A REMARK ON THE GROUP RINGS OF ORDER PRESERVING PERMUTATION GROUPS

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If G and H are two groups such that their integral group rings $Z(G)$ and $Z(H)$ are isomorphic, does it follow that G and H are isomorphic? This is the isomorphism problem and an affirmative answer is obtained in case G is a subgroup of the group of order preserving permutations of a totally ordered set.

For any totally ordered set Λ, define Orp Λ to be the group of all functions $f : \Lambda \to \Lambda$ such that f is one-to-one, onto and the inequality $x < y$ $(x, y \in \Lambda)$ implies that $xf < yf$. Following P. Hall (Lecture notes, Cambridge 1966) we denote by 0^* the class of all groups that can be embedded as subgroups of Orp Λ, for some totally ordered set Λ. Thus every lattice ordered group is an 0^* group (Holland [2]). An alternative definition of the class 0^* is given by: $G \in 0^*$ if and only if G can be totally ordered so that for any $a, b, c \in G$, $a < b$ implies that $ac < bc$ (Conrad [1]). Our main result is:

THEOREM. _If_ $Z(G) \cong Z(H)$ _and_ $G \in 0^*$, _then_ $G \cong H$.

We remark that the ring Z is used only for convenience. The Theorem holds if Z is replaced by any ring R with identity, without zero-divisors and whose group of units is a torsion group.

Proofs:

LEMMA. _If_ $G \in 0^*$ _and_ x _is a unit of_ $Z(G)$, _then_ $x = \pm g$ _for some_ $g \in G$. _Also_ $Z(G)$ _has no zero divisors_.

Proof. Let y be the multiplicative inverse of x, and choose an ordering '<' of G as above. Write $x = \sum_{i=1}^{n} \alpha_i g_i$, $y = \sum_{i=1}^{m} \beta_i h_i$ with $g_1 < g_2 < \ldots < g_n$, $h_1 < h_2 < \ldots < h_m$, and all α_i, β_j different from 0. Let s and t be such that $g_1 h_s$ is the least element of $\{g_1 h_1, g_1 h_2, \ldots, g_1 h_m\}$ and $g_n h_t$ is the greatest element of $\{g_n h_1, g_n h_2, \ldots, g_n h_m\}$. It follows that $g_1 h_s < g_i h_j$ for all $(i, j) \neq (1, s)$ and $g_i h_j < g_n h_t$ for all $(i, j) \neq (n, t)$.

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Thus

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j (g_i h_j) \neq 1$$

unless $n = m = 1$, and hence $x = \alpha_1 g_1$ with $\alpha_1 = \pm 1$. A similar argument shows that $Z(G)$ has no zero divisors.

Proof of the Theorem: We first show that $H \in 0^*$. Since $Z(G) \cong Z(H)$, the group U_G of units of $Z(G)$ is isomorphic to U_H, the group of units of $Z(H)$. Thus H is isomorphic to a subgroup of U_G. H is torsion-free for if $h \in H$ is of order $k > 1$, then $(h-1)(1+h+\ldots+h^{k-1}) = 0$, and this implies that $Z(H)$ and hence $Z(G)$ has zero-divisors. Clearly U_G is the direct product $Z_2 \times G$ where Z_2 is the cyclic group of order two. It follows that H is isomorphic to a subgroup of G and so $H \in 0^*$.

Thus the only units of $Z(H)$ are $\pm h$, $h \in H$ and $U_H = Z_2 \times H$ and it follows that $G \cong H$.

REFERENCES

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