \hat{f}(-k) = 1$, $D(x) = 0$ so that

$$\hat{f}(k)e^{ikx} + \hat{f}(-k)e^{-ikx} - \pi^{-1}D(x) = 2\cos kx.$$

This completes the proof of Theorem 1.

Theorem 1 contains as a special case the following extended version of Fejér's Theorem (cf. Zygmund [**14**, p. 107, Theorem 9.3]).

COROLLARY 1. *Let $f \in V_p(1 < p < \infty)$ and let $x \in [0, 2\pi]$. Then*

$$\lim_{n\to\infty} (n+1)^{-1} \sum_{k=\nu}^{n+\nu} A_k(x) = 0$$

uniformly in $\nu = 0, 1, 2, \ldots$.

Using an argument similar to the proof of Theorem 1, we can prove the following:

THEOREM 2. *If $\Lambda = (\lambda_{n,k})$ is an admissible matrix such that $\{e^{ikt}\}$ is summable Λ to zero for all $t \not\equiv 0 \pmod{2\pi}$ then for every $f \in V_p(1 < p < \infty)$ and for every*

https://doi.org/10.4153/CJM-1976-071-4 Published online by Cambridge University Press

$x \in [0, 2\pi]$, the sequences $\{\hat{f}(k)e^{ikx} - \hat{f}(-k)e^{-ikx}\}$ and $\{\hat{f}(\pm k)e^{\pm ikx} - (2\pi)^{-1}D(x)\}$ both are summable F_Λ to zero. The converse is also true in the sense of Theorem 1.

4. If $f \in V_p(1 < p < \infty)$, then from Theorem A, the convolution of f defined by

$$f^*(x) = (2\pi)^{-1}\int_0^{2\pi} f(x+t)d\overline{f(t)}$$

exists for every point x of continuity of f and for the values of $1 \leqq p < 2$ only. If x_j is a point of discontinuity of f, then we define $f^*(x_j) = \lim_{x\to x_j} f^*(x)$. It is easily seen (cf. Zygmund [**14**, p. 108]) that $f^* \in V_p(1 \leqq p < 2)$ and its Fourier-Young series is $\sum_{-\infty}^{\infty} |\hat{f}(k)|^2 e^{ikx}$. Since

$$f^*(x) = (2\pi)^{-1}\int_0^{2\pi} f(x+t)d\overline{h(t)} + (2\pi)^{-1}\sum_{j=0}^{\infty} f(x+x_j)\overline{d(x_j)},$$

It follows that

$$f^*(+0) - f^*(-0) = (2\pi)^{-1}\sum_{j=0}^{\infty} |d(x_j)|^2$$

where summation is over all points of discontinuity of f in $[0, 2\pi]$ and h is defined above in (3). Hence applying Theorem 1 at $x = 0$ for the Fourier-Young series of f^*, we deduce the following generalization of a theorem of Wiener [**12**].

THEOREM 3. *If $\Lambda = (\lambda_{n,k})$ is an admissible matrix such that $\{\cos kt\}$ is summable Λ (or F_Λ) to zero for all $t \not\equiv 0 \pmod{2\pi}$, then for every $f \in V_p(1 < p < 2)$ the sequence*

$$\left\{|\hat{f}(k)|^2 + |\hat{f}(-k)|^2 - (2\pi^2)^{-1}\sum_{j=0}^{\infty} |d(x_j)|^2\right\}$$

is summable Λ (respectively, F_Λ) to zero. The converse is also true in the sense of Theorem 1.

Applying Theorem 3 and Schwarz's inequality, we obtain the following extended version of a theorem of Wiener [**12**].

THEOREM 4. *If $\Lambda = (\lambda_{n,k})$, is a positive admissible matrix such that $\{\cos kt\}$ is summable Λ (or F_Λ) to zero for all $t \not\equiv 0 \pmod{2\pi}$, then for every $f \in V_p(1 < p < 2)$, the following statements are equivalent:*

(1) *f is continuous.*
(2) *$\{|\hat{f}(k)|^2 + |\hat{f}(-k)|^2\}$ is summable Λ (respectively, F_Λ) to zero.*
(3) *$\{|\hat{f}(k)| + |\hat{f}(-k)|\}$ is summable Λ (respectively, F_Λ) to zero.*

If f is a real-valued function of V_p, then $|\hat{f}(k)| = |\hat{f}(-k)|$. Hence under the hypothesis of Theorem 4, the statements (1), (2) and (3) will be equivalent to the statement that the sequence $\{|\hat{f}(k)|^2\}$ or $\{|\hat{f}(-k)|\}$ is summable Λ (respectively, F_Λ) to zero.

Since f^* exists for the values of $1 \leqq p < 2$ only, the theorems of Wiener [**12**], Keogh and Petersen [**6**], Lozinskiĭ [**8**], Matveyev (cf. Bari [**1**, p. 256, Exc. 9]) and of Siddiqi [**10**] can be extended to the class $V_p(1 \leqq p < 2)$ by making special choices of the matrix $\Lambda = (\lambda_{n,k})$.

5. Now we shall give some applications of our theorems. Recently, DeLeeuw and Katznelson [**2**] have given some results for the convergence of $\{|\hat{f}(n)|\}$ of functions of the class V_1 only. More precisely, they [**2**] proved the following theorem.

THEOREM C. *If* $\{|\hat{f}(n)|\}$ *converges to zero then* $\{|\hat{f}(-n)|\}$ *converges to zero as* $n \to \infty$ *for all* $f \in V_1$.

They [**2**] also gave an example of a function $f \in V_1$ for which

$$\overline{\lim_{n\to\infty}}\, |\hat{f}(n)| \neq \overline{\lim_{n\to\infty}}\, |\hat{f}(-n)|.$$

Applying Theorem 4, we can extend Theorem C into strictly larger class V_p in the following way.

THEOREM 5. *Let* $\Lambda = (\lambda_{n,k})$ *be a positive admissible matrix such that* $\{\cos kt\}$ *is summable* Λ (*or* F_Λ) *to zero for all* $t \not\equiv 0 \pmod{2\pi}$. *Then for every continuous function* f *of the class* $V_p(1 \leqq p < 2)$, *the sequence* $\{|\hat{f}(k)|\}$ *is summable* Λ (*respectively,* F_Λ) *to zero if and only if* $\{|\hat{f}(-k)|\}$ *is summable* Λ (*respectively,* F_Λ) *to zero.*

Proof. If $f \in V_p$ and is continuous, then from Theorem 4, we have that the sequence defined by

$$\{|\hat{f}(k)| + |\hat{f}(-k)|\} \tag{12}$$

is summable Λ (respectively, F_Λ) to zero. The equivalence of the summability of the sequences $\{|\hat{f}(k)|\}$ and $\{|\hat{f}(-k)|\}$ follows immediately from the summability of the sequence (12).

Remark. If we drop the hypothesis of continuity in Theorem 5, we can establish a criterion for the summability of the sequences $\{|\hat{f}(k)|\}$ and $\{|\hat{f}(-k)|\}$ to a number different from zero.

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Université de Moncton,
Moncton, New Brunswick