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A FAST ALGORITHM FOR CONSTRUCTING ORTHOGONAL MULTIWAVELETS

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Abstract

Multiwavelets possess some nice features that uniwavelets do not. A consequence of this is that multiwavelets provide interesting applications in signal processing as well as in other fields. As is well known, there are perfect construction formulas for the orthogonal uniwavelet. However, a good formula with a similar structure for multiwavelets does not exist. In particular, there are no effective methods for the construction of multiwavelets with a dilation factor a ($a \geq 2$, $a \in Z$). In this paper, a procedure for constructing compactly supported orthonormal multiscaling functions is first given. Based on the constructed multiscaling functions, we then propose a method of constructing multiwavelets, which is similar to that for constructing uniwavelets. In addition, a fast numerical algorithm for computing multiwavelets is given. Compared with traditional approaches, the algorithm is not only faster, but also computationally more efficient. In particular, the function values of several points are obtained simultaneously by using our algorithm once. Finally, we give three examples illustrating how to use our method to construct multiwavelets.

1. Introduction

In recent years, multiscaling functions and multiwavelets have been studied extensively. Goodman, Lee and Tang [11] established a characterisation of multiscaling functions and corresponding multiwavelets. Chui and Wang [3] introduced semi-orthogonal spline multiwavelets. Examples of cubic and quintic finite elements and their corresponding multiwavelets were studied by Strang and Strela [16]. In [2], Chui and Lian introduced a scheme for constructing symmetric and antisymmetric compactly supported orthogonal multiscaling functions and multiwavelets. Geronimo, Hardin and Massopust [10] used fractal interpolation to construct orthogonal multiscaling functions, and their corresponding multiwavelets were given in [7]. In [8],

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Donovan, Geronimo and Hardin showed that there exist compactly supported orthogonal polynomial spline multiscaling functions with arbitrarily high regularity. Interpolation multiscaling functions and multiwavelets were investigated in [19] and in [14]. For the bivariate setting, one easy way to get a multiscaling function is to use the tensor product of two univariate multiscaling functions. Although this method is simple, the bivariate tensor product multiscaling function has $r \times r$ components if the bivariate multiscaling function has r components and this hence increases complexity in applications. The most interesting research on multiscaling functions and multiwavelets is therefore that dealing with the construction of nonseparable multiscaling functions and multiwavelets. Tymczak *et al.* [18] gave a method of construction for nonseparable orthogonal multiscaling functions and multiwavelets. At present, the study of multiwavelets is still of considerable interest to many researchers (see, for example, [1, 9, 12]).

As is well known, Daubechies [4] obtained perfect constructing formulas for the uniwavelet. Since multiwavelets are vector-valued functions, the construction of multiwavelets is more difficult than that of uniwavelets. Multiwavelets can possess simultaneously many desirable properties, such as being compactly supported, orthonormality, interpolating, and very importantly, symmetry or antisymmetry. However, for the uniwavelet, some of these properties are impossible or incompatible. From this respect, applications of multiwavelets are more extensive than those of uniwavelets. Therefore, finding good construction approaches for multiwavelets is very significant both in theory and in applications. Donovan, Geronimo and Hardin [7] discussed the above problem by using fractal interpolation functions, but their construction procedure is very complicated. The main objective of this paper is to give a way of constructing compactly supported multiscaling functions and the associated multiwavelets.

The paper is organised as follows. In Section 2, we briefly recall the concept of multiresolution analysis of multiplicity r. In Section 3, we give a constructing procedure for compactly supported orthogonal multiscale functions and multiwavelets. In Section 4, based on the construction algorithm given in Section 3, some examples of constructing multiwavelets are given. In Section 5, based on multiresolution analysis and matrix theory, we give a fast numerical algorithm for computing multiwavelets.

2. Multiresolution analysis with multiplicity r

Multiwavelets are associated with multiresolution analysis of multiplicity r, that is, multiwavelets can be constructed by multiresolution analysis with multiplicity r.

Let

$$\Phi(x) = (\phi_1, \phi_2, \dots, \phi_r)^T, \quad \phi_1, \phi_2, \dots, \phi_r \in L^2(R),$$

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satisfy the following two-scale matrix equation:

$$\Phi(x) = \sum_{k=0}^{M} P_k \Phi(ax - k), \tag{2.1}$$

where M is a positive integer and some $r \times r$ matrices $\{P_k\}$ are called the two-scale matrix sequence. Here $\Phi(x)$ is called a multiscaling function with dilation a, $a \geq 2$, $a \in Z$, and multiplicity r.

Applying the Fourier transformation to (2.1), we obtain

$$\hat{\Phi}(w) = P(z)\hat{\Phi}(w/a), \tag{2.2}$$

where $z = e^{-iw/a}$, and

$$P(z) = \frac{1}{a}\sum_{k=0}^{M} P_k z^k. \tag{2.3}$$

We note that $P(z)$ is referred to as the two-scale matrix symbol of the two-scale matrix sequence $\{P_k\}$ of Φ.

Define a sequence of subspace $\mathbf{V}_j \subset L^2(R)$,

$$\mathbf{V}_j = \text{clos}_{L^2(R)}\langle \phi_{\ell:j,k} : 1 \leq \ell \leq r,\ k \in Z\rangle, \quad j \in Z, \tag{2.4}$$

here and afterwards; for $f_\ell \in L^2$, we will use the notation $f_{\ell:j,k} = a^{j/2} f_\ell(a^j x - k)$.

As usual, $\Phi(x)$ in (2.1) generates a multiresolution analysis $\{\mathbf{V}_j\}_{j\in Z}$ of $L^2(R)$, if $\{\mathbf{V}_j\}_{j\in Z}$ defined in (2.4) satisfies the nestedness condition $\cdots \subset \mathbf{V}_0 \subset \mathbf{V}_1 \subset \mathbf{V}_2 \cdots$.

Let $\mathbf{W}_j$, $j \in Z$, denote the orthogonal complementary subspace of $\mathbf{V}_j$ in $\mathbf{V}_{j+1}$, and let the vector-valued function $\Psi(x) = (\psi_1, \psi_2, \ldots, \psi_{(a-1)r})^T$, $\psi_\ell \in L^2$, $\ell = 1, 2, \ldots, (a-1)r$, constitute a Riesz basis for $\mathbf{W}_j$, that is,

$$\mathbf{W}_j = \text{clos}_{L^2(R)}\langle \psi_{\ell:j,k} : 1 \leq \ell \leq (a-1)r,\ k \in Z\rangle, \quad j \in Z. \tag{2.5}$$

From condition (2.5), it is clear that $\psi_1(x), \psi_2(x), \ldots, \psi_{(a-1)r}(x)$ are in $\mathbf{W}_0 \subset \mathbf{V}_1$. Hence there exists a sequence of matrices $\{Q_k\}_{k\in Z}$ such that

$$\Psi(x) = \sum_{k=0}^{M} Q_k \Phi(ax - k). \tag{2.6}$$

From the two-scale relation (2.6), we obtain $\hat{\Psi}(w) = Q(z)\hat{\Phi}(w/a)$, where $Q(z) = (1/a)\sum_{k=0}^{M} Q_k z^k$.

For column vector functions Λ and Γ with elements in $L^2(R)$, define

$$\langle \Lambda, \Gamma \rangle = \int_R \Lambda(x)\Gamma(x)^T\, dx.$$

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We call $\Phi(x) = (\phi_1, \phi_2, \dots, \phi_r)^T$ an orthogonal multiscaling function if

$$\langle \Phi(\cdot), \Phi(\cdot - n)\rangle = \delta_{0,n} I_r, \quad n \in Z.$$

Here $\Psi(x) = (\psi_1, \psi_2, \dots, \psi_{(a-1)r})^T$ will be said to be the orthogonal multiwavelets associated with multiscaling functions Φ if $\Psi(x)$ satisfy the following equations:

$$\langle \Phi(\cdot), \Psi(\cdot - n)\rangle = \langle \Psi(\cdot), \Phi(\cdot - n)\rangle = O_{r\times(a-1)r},$$
$$\langle \Psi(\cdot), \Psi(\cdot - n)\rangle = \delta_{0,n} I_{(a-1)r}, \quad n \in Z,$$

where $O_{r\times(a-1)r}$ and $I_{(a-1)r}$ denote the zero matrix and identity matrix, respectively.

LEMMA 2.1. *Let $\eta = (\eta_1, \eta_2, \dots, \eta_r)^T$, where $\eta_1, \eta_2, \dots, \eta_r \in L^2$. Then $\{\eta_\ell(x - k) : 1 \le \ell \le r,\ k \in Z\}$ is a family of orthogonal functions if and only if*

$$\sum_{k\in Z} \hat{\eta}(\omega + 2k\pi)\hat{\eta}(\omega + 2k\pi)^* = I_r. \tag{2.7}$$

Here and throughout, the asterisk denotes complex conjugation of the transpose.

PROOF. Let η be a family of orthogonal functions. For every $n \in Z$, we have

$$\begin{aligned}\delta_{0,n} I_r = \langle \eta(\cdot), \eta(\cdot - n)\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\eta}(w)\hat{\eta}(w)^* e^{inw}\, dw \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{2\pi}\int_{2k\pi}^{2(k+1)\pi} \hat{\eta}(w)\hat{\eta}(w)^* e^{inw}\, dw \\ &= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{k\in Z}\hat{\eta}(w + 2k\pi)\hat{\eta}(w + 2k\pi)^*\right] e^{inw}\, dw,\end{aligned}$$

which implies (2.7) holds. The converse is obvious.

LEMMA 2.2. *Let $\Phi(x)$ be a multiscaling function satisfying* (2.1) *and $P(z)$ defined in* (2.3) *be a two-scale matrix symbol. Then*

(i) *$\Phi(x)$ is compactly supported, with* $\operatorname{supp}\Phi(x) \subset [0, M/(a-1)]$,
(ii) *$P(1)$ has eigenvalue* 1, *and $[P(1)]^n$ converges as $n \to \infty$, and*
(iii) *the vector $u = \hat{\Phi}(0)$ is an eigenvector corresponding to the eigenvalue* 1 *of* $P(1)$.

PROOF. Similar to the case of $a = 2$ (see [15]), (i) can be proved. Next, we prove (ii) and (iii).

It is clear from (2.2) that $P(1)$ has eigenvalue 1 with eigenvector $u = \hat{\Phi}(0)$. So in order to prove (ii) and (iii), it is sufficient to prove that $[P(1)]^n$ converges as $n \to \infty$. This is, however, obvious, since it follows from (2.2) that $u = \lim_{\ell\to\infty}\{[P(1)]^\ell u\}$, which implies that $[P(1)]^n$ must converge as $n \to \infty$.

LEMMA 2.3. *Let* Φ *be a multiscaling function satisfying* (2.1). *If both* P_0, P_M *are not nilpotent, then* $\operatorname{supp}\Phi = [0, M/(a-1)]$.

The case of $a = 2$ in Lemma 2.3 has been thoroughly studied by Massopust, Ruch and Van Fleet [13]. By means of an idea of these aforementioned authors, Lemma 2.3 can be proved for $a \geq 3$.

3. Construction of orthonormal multiwavelets

THEOREM 3.1. *Let* $\Phi(x)$ *be the orthogonal multiscaling functions defined in* (2.1), $P(z)$ *be the two -scale matrix symbol defined in* (2.3) *and* ω_j, $j = 1, 2, \ldots, a$ *be roots of the equation* $z^a - 1 = 0$. *Then*

$$\sum_{j=1}^{a} P(\omega_j z)P(\omega_j z)^* = I_r, \quad |z| = 1. \tag{3.1}$$

Equation (3.1) *is equivalent to the following equation*:

$$\sum_{i=0}^{M} P_i P_{i+ak}^* = a\delta_{k,0} I_r, \quad |z| = 1. \tag{3.2}$$

Further, suppose $\Psi = (\psi_1, \psi_2, \ldots, \psi_{(a-1)r})^T$ *are the orthogonal multiwavelets associated with* Φ *and* $Q(z)$ *is a two-scale matrix symbol. Then*

$$\sum_{j=1}^{a} P(\omega_j z)Q(\omega_j z)^* = 0 \quad \textit{and} \tag{3.3}$$

$$\sum_{j=1}^{a} Q(\omega_j z)Q(\omega_j z)^* = I_{(a-1)r}. \tag{3.4}$$

Equations (3.3) *and* (3.4) *are equivalent to the following equations, respectively*:

$$\sum_{i=0}^{M} P_i Q_{i+ak}^T = 0 \quad \textit{and} \quad \sum_{i=0}^{M} Q_i Q_{i+ak}^T = a\delta_{0,k} I_{(a-1)r}.$$

PROOF. By Lemma 2.1, we have $\sum_{\ell\in Z} |\hat{\Phi}(\omega + 2\pi\ell)|^2 = I_r$, for all ω. Hence

$$\begin{aligned} I_r &= \sum_{\ell} |\hat{\Phi}(a\omega + 2\pi\ell)|^2 \\ &= \sum_{\ell=ak} |P(e^{-i(\omega+2k\pi)})|^2 |\hat{\Phi}(\omega + 2k\pi)|^2 \end{aligned}$$

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$$
\begin{aligned}
&+ \sum_{\ell=ak+1} |P(e^{-i(\omega+2k\pi+2\pi/a)})|^2 \left|\hat{\Phi}(\omega+2k\pi+2\pi/a)\right|^2 + \cdots \\
&+ \sum_{\ell=ak+a-1} |P(e^{-i(\omega+2k\pi+2\pi/(a-1))})|^2 \left|\hat{\Phi}(\omega+2k\pi+2\pi/(a-1))\right|^2 \\
= &\ |P(e^{-i\omega})|^2 \sum_k |\hat{\Phi}(\omega+2k\pi)|^2 \\
&+ |P(e^{-i(\omega+2\pi/a)})|^2 \sum_k \left|\hat{\Phi}(\omega+2k\pi+2\pi/a)\right|^2 + \cdots \\
&+ |P(e^{-i(\omega+2\pi/(a-1))})|^2 \sum_k \left|\hat{\Phi}(\omega+2k\pi+2\pi/(a-1))\right|^2 \\
= &\ |P(e^{-i\omega})|^2 + |P(e^{-2\pi/a}e^{-i\omega})|^2 + \cdots + |P(e^{-2(a-1)\pi/a}e^{-i\omega})|^2,
\end{aligned}
$$

which implies (3.1) holds. Similarly, applying the Poisson formula

$$\sum_\ell \hat{\Phi}(\omega+2\pi\ell)\overline{\hat{\Psi}(\omega+2\pi\ell)} = 0$$

for all ω, (3.3) and (3.4) can be proven.

We assume that the functions in Φ are $(r-1)$-times differentiable. Analogous to Hermite cardinal spline interpolation, $\mathbf{\Phi}(x) = (\phi_1, \phi_2, \dots, \phi_r)^T$ with common support is said to be an interpolation, if it satisfies the following condition:

$$\begin{cases} \mathbf{\Phi}^{(j-1)}(k+k_0) = \phi_j^{(j-1)}(k_0)\delta_{k,0}\mathbf{e}_j, & 1 \le j \le r, k \in Z, \\ \phi_j^{(j-1)}(k_0) \ne 0, & 1 \le j \le r, \end{cases} \tag{3.5}$$

where k_0 denotes some integer inside the support of ϕ_j, $1 \le j \le r$, and $\mathbf{e}_1 = (1, 0, \dots, 0)^T$, $\mathbf{e}_2 = (0, 1, \dots, 0)^T, \dots, \mathbf{e}_r = (0, \dots, 0, 1)^T$.

THEOREM 3.2. *Let $\mathbf{\Phi}(x)$ be a multiscaling function with dilation a and multiplicity r as in* (2.1) *and satisfy* (3.5) *for some positive integer k_0, $1 \le k_0 \le [M/(a-1)]$. Then*

$$P_{ak+k_0} = \delta_{k,0}P_{k_0}, \quad k \in Z, \tag{3.6}$$

with

$$P_{k_0} = \mathrm{diag}(1, 1/a, \dots, 1/a^{r-1}). \tag{3.7}$$

PROOF. Taking the $(j-1)$th derivative of (2.1) and applying the interpolation condition (3.5), we have $P_{ak+k_0}e_j = (1/a^{j-1})\delta_{k,0}e_j$, $1 \le j \le r$, which implies (3.6) and (3.7).

THEOREM 3.3. *Let $\Phi(x) = (\phi_1, \phi_2, \dots, \phi_r)^T$ be a multiscaling function with dilation a as in* (2.1) *and let $P(z)$ be a two-scale matrix symbol. If* $\operatorname{supp}\phi_i = [h_i, g_i]$, $1 \le i \le r$, *then*

(i) *ϕ_{2j-1} are symmetric and ϕ_{2j} antisymmetric for all j in the following sense*:

$$\phi_i(x) = (-1)^{i-1}\phi_i(h_i + g_i - x), 1 \le i \le r, \tag{3.8}$$

if and only if the entries $P_{i,j}$ of the matrix $P(z)$ satisfy

$$P_{i,j}(z) = (-1)^{i+j} z^{a(h_i+g_i)-(h_j+g_j)} P_{i,j}(\overline{z}), \quad 1 \le i, j \le r. \tag{3.9}$$

(ii) *For any preassigned integer r_1, $1 \le r_1 \le r$, the scaling functions $\phi_1, \phi_2, \dots, \phi_{r_1}$ are symmetric and the remainder ones $\phi_{r_1+1}, \dots, \phi_r$ are antisymmetric in the sense*

$$\begin{aligned} \phi_i(x) &= \phi_i(h_i + g_i - x), \quad i = 1, 2, \dots, r_1, \\ \phi_i(x) &= -\phi_i(h_i + g_i - x), \quad i = r_1, r_1 + 1, \dots, r, \end{aligned}$$

if and only if the entries $P_{i,j}$ of the matrix $P(z)$ satisfy

$$P_{i,j} = \begin{cases} z^{a(h_i+g_i)-(h_j+g_j)} P_{i,j}(\overline{z}), & 1 \le i, j \le r_1 \text{ or } r_1 + 1 \le i, j \le r; \\ -z^{a(h_i+g_i)-(h_j+g_j)} P_{i,j}(\overline{z}), & 1 \le i \le r_1 \text{ and } r_1 + 1 \le j \le r; \\ & \text{or } r_1 + 1 \le i \le r \text{ and } 1 \le j \le r_1. \end{cases}$$

(iii) *If $a(h_i + g_i) - (h_j + g_j) < 0$ or is not an integer for $1 \le i, j \le r$, then $P_{i,j} = 0$.*

PROOF. If $\phi_1, \phi_2, \dots, \phi_r$ satisfy (3.8), then

$$\begin{aligned} \Phi(x) &= (\phi_1(x), \phi_2(x), \dots, \phi_r(x))^T \\ &= S_r(\phi_1(h_1 + g_1 - x), \phi_2(h_2 + g_2 - x), \dots, \phi_r(h_r + g_r - x))^T, \end{aligned}$$

where

$$S_r = \operatorname{diag}(1, -1, \dots, (-1)^{r-1}). \tag{3.10}$$

Hence, by (3.10), we have for $i, j = 0, 1, \dots, a$ and $s = 1, 2, \dots, a - 1$

$$\begin{cases} \hat{\Phi}(\omega) = S_r D_r(z^a)\overline{\hat{\Phi}(\omega)}, \\ D_r(z) = \operatorname{diag}(z^{h_1+g_1}, z^{h_2+g_2}, \dots, z^{h_r+g_r}). \end{cases}$$

Successively using (2.2), we obtain

$$P(z)\hat{\Phi}(\omega/a) = S_r D_r(z^a)\overline{P(z)} D_r(\overline{z}) S_r \hat{\Phi}(\omega/a).$$

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Since $\{\phi_\ell(x-k) : 1 \le \ell \le r, k \in Z\}$ is a Riesz basis of V_0,

$$P(z) = S_r D_r(z^a)\overline{P(z)} D_r(\overline{z}) S_r,$$

or equivalently,

$$S_r P(z) S_r = D_r(z^a)\overline{P(z)} D_r(\overline{z}),$$

which implies (3.9) holds. This completes the proof of Theorem 3.3.

COROLLARY 3.4. *If* $\operatorname{supp}\phi_1 = \operatorname{supp}\phi_2 = \cdots = \operatorname{supp}\phi_r = [0, M/(a-1)]$, *then* ϕ_{2j-1} *are symmetric and* ϕ_{2j} *antisymmetric for all* j *if and only if* $P_k = S_r P_{M-k} S_r$.

In fact, since $\operatorname{supp}\phi_i = [0, M/(a-1)]$, $a(h_i + g_i) - (h_j + g_j) \equiv M$, we obtain $P(z) = z^M S_r P(\overline{z}) S_r$ by (3.9). Hence Corollary 3.4 holds.

As we know, for a multiscaling function $\Phi(x)$, if $\operatorname{supp}\Phi(x) = [0, M]$, then $\operatorname{supp}\Phi'(x) = [0, \lceil M/a \rceil]$, where

$$\Phi'(x) = [\Phi^T(ax), \Phi^T(ax-1), \ldots, \Phi^T(ax-a+1)]^T.$$

Hence, without loss of generality, we only investigate the construction of multiwavelets with $a+1$ coefficients, that is, $\Phi^T(x)$ satisfies the following equation:

$$\Phi(x) = \sum_{k=0}^{a} P_k \Phi(ax-k). \tag{3.11}$$

LEMMA 3.5. *Let* B *be an* $n \times n$ *positive definite matrix. Then there exist* 2^n *symmetric matrices* A *which are distinct such that* $A^2 = B$.

PROOF. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be n eigenvalues of B. Since B is a positive definite matrix, $\lambda_i > 0$, $i = 1, 2, \ldots, n$. There exists an orthogonal matrix U, such that $B = U^* \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) U$. It is clear that

$$A = U^* \operatorname{diag}\left(\pm\sqrt{\lambda_1}, \pm\sqrt{\lambda_2}, \ldots, \pm\sqrt{\lambda_n}\right) U$$

satisfies the matrix equation $A^2 = B$ and that A is symmetric. Obviously, the number of the above A is 2^n.

In the application of multiwavelets, certain special properties are desirable, such as interpolating. In the two-scale matrix sequence $\{P_k\}$, associated with multiwavelets with these properties, there must exist some P_i, $0 \le i \le a$, such that the matrix $(aI - P_i P_i^T)^{-1} P_i P_i^T$ is a positive definite matrix. In fact, by Theorem 3.2, if $\Phi(x)$ is an orthogonal interpolatory multiscaling function, then there exists some positive integer k_0 such that $P_{k_0} = \operatorname{diag}(1, 1/a, \ldots, 1/a^{r-1})$. It is clear that the matrix $(aI - P_{k_0} P_{k_0}^T)^{-1} P_{k_0} P_{k_0}^T$ is a positive definite matrix.

LEMMA 3.6. *Let $\Phi(x)$ be the orthogonal compactly supported multiscaling function with dilation a and multiplicity r satisfying* (3.11). *Assume that there exists some P_i, $0 \le i \le a$, such that the matrix H defined in the following equation is a positive definite matrix:*

$$H^2 = (aI_r - P_i P_i^T)^{-1} P_i P_i^T. \tag{3.12}$$

Let H_s $(s = 1, 2, \dots, a-1)$ be $(a-1)$ distinct symmetric matrices satisfying (3.12). *Define*

$$q_j^{(s)} = \begin{cases} H_s P_j, & j \neq i; \\ -H_s^{-1} P_j, & j = i, \end{cases} \quad j = 0, 1, \dots, a;\ s = 1, 2, \dots, a-1. \tag{3.13}$$

Then

$$P_0 (q_a^{(s)})^T = 0, \tag{3.14}$$

$$P_0 (q_0^{(s)})^T + P_1 (q_1^{(s)})^T + \cdots + P_a (q_a^{(s)})^T = 0, \tag{3.15}$$

$$(q_0^{(\ell)})(q_a^{(s)})^T = 0, \quad \ell, s = 1, 2, \dots, a-1, \tag{3.16}$$

$$(q_0^{(s)})(q_0^{(s)})^T + (q_1^{(s)})(q_1^{(s)})^T + \cdots + (q_a^{(s)})(q_a^{(s)})^T = aI_r. \tag{3.17}$$

PROOF. For convenience, let $i = 1$. Equations (3.14) and (3.16) can be proven using (3.3).

For (3.15) and (3.17), we have from (3.3) that

$$\begin{aligned}
\sum_{\ell=0}^{a} P_\ell (q_\ell^{(s)})^T &= P_0 P_0^T H_s - P_1 P_1^T (H_s^{-1}) + \cdots + P_a P_a^T H_s \\
&= [P_0 P_0^T + P_2 P_2^T + \cdots + P_a P_a^T] H_s - P_1 P_1^T (H_s)^{-1} \\
&= [aI_r - P_1 P_1^T] H_s - P_1 P_1^T (H_s)^{-1} \\
&= [(aI_r - P_1 P_1^T)(H_s)^2 - P_1 P_1^T](H_s)^{-1} = 0,
\end{aligned}$$

$$\begin{aligned}
\sum_{\ell=0}^{a} q_\ell^{(s)} (q_\ell^{(s)})^T &= H_s P_0 P_0^T H_s + (H_s)^{-1} P_1 P_1^T (H_s)^{-1} + \cdots + H_s P_a P_a^T H_s \\
&= H_s [P_0 P_0^T + P_2 P_2^T + \cdots + P_a P_a^T] H_s + (H_s)^{-1} P_1 P_1^T (H_s)^{-1} \\
&= H_s [aI_r - P_1 P_1^T] H_s + (H_s)^{-1} P_1 P_1^T (H_s)^{-1} \\
&= (H_s)^{-1} [(H_s)^2 (aI_r - P_1 P_1^T)(H_s)^2 - P_1 P_1^T](H_s)^{-1} \\
&= (H_s)^{-1} [(H_s)^2 P_1 P_1^T + P_1 P_1^T](H_s)^{-1} \\
&= (H_s)^{-1} [(H_s)^2 + I_r] P_1 P_1^T (H_s)^{-1} \\
&= H_s [P_1 P_1^T + (H_s)^{-2} P_1 P_1^T](H_s)^{-1} \\
&= H_s a I_r (H_s)^{-1} = aI_r.
\end{aligned}$$

This completes the proof of Lemma 3.6.

In the setting of Lemma 3.6, we can generate $a-1$ sequences $\{q_k^{(s)}\}$, $s = 1, 2, \ldots, a-1$. We construct the following functions in terms of these sequences:

$$\psi_s(x) = \sum_{k=0}^{a} q_k^{(s)} \Phi(ax-k), \quad s = 1, 2, \ldots, a-1. \tag{3.18}$$

On the premise of no change in the vector function $\Phi(x)$, applying Gram-Schmidt orthonormalisation to the vectors of a functions $\Phi(x)$, $\psi_s(x)$, $s = 1, 2, \ldots, a-1$, and generating $a-1$ new functions $\Psi_s(x)$, $s = 1, 2, \ldots, a-1$, we can conclude that there must exist $a-1$ sequences $\{Q_k^{(s)}\}$, $s = 1, 2, \ldots, a-1$, such that

$$\Psi_s(x) = \sum_{k=0}^{a} Q_k^{(s)} \Phi(ax-k), \quad s = 1, 2, \ldots, a-1. \tag{3.19}$$

Hence we have the following theorem.

THEOREM 3.7. *In the setting of Lemma* 3.6, *let* $\Psi_s(x)$, $s = 1, 2, \ldots, a-1$ *be defined as in* (3.19). *Define* $\Psi(x) = [\Psi_1(x)^T, \Psi_2(x)^T, \ldots, \Psi_{a-1}(x)^T]^T$. *Then* $\Psi(x)$ *is a compactly supported orthogonal multiwavelet with dilation a associated with* $\Phi(x)$, *and satisfies the following two-scale matrix equation*:

$$\Psi(x) = \sum_{k=0}^{a} \left[(Q_k^{(1)})^T, (Q_k^{(2)})^T, \ldots, (Q_k^{(a-1)})^T\right]^T \Phi(ax-k). \tag{3.20}$$

COROLLARY 3.8. *In the setting of Lemma* 3.6,

(i) *If the dilation factor* $a = 2$, *then* $\psi_1(x)$ *defined in* (3.18) *is a compactly supported orthogonal multiwavelet with dilation* 2 *associated with* $\Phi(x)$.

(ii) *If the dilation factor* $a = 3$, *and* $\Psi_s(x) = \sum_{k=0}^{3} Q_k^{(s)} \Phi(ax-k)$, $s = 1, 2$, *then* $\Psi(x) = [\Psi_1(x)^T, \Psi_2(x)^T]^T$ *is a compactly supported orthogonal multiwavelet with dilation* 3 *associated with* $\Phi(x)$, *and satisfies* (3.20) *in which*

$$\begin{cases} Q_k^{(1)} = q_k^{(1)}, \\ Q_k^{(2)} = \dfrac{1}{2}\left[q_k^{(2)} - \displaystyle\sum_{h=0}^{3} q_h^{(2)} (Q_h^{(1)})^T Q_k^{(1)}\right], \end{cases} \quad k = 0, 1, 2, 3. \tag{3.21}$$

4. Example

We will illustrate by a specific example how to construct orthogonal multiwavelets based on our method.

EXAMPLE 1 (Construction of orthogonal multiwavelets). Let $\Phi(x) = (\phi_1, \phi_2)^T$, $\operatorname{supp}\Phi(x) = [0, 2]$, be 3-coefficient orthogonal multiscaling functions satisfying the following equations [17]:

$$\Phi(x) = P_0\Phi(2x) + P_1\Phi(2x-1) + P_2\Phi(2x-2),$$

where

$$P_0 = \begin{bmatrix} 0 & (2+\sqrt{7})/4 \\ 0 & (2-\sqrt{7})/4 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}, \quad P_2 = \begin{bmatrix} (2-\sqrt{7})/4 & 0 \\ (2+\sqrt{7})/4 & 0 \end{bmatrix}.$$

Suppose $i = 1$. Using (3.12) and (3.13) in Corollary 3.8, we obtain

$$H = \begin{bmatrix} (7+\sqrt{7})/14 & (7-\sqrt{7})/14 \\ (7-\sqrt{7})/14 & (7+\sqrt{7})/14 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} 0 & 3/4 \\ 0 & 1/4 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} -(2+\sqrt{7})/4 & -(2-\sqrt{7})/4 \\ -(2-\sqrt{7})/4 & -(2+\sqrt{7})/4 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1/4 & 0 \\ 3/4 & 0 \end{bmatrix}.$$

From Corollary 3.8, $\Psi(x) = \sum_{k=0}^{2} Q_k\Phi(2x-k)$ are orthogonal multiwavelets associated with $\Phi(x)$.

EXAMPLE 2 (Trivial example – construction of an orthogonal uniwavelet). Let ϕ_3^D be a Daubechies scaling function [5], that is,

$$\phi_3^D(x) = \frac{1+\sqrt{3}}{4}\phi_3^D(2x) + \frac{3+\sqrt{3}}{4}\phi_3^D(2x-1) + \frac{3-\sqrt{3}}{4}\phi_3^D(2x-2) + \frac{1-\sqrt{3}}{4}\phi_3^D(2x-3).$$

Since $\phi_3^D(x)$ is a 4-coefficient orthogonal scaling function, using Lemma 2.2, and letting $\Phi(x) = (\phi_3^D(2x), \phi_3^D(2x-1))^T$, then

$$\Phi(x) = \begin{bmatrix} (1+\sqrt{3})/4 & (3+\sqrt{3})/4 \\ 0 & 0 \end{bmatrix}\Phi(2x) + \begin{bmatrix} (3-\sqrt{3})/4 & (1-\sqrt{3})/4 \\ (1+\sqrt{3})/4 & (3+\sqrt{3})/4 \end{bmatrix}\Phi(2x-1) + \begin{bmatrix} 0 & 0 \\ (3-\sqrt{3})/4 & (1-\sqrt{3})/4 \end{bmatrix}\Phi(2x-2)$$

is an orthogonal multiscaling function with multiplicity 2. Using Corollary 3.8, we obtain

$$\Psi(x) = \begin{bmatrix} \psi_3^D(2x) \\ \psi_3^D(2x-1) \end{bmatrix} = \begin{bmatrix} (\sqrt{3}-1)/4 & (3-\sqrt{3})/4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_3^D(4x) \\ \phi_3^D(4x-1) \end{bmatrix}$$

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$$+\begin{bmatrix} -(3+\sqrt{3})/4 & (1+\sqrt{3})/4 \\ -(\sqrt{3}-1)/4 & -(3-\sqrt{3})/4 \end{bmatrix}\begin{bmatrix} \phi_3^D(4x-2) \\ \phi_3^D(4x-3) \end{bmatrix}$$
$$+\begin{bmatrix} 0 & 0 \\ (3+\sqrt{3})/4 & -(1+\sqrt{3})/4 \end{bmatrix}\begin{bmatrix} \phi_3^D(4x-4) \\ \phi_3^D(4x-5) \end{bmatrix}.$$

Hence

$$\psi_3^D(x) = \frac{\sqrt{3}-1}{4}\phi_3^D(2x) + \frac{3-\sqrt{3}}{4}\phi_3^D(2x-1) - \frac{3+\sqrt{3}}{4}\phi_3^D(2x-2) + \frac{1+\sqrt{3}}{4}\phi_3^D(2x-3).$$

EXAMPLE 3. (Construction of orthogonal multiwavelets with dilation 3 and multiplicity 3). Let $\Phi(x) = (\phi_1, \phi_2, \phi_3)^T$ satisfy $\Phi(x) = P_0\Phi(3x) + P_1\Phi(3x-1) + P_2\Phi(3x-2)$. By Lemma 2.2, $\operatorname{supp}\Phi(x) \subset [0,1]$. Suppose both ϕ_1 and ϕ_3 are symmetric and ϕ_2 is antisymmetric. Then $\Phi(x)$ satisfies the interpolation condition (3.5) with $k_0 = 1$. Then in view of (3.6) and (3.7), we obtain

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/9 \end{bmatrix}.$$

Let $P_0 = [x_{i,j}]_{i,j=1}^3$. Since $P_k = S_3 P_{M-k} S_3$, we have $P_0 = S_3 P_0 S_3$ and $P_3 = P_0$. Applying (3.2), we obtain

$$P_0 = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ -1/2 & \sqrt{7}/6 & 1 \\ 0 & 0 & 11/9 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ 1/2 & \sqrt{7}/6 & -1 \\ 0 & 0 & 11/9 \end{bmatrix}.$$

Taking $i = 1$ and using (3.12), we obtain

$$H^2 = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/26 & 0 \\ 0 & 0 & 1/242 \end{bmatrix}.$$

Hence

$$H_1 = \begin{bmatrix} \sqrt{2}/2 & 0 & 0 \\ 0 & \sqrt{26}/26 & 0 \\ 0 & 0 & \sqrt{2}/22 \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} \sqrt{2}/2 & 0 & 0 \\ 0 & \sqrt{26}/26 & 0 \\ 0 & 0 & -\sqrt{2}/22 \end{bmatrix}.$$

Using (3.13), we have

$$q_0^{(1)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{26}}{52} & \frac{\sqrt{182}}{156} & \frac{\sqrt{26}}{26} \\ 0 & 0 & \frac{\sqrt{2}}{18} \end{bmatrix}, \quad q_1^{(1)} = \begin{bmatrix} -\sqrt{2} & 0 & 0 \\ 0 & -\frac{\sqrt{26}}{3} & 0 \\ 0 & 0 & -\frac{11\sqrt{2}}{9} \end{bmatrix},$$

$$q_2^{(1)} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{\sqrt{26}}{52} & \frac{\sqrt{182}}{156} & -\frac{\sqrt{26}}{26} \\ 0 & 0 & \frac{\sqrt{2}}{18} \end{bmatrix}, \qquad q_0^{(2)} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{\sqrt{26}}{52} & \frac{\sqrt{182}}{156} & \frac{\sqrt{26}}{26} \\ 0 & 0 & -\frac{\sqrt{2}}{18} \end{bmatrix},$$

$$q_1^{(2)} = \begin{bmatrix} -\sqrt{2} & 0 & 0 \\ 0 & -\frac{\sqrt{26}}{3} & 0 \\ 0 & 0 & \frac{11\sqrt{2}}{9} \end{bmatrix}, \qquad q_2^{(2)} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{\sqrt{26}}{52} & \frac{\sqrt{182}}{156} & -\frac{\sqrt{26}}{26} \\ 0 & 0 & -\frac{\sqrt{2}}{18} \end{bmatrix}.$$

Finally, we obtain orthogonal multiwavelets by (3.20) and (3.21).

5. A fast algorithm for computing multiwavelets

In this section, by means of the idea of Daubechies and Lagarias [6], we present a fast numerical algorithm for computing the function value of multiwavelets at an arbitrary point. For generality, we assume that the scale factor of the multiscaling function is an integer a, $a \geq 2$.

Let $\mathbf{\Phi}(x) = (\phi_1, \phi_2, \dots, \phi_r)^T$ be multiscaling functions which satisfy the following equation:

$$\mathbf{\Phi}(x) = \sum_{k=0}^{M} P_k \mathbf{\Phi}(ax - k). \tag{5.1}$$

Let $m = \lceil M/(a-1) \rceil$. Define a vector-valued function $\mathbf{A} \in R^{(m-1)r}$, by

$$\begin{aligned} \mathbf{A} &= [\mathbf{\Phi}^T(1), \mathbf{\Phi}^T(2), \dots, \mathbf{\Phi}^T(m-1)]^T \\ &= [\phi_1(1), \phi_2(1), \dots, \phi_r(1), \phi_1(2), \phi_2(2), \dots, \phi_r(2), \dots, \\ &\qquad \phi_1(m-1), \phi_2(m-1), \dots, \phi_r(m-1)]^T. \end{aligned}$$

Taking (5.1) into account, we have

$$\mathbf{A} = M\mathbf{A}, \tag{5.2}$$

where M is an $(m-1)r \times (m-1)r$ matrix defined by $(M)_{i,j} = P_{ai-j}$, $i, j = 1, 2, \dots, m-1$. Hence, under the normalisation condition $\sum_{\ell=1}^{r} \sum_{k=1}^{m-1} \phi_\ell(k) = 1$, (5.2) has a unique solution.

Define a vector-valued function $\mathbf{H}(x) \in R^{rm}$, $x \in [0, 1]$ by

$$\begin{aligned} \mathbf{H}(x) = [&\phi_1(x), \phi_2(x), \dots, \phi_r(x), \phi_1(x+1), \phi_2(x+1), \dots, \\ &\phi_r(x+1), \dots, \phi_1(x+m-1), \phi_2(x+m-1), \dots, \phi_r(x+m-1)]^T. \end{aligned}$$

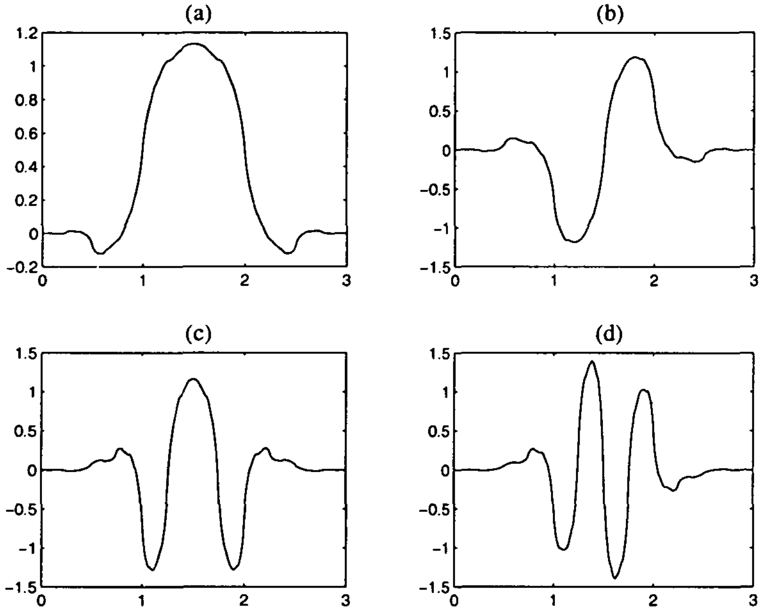

FIGURE 1. The function $\Phi(x)$ and its corresponding orthogonal multiwavelets $\Psi(x)$: (a) $\phi_1(x)$, (b) $\phi_2(x)$, (c) $\psi_1(x)$ and (d) $\psi_2(x)$.

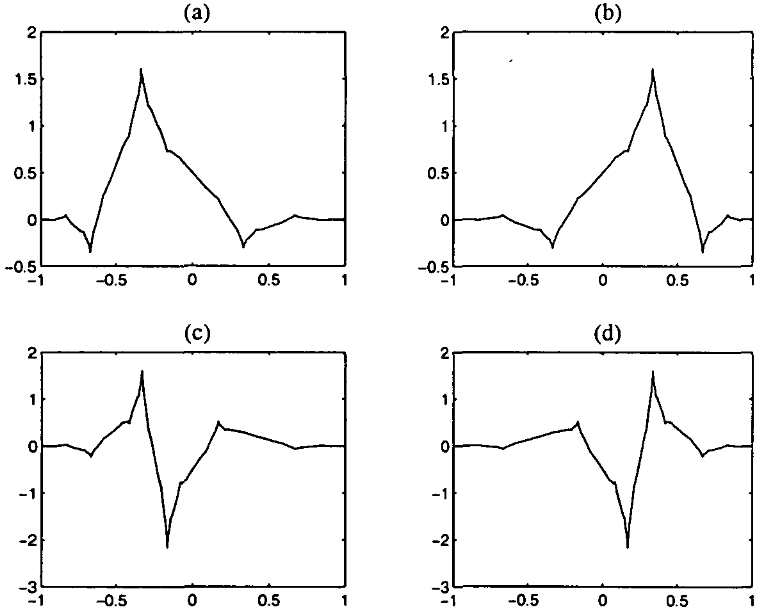

FIGURE 2. The function $\Phi(x)$ and its corresponding orthogonal multiwavelets $\Psi(x)$ from Example 1: (a) $\phi_1(x)$, (b) $\phi_2(x)$, (c) $\psi_1(x)$ and (d) $\psi_2(x)$.

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Obviously, we have

$$\mathbf{H}(0) = [\overbrace{0, 0, \ldots, 0}^{r}, \mathbf{A}^T]^T. \tag{5.3}$$

Define $rm \times rm$ matrices $T_0, T_1, \ldots, T_{a-1}$ by

$$(T_\ell)_{i,j} = P_{ai-j-(a-1)+\ell}, \quad \ell = 0, 1, \ldots, a-1;\ i, j = 1, 2, \ldots, m. \tag{5.4}$$

For every $x \in [0, 1]$, we have, from (5.1),

$$\mathbf{H}((x+\ell)/a) = T_\ell \mathbf{H}(x), \quad \ell = 0, 1, \ldots, a-1. \tag{5.5}$$

For any $x \in [0, 1]$, its a-adic fractional expression is

$$x = \sum_{j=1}^{\infty} d_j a^{-j}, \quad d_j \in \{0, 1, \ldots, a-1\}.$$

Define the shift operator τ on x by $\tau x = \sum_{j=2}^{\infty} d_j a^{-j+1}$. It follows from (5.5) that

$$\mathbf{H}(x) = T_{d_1}\mathbf{H}(\tau x). \tag{5.6}$$

Let $x = 0.d_1 d_2 \ldots d_m$. Repeat using (5.6), then we have $\mathbf{H}(x) = T_{d_1} T_{d_2} \cdots T_{d_m} \mathbf{H}(0)$.

According to the above discussion, we can obtain the following algorithm:

(1) By (5.4), construct $rm \times rm$ matrices $T_0, T_1, \ldots T_{a-1}$.

(2) For every $x \in [0, m]$, there must exist an integer k such that $x \in [k, k+1)$. Let $s = x - k$. Since $s \in [0, 1]$, s can be approximated by $\sum_{j=1}^{m} d_j a^{-j}$.

(3) Compute $\mathbf{H} = T_{d_1} T_{d_2} \cdots T_{d_m} \mathbf{H}(0)$, where $\mathbf{H}(0)$ is defined in (5.3).

(4) The components of the vector $\mathbf{H}$, $h_{rk+1}, h_{rk+2}, \ldots, h_{rk+r}$, are the approximate values of $\phi_1(x), \phi_2(x), \ldots, \phi_r(x)$, respectively.

REMARK. The first r components of the vector $\mathbf{H}$, $h_1, \ldots, h_r$, are the approximate values of $\phi_1(s), \ldots, \phi_r(s)$, respectively; the second r components of the vector $\mathbf{H}$, $h_{r+1}, \ldots, h_{r+r}$, are the approximate values of $\phi_1(1+s), \ldots, \phi_r(1+s)$, respectively; the final r components of the vector $\mathbf{H}$, $h_{(m-2)r+1}, \ldots, h_{(m-2)r+r}$, are the approximate values of $\phi_1(m-1+s), \ldots, \phi_r(m-1+s)$, respectively, that is, the function values of several points are obtained simultaneously using the algorithm once.

Let $\Phi(x) = (\phi_1(x), \phi_2(x))^T$, supp $\Phi(x) = [0, 3]$, be an orthogonal multiscaling function satisfying the following equations [2]:

$$\Phi(x) = P_0\Phi(2x) + P_1\Phi(2x-1) + P_2\Phi(2x-2) + P_3\Phi(2x-3),$$

where

$$P_0 = \begin{bmatrix} \frac{10-3\sqrt{10}}{40} & \frac{5\sqrt{6}-2\sqrt{15}}{40} \\ \frac{5\sqrt{6}-3\sqrt{15}}{40} & \frac{5-3\sqrt{10}}{40} \end{bmatrix}, \quad P_1 = \begin{bmatrix} \frac{30+3\sqrt{10}}{40} & \frac{5\sqrt{6}-2\sqrt{15}}{40} \\ -\frac{5\sqrt{6}+7\sqrt{15}}{40} & \frac{15-3\sqrt{10}}{40} \end{bmatrix},$$

$$P_2 = \begin{bmatrix} \frac{30+3\sqrt{10}}{40} & -\frac{5\sqrt{6}-2\sqrt{15}}{40} \\ \frac{5\sqrt{6}+7\sqrt{15}}{40} & \frac{15-3\sqrt{10}}{40} \end{bmatrix}, \quad P_3 = \begin{bmatrix} \frac{10-3\sqrt{10}}{40} & -\frac{5\sqrt{6}-2\sqrt{15}}{40} \\ -\frac{5\sqrt{6}-3\sqrt{15}}{40} & \frac{5+3\sqrt{10}}{40} \end{bmatrix}.$$

The corresponding orthogonal multiwavelets satisfy the following equations:

$$\Psi(x) = Q_0\Phi(2x) + P_1\Phi(2x-1) + P_2\Phi(2x-2) + P_3\Phi(2x-3),$$

where

$$Q_0 = \begin{bmatrix} \frac{5\sqrt{6}-2\sqrt{15}}{40} & -\frac{10-3\sqrt{10}}{40} \\ -\frac{5-3\sqrt{10}}{40} & \frac{5\sqrt{6}-3\sqrt{15}}{40} \end{bmatrix}, \quad Q_1 = \begin{bmatrix} -\frac{5\sqrt{6}-2\sqrt{15}}{40} & \frac{30+3\sqrt{10}}{40} \\ \frac{15-3\sqrt{10}}{40} & \frac{5\sqrt{6}+7\sqrt{15}}{40} \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} -\frac{5\sqrt{6}-2\sqrt{15}}{40} & -\frac{30+3\sqrt{10}}{40} \\ -\frac{15-3\sqrt{10}}{40} & \frac{5\sqrt{6}+7\sqrt{15}}{40} \end{bmatrix}, \quad Q_3 = \begin{bmatrix} \frac{5\sqrt{6}-2\sqrt{15}}{40} & \frac{10-3\sqrt{10}}{40} \\ \frac{5-3\sqrt{10}}{40} & \frac{5\sqrt{6}-3\sqrt{15}}{40} \end{bmatrix}.$$

Using our algorithm, we draw the graphs of the above scaling functions and wavelets, as seen in Figure 1 and the graphs of the scaling functions and wavelets of Example 1 as seen in Figure 2, respectively.

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