FREE ELASTICAE AND WILLMORE TORI IN WARPED PRODUCT SPACES

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0. Abstract. We use the principle of symmetric criticality to connect the Willmore variational problem for surfaces in a warped product space with base a circle, and the free elastica variational problem for curves on its fiber. In addition we obtain a rational one-parameter family of closed helices in the anti De Sitter 3-space which are critical points of the total squared curvature functional. This means they are free elasticae. Also they are spacelike; this allows us to construct a corresponding family of spacelike Willmore tori in a certain kind of spacetime close to the Robertson-Walker spaces.

1. Introduction. The Willmore functional is defined, on the class of Riemannian (or Lorentzian) surfaces with or without boundary of a semi-Riemannian manifold $(N, \bar{g})$, by

$$\mathcal{W}(S) = \int_S (< H, H > + R) dA + \int_{\partial S} \kappa_g dt,$$

where ∂S denotes the boundary of the surface S, H its mean curvature vector field, R is the sectional curvature of M on the tangent plane of S and κ_g denotes the curvature function of ∂S in M.

It is known (see **[8]**) that this functional is an invariant under conformal changes of the ambient metric $\bar{g}$. The critical points of $\mathcal{W}$ are called Willmore surfaces. For example, the closed (compact and without boundary) surfaces with H vanishing identically, in particular minimal and maximal surfaces, are obvious examples of Willmore surfaces. Articles showing different methods of getting examples of non minimal Willmore surfaces in spheres are known in the literature (see for example **[1, 3, 7, 9, 15]** etc.).

In **[2]**, the author gave the first non trivial known examples of Lorentzian Willmore tori in the standard anti De Sitter 3-space. Other examples of Lorentzian Willmore tori in semi-Riemannian space forms of low dimension were obtained in **[5]**.

On the other hand, we consider the *total squared curvature* functional acting on closed curves (or curves satisfying given first order boundary data) in a semi-Riemannian manifold (M, g). The extremal points of this functional are called *free elastic* curves in (M, g) (see **[11]** for details of elastic, non necessarily free, curves in real-space-forms).

It is not difficult to understand that both variational problems are strongly related through the *principle of symmetric criticality* **[14]**. U. Pinkall used this idea in **[15]**, it was also exploited in different contexts (see for instance **[1, 2, 7, 5, 12]** etc.).

In this paper, once more we exploit this argument to get examples of Riemannian and Lorentzian Willmore tori in spaces shaped as warped products with base a standard circle. These examples come from free elastic curves on their fibers, (see Theorem 1). The family of these spaces includes the Robertson-Walker spaces with closed base.

Moreover we are interested in obtaining non trivial examples of spacelike Willmore tori in spacetimes. To do this, we deal in most of the paper with the standard anti De Sitter

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3-space, $H_1^3(-1)$. Therefore, we use the Hopf fibration of $H_1^3(-1)$ on the hyperbolic 2-plane, $H^2(-4)$, to characterize the helices of $H_1^3(-1)$ as geodesics of the *Hopf tubes* (that is complete horizontal lifts of curves) on curves of constant curvature in $H^2(-4)$. Then we determine the isometry type of the *Hopf tori* (complete horizontal lifts of closed curves) (see Theorem 2) and so use it to characterize the closed helices of $H_1^3(-1)$.

In Section 6, we deal with the free elastica variational problem for closed helices in $H_1^3(-1)$. We obtain all the solutions of the Euler-Lagrange equations that are closed helices. Therefore we get a rational one-parameter family of solutions (see Theorem 3). Lastly we use these solutions to construct spacelike (that is Riemannian) Willmore tori in the spacetime $S^1 \times_f H_1^3(-1)$, where f denotes any positive function on the unit circle S^1.

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2. Willmore tori and free elasticae. Let (M, g) be a n dimensional semi-Riemannian manifold, with metric g. We denote by ∇ the semi-Riemannian connection. Let $f : S^1 \longrightarrow R$ be a positive function on the unit circle. On $N = S^1 \times M$ we consider the semi-Riemannian metric $\bar{g}_\epsilon = \epsilon dt^2 + f^2 g$, with obvious meaning and $\epsilon = \pm 1$. Then $(N, \bar{g}_\epsilon)$ is called the warped product with warping function f, base or leaf $(S^1, \epsilon dt^2)$ and fiber (M, g). When the metrics on the base and the fiber are understood, then we still use $\epsilon S^1 \times_f M$ to denote $(N, \bar{g}_\epsilon)$, (see **[13]** for details about this subject). We notice that the index of $(N, \bar{g}_\epsilon)$ coincides with the index of (M, g) if $\epsilon = 1$ while it increases this index by one if $\epsilon = -1$.

Let γ be a non null curve (arclength parametrized) in (M, g) with length $L > 0$, Frenet frame $\{T = \gamma', \xi_2, \ldots, \xi_n\}$ and curvatures $\{\kappa_1, \ldots, \kappa_{n-1}\}$. We consider the tube $\mathcal{T}_\gamma = S^1 \times \gamma$ which is immersed in $(N, \bar{g}_\epsilon)$. It can be parametrized by $\Phi(s, t) = (e^{it}, \gamma(s))$ and the volume form of the induced metric is

$$dA = \epsilon f(t)\, dt\, ds.$$

THEOREM 1. *Let $\gamma : [0, L] \longrightarrow M$ be a non-null curve. The tube $\mathcal{T}_\gamma = S^1 \times \gamma$ is a Willmore surface in $(N, \bar{g}_\epsilon)$ if and only if γ is a free elastica in (M, g). In particular $\mathcal{T}_\gamma = S^1 \times \gamma$ is a Willmore torus in $(N, \bar{g}_\epsilon)$ if and only if γ is a closed free elastica in (M, g).*

Proof. To compute the Willmore functional of $\mathcal{T}_\gamma = S^1 \times \gamma$ in $(N, \bar{g}_\epsilon)$, we first calculate its second fundamental form, say σ. We use parametrization Φ and well known relationship for warped products, **[13]**, to obtain

$$\sigma(\Phi_t, \Phi_t) = \sigma(\Phi_t, \Phi_s) = 0,$$
$$\sigma(\Phi_s, \Phi_s) = \kappa_1 \xi_2.$$

Now the squared mean curvature function of $\mathcal{T}_\gamma = S^1 \times \gamma$ in $(N, \bar{g}_\epsilon)$ is

$$\bar{g}_\epsilon(H, H) = \frac{\delta_2 \kappa_1^2}{4f^2},$$

where δ_2 denotes the causal character of ξ_2.

The second term in the Willmore integral is the sectional curvature of $(N, \bar{g}_\epsilon)$ relative to the tangent plane of $\mathcal{T}_\gamma = S^1 \times \gamma$. It is not difficult to see that $R = -\frac{f''}{f}$. Therefore

$$\mathcal{W}(\mathcal{T}_\gamma) = \int_\gamma \int_0^{2\pi} \left(\frac{\delta_2 \kappa_1^2(s)}{4f^2(t)} - \frac{f''(t)}{f(t)} \right) f(t)\, dt\, ds + \int_{\partial \mathcal{T}_\gamma} \kappa_g\, dt.$$

It is clear that κ_g vanishes identically; in fact if $\partial\mathcal{T}_\gamma$ is not empty (this happens if γ is not closed), then it is made up of a couple of leaves and the leaves are geodesics in $(N, \bar{g}_\epsilon)$. We also use the lemma of Hopf to get

$$\mathcal{W}(\mathcal{T}_\gamma) = \frac{\delta_2}{4} \left(\int_0^{2\pi} \frac{1}{f(t)} dt \right) \int_\gamma \kappa_1^2 ds.$$

It is obvious that S^1 acts on $\mathcal{T}_\gamma = S^1 \times \gamma$ through homotheties. If we consider the manifold of immersions of a torus in $(N, \bar{g}_\epsilon)$, the symmetric points under the above mentioned S^1 action on this manifold are immersions of the type $\mathcal{T}_\gamma = S^1 \times \gamma$ for a given non-null closed curve in (M, g). Hence one can apply the principle of symmetric criticality [**14**] to get the statement of the Theorem. □

Remark 1. It should be noticed that $\mathcal{T}_\gamma = S^1 \times \gamma$ never has constant mean curvature unless both κ_1 and f are constants. Moreover given γ in (M, g), one can choose f in such a way that $\mathcal{W}(\mathcal{T}_\gamma) < 2\pi^2$ and this fact contrasts with the well known Willmore conjecture.

The complete classification of closed free elasticae in the standard 2-sphere was achieved by J. Langer and D. A. Singer [**11**]. It can be briefly described as follows. Up to rigid motions in the unit sphere, the family of closed free elasticae is made up of the m-fold cover γ_o^m of a geodesic γ_o, say the equator, and an integer two-parameter family $\{\gamma_{m,n} \mid (m, n) \in Z^2; 0 < m < n\}$, where $\gamma_{m,n}$ indicates that it closes up after n periods and m trips around the equator γ_o (see [**11**] for details).

Corollary 1. *For any positive function f on the unit circle, there exist infinitely many Willmore tori in the 3-dimensional warped product $\epsilon S^1 \times_f S^2$. This family includes $\{\mathcal{T}_{\gamma_o^m} \mid m \in Z - \{0\}\}$ and $\{\mathcal{T}_{\gamma_{m,n}} \mid (m, n) \in Z^2; 0 < m < n\}$. These tori are Riemannian or Lorentzian according as $\epsilon = 1$ or $\epsilon = -1$ respectively.*

If $M = H^2$ is the hyperbolic 2-plane endowed with its canonical metric of constant Gaussian curvature, say -1, then the complete classification of free elasticae is also achieved by [**11**]. Besides the m fold cover η_o^m of the so called *hyperbolic equator*, η_o (that is a geodesic circle of radius $\sinh^{-1}(1)$ in H^2), there exist an integer 2-parameter family of free elasticae, $\{\eta_{m,n} \mid m > 1$ *and* $\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}\}$, (see once more [**11**] for a geometrical description of this family).

Corollary 2. *For any positive function f on the unit circle, there exist infinitely many Willmore tori in the 3-dimensional warped product $\epsilon S^1 \times_f H^2$. This family includes $\{\mathcal{T}_{\gamma_o^m} \mid m \in Z - \{0\}\}$ and $\{\mathcal{T}_{\eta_{m,n}} \mid m > 1$ and $\frac{1}{2} < \frac{m}{n} < \frac{\sqrt{2}}{2}\}$. These tori are Riemannian or Lorentzian according as $\epsilon = 1$ or $\epsilon = -1$ respectively.*

3. The hyperbolic Hopf fibration. The 4-dimensional pseudo-Euclidean space with index 2, $\mathbb{R}^4_2$, can be identified with $C^2 = \{z = (z_1, z_2) \mid z_1, z_2 \in C\}$ endowed with the usual inner product $< z, w >= \mathrm{Re}(z_1\bar{w}_1 - z_2\bar{w}_2)$. The 3-dimensional anti De Sitter space is the hyperquadric $H^3_1(-1) = \{z \in R^4_2 \mid < z, z >= -1\}$, and the induced metric defines a Lorentzian structure with constant sectional curvature -1, on $H^3_1(-1)$. The circle of radius one, S^1, regarded as the set of unit complex numbers, acts naturally (multiplication coordinate to coordinate) on $H^3_1(-1)$. The space of orbits, under this action, can be identified with the hyperbolic 2-plane of Gaussian curvature -4, say $H^2(-4)$. The natural projection $\Pi : H^3_1(-1) \longrightarrow H^2(-4)$ gives a semi-Riemannian submersion.

A global unit timelike vector field V, can be defined on $H^3_1(-1)$, by putting $V_z = iz$ for all $z \in H^3_1(-1)$, (of course $i = \sqrt{-1}$). The V flow is made up of fibers, which are unit circles with negative defined metric. We will use the standard notation and terminology of **[13]**, relative to semi-Riemannian submersions. In particular one has the splitting $T_z = \mathcal{V}_z \oplus \mathcal{H}_z$, $z \in H^3_1(-1)$, where T_z is the tangent 3-space to $H^3_1(-1)$ in z, $\mathcal{V}_z = \mathrm{Span}(\mathcal{V}_z)$ is the vertical line and $\mathcal{H}_z$ is the horizontal subspace ($i\mathcal{H}_z = \mathcal{H}_z$). Recall that $\mathcal{V}_z = \mathrm{Ker}(d\Pi_z)$ and $d\Pi_z$ restricted to $\mathcal{H}_z$ gives an isometry between $\mathcal{H}_z$ and the tangent plane to $H^2(-4)$ at $\Pi(z)$. Overbars will denote the horizontal lifts of corresponding objects on $H^2(-4)$. The semi-Riemannian connections $\bar{\nabla}$ and ∇ of $H^3_1(-1)$ and $H^2(-4)$ respectively satisfy

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \overline{\nabla_X Y} + (g_o(JX, Y) \circ \Pi)V \tag{1}$$
$$\bar{\nabla}_{\bar{X}}V = \bar{\nabla}_V\bar{X} = i\bar{X} \tag{2}$$
$$\bar{\nabla}_V V = 0, \tag{3}$$

where J and g_o denote the standard complex structure and metric of $H^2(-4)$ respectively. Notice that the third equation gives the geodesic character of the fibers in $H^3_1(-1)$.

The mapping $\Pi : H^3_1(-1) \longrightarrow H^2(-4)$ is also a principal fibre bundle on $H^2(-4)$ with structure group S^1 (a *circle bundle*). We define a connection on this bundle by assigning to each $z \in H^3_1(-1)$ the horizontal 2-plane $\mathcal{H}_z$. The Lie algebra $u(1)$ of $S^1 = \mathcal{U}(1)$ is identified with R, so V is the *fundamental* vector field 1^* corresponding to $1 \in u(1)$.

We denote by ω and Ω the connection 1-form and the curvature 2-form of this connection respectively. It is well known that there is a unique R valued 2-form Θ on $H^2(-4)$ such that $\Omega = \Pi^*(\Theta)$. We also put dA to denote the canonical volume form on $H^2(-4)$, in particular $dA(X, JX) = 1$ for any unit vector field X on $H^2(-4)$. It is clear that $\Theta(X, JX) = \Omega(\bar{X}, i\bar{X})$ and so we can use the structure equation, the horizontality of $\bar{X}$, and $i\bar{X}$ and the formula (1) to obtain

$$\Omega(\bar{X}, i\bar{X}) = d\omega(\bar{X}, i\bar{X}) = -\omega([\bar{X}, i\bar{X}]) = -2\omega(V) = -2,$$

and consequently

$$\Theta = -2dA. \tag{4}$$

4. Hopf tubes and Hopf tori. Let $\alpha : [0, L] \longrightarrow H^2(-4)$ be an immersed curve with length $L > 0$. We always assume that α is parametrized by arclength. The complete lift $C_\alpha = \Pi^{-1}(\alpha)$ will be called the *Hopf tube* associated with α and it can be parametrized as follows. We start

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from a horizontal lift $\bar{\alpha} : [0, L] \longrightarrow H_1^3(-1)$ of α and then we get all the horizontal lifts of α by acting S^1 over $\bar{\alpha}$. Therefore we have $\Psi : [0, L] \times R \longrightarrow H_1^3(-1)$ with

$$\Psi(s, t) = e^{it}\bar{\alpha}(s).$$

It is not difficult to see that C_α is a Lorentzian flat surface which is isometric to $[0, L] \times S^1$ (where the second factor is endowed with its negative defined standard metric). In particular, if α is closed, then C_α is a Lorentzian flat torus (the *Hopf torus* associated with α). It will be embedded in $H_1^3(-1)$ if α is so in $H^2(-4)$, and its isometry type depends not only on L but also on the area $A > 0$ in $H^2(-4)$ enclosed by α.

THEOREM 2. *Let α be a closed immersed curve in $H^2(-4)$ of length L and enclosing an area A. The corresponding Hopf torus C_α is isometric to L^2/Γ, where Γ is the lattice in the Lorentzian plane $L^2 = R_1^2$, generated by $(0, 2\pi)$ and $(L, 2A)$.*

Proof. Let $\bar{\alpha}$ be any horizontal lift of α and $\Psi : R^2 \longrightarrow C_\alpha \subset H_1^3(-1)$ the semi-Riemannian covering defined by $\Psi(s, t) = e^{it}\bar{\alpha}$. The lines parallel to the t axis in L^2 are mapped by Ψ onto the fibres of Π, while the lines parallel to the s axis in L^2 are mapped by Ψ onto the horizontal lifts of α. These curves are not closed because the holonomy of the involved connection, which was defined above. However the non-closedness of the horizontal lifts of closed curves is measured just for the curvature as follows, (we will apply, without major details, a well known argument which is nicely exposited in [**10** (vol. II, p. 293)]: there exists $\delta \in [-\pi, \pi)$ such that $\bar{\alpha}(L) = e^{i\delta}\alpha(0)$ (for any horizontal lift). The whole group of deck transformations of Ψ is so generated by the translations $(0, 2\pi)$ and (L, δ). Finally we have $\delta = -\int_c \Theta$, where c is any 2-chain on $H^2(-4)$ with boundary $\partial c = \alpha$. In particular, from (4), we get $\delta = 2A$ and it proves the statement. □

5. Helices. From now on, we assume that α is an arclength parametrized curve with constant curvature κ in $H^2(-4)$. Then $C_\alpha = \Pi^{-1}(\alpha)$ is a Lorentzian flat surface with constant mean curvature. Moreover it admits an obvious parametrization $\Psi(s, t)$ by means of fibers (s constant) and horizontal lifts of α (t constant). Let β be a non-null geodesic of C_α; it is determined from its *slope* g, (which is measured with respect to Ψ). It is not difficult to see that β is a helix in $H_1^3(-1)$, with curvature and torsion given respectively by

$$\rho = \frac{\epsilon(\kappa + 2g)}{1 - g^2}, \tag{5}$$

$$\nu = \frac{-\epsilon(1 + g\kappa + g^2)}{1 - g^2}, \tag{6}$$

where $\epsilon = \pm 1$ represents the causal character of β.

We also have a converse of this fact, namely, given any helix β in $H_1^3(-1)$ with curvature ρ and torsion ν, then it can be regarded as a geodesic in a certain Hopf tube of $H_1^3(-1)$. Indeed, just consider the Hopf tube $C_\alpha = \Pi^{-1}(\alpha)$, where α is a curve in $H^2(-4)$ with constant curvature $\kappa = \frac{\epsilon(\rho^2 - \nu^2 + 1)}{\rho}$, where ϵ denotes the causal character of β; then choose a geodesic in C_α with slope $g = -\frac{\epsilon + \nu}{\rho}$.

We suppose that α is closed, that is, it is a geodesic circle of a certain radius $r > 0$ in $H^2(-4)$. Then its curvature is $\kappa = -2\coth 2r$ (notice we choose orientation to get negative values for curvature). The length of α is $L = \pi \sinh 2r$ and the enclosed area in $H^2(-4)$ is $A = \frac{\pi}{2}(\cosh 2r - 1)$. As we already know the Hopf torus $C_\alpha = \Pi^{-1}(\alpha)$ comes from a lattice in L^2 which is generated by $(0, 2\pi)$ and $(L, 2A)$. Now a geodesic $\beta(s)$ of $C_\alpha = \Pi^{-1}(\alpha)$ is closed if and only if there exists $s_o > 0$ such that $\Psi^{-1}(\beta(s_o)) \in \Gamma$. Consequently

$$g = \frac{2\pi}{L}\left(q + \frac{A}{\pi}\right), \tag{7}$$

where q is a rational number.

The slope of closed helices can be also written in terms of κ as follows:

$$g = q\sqrt{\kappa^2 - 4} - \frac{1}{2}\kappa, \tag{8}$$

where $q \in Q - \{0\}$.

6. Closed helices being free elasticae. We recall that a free elastica γ of a semi-Riemannian manifold M is a critical point of the functional,

$$\mathcal{F}(\gamma) = \int_\gamma \kappa^2(s)\, ds,$$

where κ denotes the curvature function of γ. Of course this functional acts on a manifold consisting only of regular closed curves or curves which satisfy given first order boundary data, (see **[11]**). The term free was introduced in **[11]** to describe the critical points with no constraint on the arclength of the curves. The Euler-Lagrange equations for free elasticae in a semi-Riemannian manifold with constant curvature were obtained in **[4]**. These equations reduce the study of free elasticae in semi-Riemannian real-space-forms to two or three dimensional spaces. The curvature κ and the torsion τ of an extremal point of the functional $\mathcal{F}$ must satisfy the following Euler-Lagrange equations:

$$\epsilon_2\kappa'' + \epsilon_1\kappa^3 - 2\epsilon_3\kappa\tau^2 + 2\epsilon_1\epsilon_2 c\kappa = 0 \tag{9}$$

$$\kappa^2\tau = \text{constant}, \tag{10}$$

where ϵ_1, ϵ_2 and ϵ_3 denote the causal characters of the curve, its normal and its binormal respectively, and c is the curvature of the ambient space.

Now if β is a helix in $H_1^3(-1)$ with curvature $\rho > 0$ and torsion $\nu \neq 0$, then β is a free elastica if and only if

$$\rho^2 + 2\nu^2 - 2 = 0, \tag{11}$$

that is in the (ρ, ν) plane of helices in $H_1^3(-1)$, the total squared curvature has exactly one ellipse of critical points. To determine the closed helices in $H_1^3(-1)$ which are in the above ellipse, we use the discussion we made in the last section; in particular the Euler-Lagrange equations can be written in terms of κ and the slope g, as follows

$$4\kappa g^3 + (2\kappa^2 + 12)g^2 + 8\kappa g + \kappa^2 = 0. \tag{12}$$

The following theorem shows the existence of a rational one-parameter family of closed helices which are free elasticae in $H_1^3(-1)$.

THEOREM 3. *For any non-zero rational number q, there exists a closed helix β_q in $H_1^3(-1)$ which is a free elastica in $H_1^3(-1)$. Moreover all these helices are spacelike.*

Proof. First we notice that the left hand of (12) factors into $(\kappa + 2g)(2\kappa g^2 + 6g + \kappa)$. Since $\kappa^2 > 4$ and $q \neq 0$, the equation (8) gives that $\kappa + 2g \neq 0$ and so the Euler-Lagrange equations become $2\kappa g^2 + 6g + \kappa = 0$. This formula is combined with (8) to show that a closed helix β is a free elastica in $H_1^3(-1)$ if and only if regarded as a geodesic of *rational slope* q in a Hopf torus on a geodesic circle α in $H^2(-4)$ with curvature κ, both parameters give a zero of the following function,

$$F(\kappa, q) = (4q^2 + 1)\kappa\sqrt{\kappa^2 - 4} - 4q(\kappa^2 - 3). \tag{13}$$

It is not difficult to see that for any non-zero rational number q, there exists a real number $\kappa \in (-\infty, -2)$ such that $F(\kappa, q) = 0$. We choose a geodesic circle in $H^2(-4)$ with curvature κ and a geodesic in $C_\alpha = \Pi^{-1}(\alpha)$ whose slope is $g = \frac{2\pi}{L}(q + \frac{A}{\pi})$, where L and A are respectively the length of α and the enclosed area by α in $H^2(-4)$. Certainly β is a closed helix in $H_1^3(-1)$ and its curvature and torsion satisfy the Euler-Lagrange equations for free elasticae. We also see that these helices are spacelike because $1 - g^2 > 0$. $\square$

REMARK 2. It is easy to deduce from $F(\kappa, q) = 0$ that the relationship between q and κ is given by

$$q = \frac{(\kappa^2 - 3) \pm \sqrt{9 - 2\kappa^2}}{2\kappa\sqrt{\kappa^2 - 4}}, \tag{14}$$

where $4 < \kappa^2 \leq \frac{9}{2}$.

From this equation, one can see that every real number $q \neq 0$ occurs for exactly one κ, while each κ determines exactly two values of q, except when $\kappa^2 = \frac{9}{2}$ (which corresponds to $q = \frac{1}{2}$ or $q = -\frac{1}{2}$). The product of these two values of q is always $\frac{1}{4}$, therefore when one of the two is rational the other one must also be rational. Thus the corresponding Hopf tori possess transverse foliations by closed free elastic helices.

COROLLARY 3. *For any positive function f on the unit circle and any non-zero rational number q there exists a Willmore tori $T_{\beta q}$ in the 4-dimensional warped product $\epsilon S^1 \times_f H_1^3(-1)$. It is Riemannian or Lorentzian according to $\epsilon = 1$ or $\epsilon = -1$ respectively.*

REMARK 3. We can use the usual Hopf fibration of the 3-dimensional unit sphere $S^3(1)$ on the 2-sphere to get a rational one-parameter family of closed helices being free elasticae in

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$S^3(1)$, [**6**, **15**]. Therefore we can get Willmore tori in the warped product $\epsilon S^1 \times_f S^3(1)$, for any positive function f on the unit circle.

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