CONDITIONALLY CONVERGENT SPECTRAL EXPANSIONS

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We will consider a reflexive Banach space $\mathfrak{B}$, with real or complex scalars, and a bounded operator in $\mathfrak{B}$ with a real spectrum.

A self-adjoint (i.e. Hermitian) operator T in a finite-dimensional vector space $\mathfrak{B}$ has a complete set of eigenvectors; writing $E(\tau)$ for the orthogonal projection onto the subspace spanned by eigenvectors of eigenvalues in τ, T can be expressed as

(1) $$T = \int \lambda E(d\lambda).$$

For each set of real numbers τ,

$$\left.\begin{array}{l} \text{a projection } E(\tau) \text{ exists.} \\ \text{We have} \\ (2) \qquad \|E(\tau)\| < K \\ \text{and for any vector } x, \\ E(\tau)x = \lim E(\tau_n)x, \end{array}\right\}$$

if τ_n is a sequence of sets expanding to τ. If the spectrum of T is $\sigma(T)$ the spectrum of T in $E(\tau)\mathfrak{B}$ is

(3) $$\sigma(T; E(\tau)\mathfrak{B}) = \sigma(T) \cap \tau.$$

These, and related facts, are well known, or are obvious consequences of well-known results. They have been generalised to self-adjoint operators in Hilbert space (**6**), in which setting they constitute the "Spectral Theorem". In this case some proofs (see e.g. (**11**)) use the fact that, for all real polynomials p,

(4) $$\|p(T)\| \leqq \sup_{\lambda \in \sigma(T)} |p(\lambda)|,$$

which is easily proved. The inequalities (4) and

(5) $$\|p(T)\| \leqq K \sup |p(\lambda)|$$

have been investigated, for any operator in a Banach space ((**7**), (**3**)). It

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appears that, if we require (2) to hold for Borel sets τ_n, τ, then (1) and (2) are equivalent to (5).

In the spaces L^p ($1 < p < \infty$; $p \neq 2$) the most important operators — those integral and differential operators, which, in L^2, would be self-adjoint — tend to have eigenfunction expansions which converge (**12**, §§ 7.3, 12.42), (**2**), (**9**), (**10**)), but only conditionally (**12**, § 9.5). This corresponds to the statement that $E(\tau)$ should exist, and (2) hold, when τ and τ_n are *intervals* on the real line. Taking (2) in this sense, *the object of the present paper is to investigate the equivalence of* (1) *and* (2) *to the inequality*

$$\|p(T)\| \leqq K\mathbf{|}p\mathbf{|}, \tag{6}$$

where

$$\mathbf{|}p\mathbf{|} = \sup_{\lambda \in J} |p(\lambda)| + \operatorname*{var}_{J} p(\lambda). \tag{7}$$

((6) should hold for some closed real interval J, some $K < \infty$, and all real polynomials p. If this is so, J contains $\sigma(T)$.) Actually, starting from (6), I fail[1] to prove (1) but obtain the weaker result (3), together with the existence of

$$S = \int \lambda E(d\lambda). \tag{8}$$

I prove that $S - T$ is generalised nilpotent, and zero in some special cases; I can probably[1] prove that $(S - T)^2 = 0$ and that

$$\|(S - T)E([c, d])\| \leqq K(d - c) \qquad (-\infty < c < d < \infty) \tag{9}$$

but the question whether $S = T$ in general remains open.

Of course, the constants K in (2), (6) and (9) may differ.

The argument from (1) and (2) to (6) is fairly trivial (see § 5) so that the following theorem should be regarded as the main result. (For notation, see § 1).

THEOREM A. *If T is well-bounded then for any real number μ there is a unique bounded projection P_μ such that*

(i) $P_\mu \cap\cap T$;

(ii) $P_\mu(\mathfrak{B})$ *is the space of eigenvectors of* μ.

In the space $\mathfrak{C} = (I - P_\mu)\mathfrak{B}$ there is a unique bounded projection F_μ such that

(iii) $F_\mu \cap\cap (T; \mathfrak{C})$;

(iv) $\sigma(T; F_\mu\mathfrak{C}) \subseteqq (-\infty, \mu] \cap \sigma(T)$

(v) $\sigma(T; (I - F_\mu)\mathfrak{C}) \subseteqq [\mu, \infty) \cap \sigma(T)$.

[1] Dr. Ringrose disposes of these difficulties in the following paper.

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Writing G_μ for the projection $F_\mu(I - P_\mu)$ and E_μ for the projection $G_\mu + P_\mu$ we have

(vi) $||P_\mu|| \leqq 3K$, $||G_\mu|| \leqq 2K$, $||E_\mu|| \leqq 2K$, *where K is the constant of* (6).

(vii) $E_\nu G_\mu = E_\nu E_\mu = E_\nu \quad (\nu < \mu)$;

(viii) $\lim_{\nu \to \mu - 0} E_\nu x = G_\mu x \quad (x \in \mathfrak{B})$;

(ix) $\lim_{\nu \to \mu + 0} E_\nu(x) = E_\mu x \quad (x \in \mathfrak{B})$;

(x) $E_\lambda = 0 (\lambda < a)$; $E_\lambda = I(\lambda \geqq b)$, *where $J = [a, b]$ is the interval mentioned in* (7).

The Spectral Theorem is deduced from Theorem A in § 6. Unfortunately, this case (where T is self-adjoint) is the only one in which I can verify (6) directly.

1. Notation

The word "operator" means "linear operator", wherever it appears.

My only non-standard notation: T is *well-bounded* if (6) is satisfied (for some real interval J, some number $K < \infty$, and all real polynomials p).

For most of our terminology and notation and for facts which we take for granted the reader can consult any text on functional analysis; for example (**10**).

The following remarks may help the reader: ϕ denotes the empty set, $[a, b]$ a closed interval; $T \cap S$ means that T and S commute (in an obvious sense, since all our operators are bounded), $T \cap\cap S$ means that T commutes with every bounded operator which commutes with S; if A and B are subsets of a Banach space I write $A + B$ for the set of vectors $a + b$ $(a \in A, b \in B)$; for any operator E, $E\mathfrak{B}$ denotes the range of E (thus if E is a projection, $(I - E)\mathfrak{B}$ is the nullspace of E); the adjoint T^* of T can be defined by the equation

$$(Tx, y) = (x, T^*y) \qquad (x \in \mathfrak{B}, y \in \mathfrak{B}^*)$$

(note that using the alternative definition would not affect our arguments); $\int f(\lambda)E(d\lambda)$ means the same as $\int f(\lambda)dE_\lambda$; for a sequence of operators T_n and a limit operator T, we say that $T_n \to T$ strongly if $T_n x \to Tx$ for all $x \in \mathfrak{B}$; $\sigma(T)$, the spectrum of T, is the set of scalars λ for which $T - \lambda I$ fails to have an inverse (in the algebra of bounded linear operators on $\mathfrak{B}$ to $\mathfrak{B}$); if $p(\lambda) = a_0 + a_1\lambda + \cdots + a_n\lambda^n$ is a polynomial we write $p(T) = a_0 I + a_1 T + \cdots + a_n T^n$.

2. Operational Calculus

The following result is our basic tool.

Lemma 2.1. *Let T be well-bounded. Then the correspondence*

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$$p(\lambda) \to p(T)$$

can be extended (in a unique way) from the set of polynomials to the set of all absolutely continuous real functions, with (6) *remaining true. For the extended correspondence we have*

(i) $p(\lambda)q(\lambda) \to p(T)q(T)$

(ii) $cp(\lambda) \to cp(T)$

(iii) $p(\lambda) + q(\lambda) \to p(T) + q(T)$

(iv) $p(T^*) = (p(T))^*$

(v) $p(T) \cap\cap T$.

Proof. If p is absolutely continuous, choose (by approximating to p', in L_1', by a polynomial), polynomials p_n such that $\| p_n - p \| \to 0$. Then

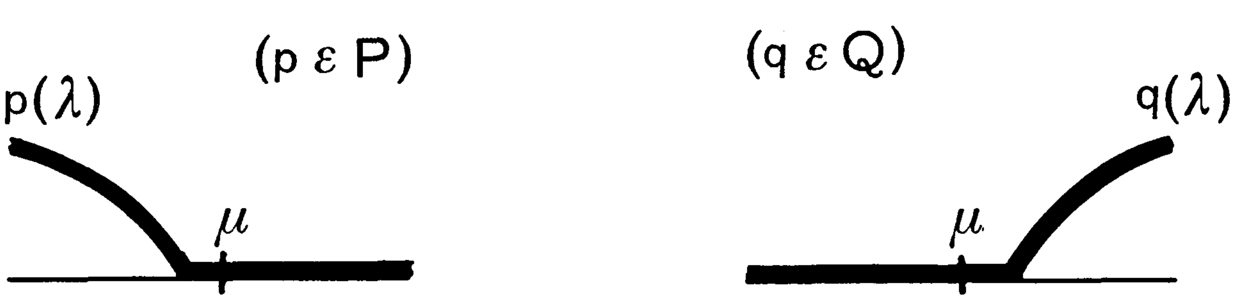
$$||p_n(T) - p_m(T)|| \leqq K \| p_n - p_m \| \to 0 \text{ as } m, n \to \infty,$$

so that $p_n(T)$ converges in operator norm to an operator (independent of the choice of p_n) which will be called $p(T)$. Clearly (6) is true. Since (i) to (v) are true for polynomials p they must also, for reasons of continuity, be true for absolutely continuous functions.

We can now clarify the role of the interval J, by showing that *J contains the spectrum of T*. In fact, if $\nu \notin J$, the function $(\lambda - \nu)^{-1}$ is absolutely continuous over J; this function thus corresponds to some operator which, by (i), must be the inverse of $T - \nu I$.

Let μ be any real number. Write P (or Q) for the class of real functions, each of which is absolutely continuous and is zero throughout some neighbourhood of $[\mu, \infty)$ (or of $(-\infty, \mu]$). We will consider the subspace $\mathfrak{B}_\mu$ (or $\mathfrak{B}'_\mu$) (not in general closed) composed of elements $p(T)x$ $(x \in \mathfrak{B},\ p \in P)$ (or $q(T)x$ $(x \in \mathfrak{B}, q \in Q)$).

Diagram 1



Lemma 2.2. *$\mathfrak{B}_\mu$ is a subspace.*

Proof. If $p, r \in P$ we can find $s \in P$ such that

$$s(\lambda)p(\lambda) \equiv p(\lambda), \qquad s(\lambda)r(\lambda) \equiv r(\lambda).$$

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Thus

$$p(T)x + r(T)y = s(T)(p(T)x + r(T)y) \in \mathfrak{B}_\mu.$$

Also $k(p(T)x) = (kp(T))x \in \mathfrak{B}_\mu$, for any real number k.

LEMMA 2.3. *$\mathfrak{B}'_\mu$ is a subspace.*

PROOF. Similar to Lemma 2.2.

LEMMA 2.4. *$\mathfrak{B}_\mu$ and $\mathfrak{B}'_\mu$ are disjoint.*

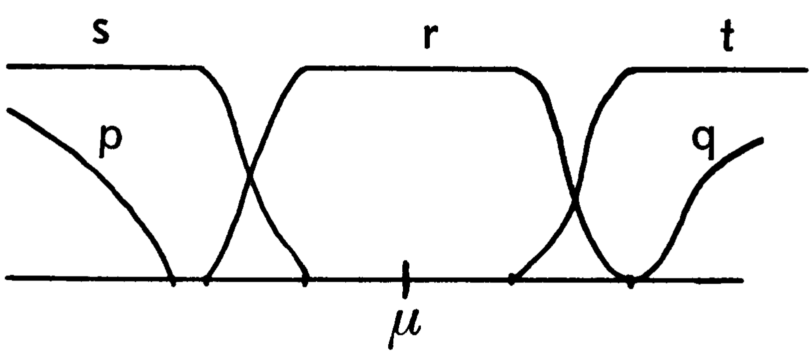
PROOF. Let $p \in P$, $q \in Q$ and suppose that

$$z = p(T)x = q(T)y.$$

We can choose absolutely continuous functions r, s, t such that $s \in P$, $t \in Q$,

$$p(\lambda)r(\lambda) \equiv q(\lambda)r(\lambda) \equiv 0,$$
$$s(\lambda) + r(\lambda) + t(\lambda) \equiv 1.$$

Diagram 2



Clearly

$$s(\lambda)q(\lambda) \equiv t(\lambda)p(\lambda) \equiv 0.$$

Thus

$$\begin{aligned} z &= s(T)z + r(T)z + t(T)z \\ &= s(T)q(T)y + r(T)q(T)y + t(T)p(T)x \\ &= 0. \end{aligned}$$

LEMMA 2.5. *If x is an eigenvector of μ, if $p \in P$ and $q \in Q$, then $p(T)x = q(T)x = 0$.*

PROOF. If $Tx = \mu x$, then the formula

$$r(T)x = r(\mu)x$$

is true for all polynomials r and hence, by Lemma 2.1, for all absolutely continuous functions. Thus

$$p(T)x = p(\mu)x = 0 \qquad (p \in P)$$
$$q(T)x = q(\mu)x = 0 \qquad (q \in Q).$$

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LEMMA 2.6. *Suppose that* $x = u + v + w$ *where* $u \in \mathfrak{B}_\mu$, *v is an eigenvector of* μ, *and* $w \in \mathfrak{B}'_\mu$. *Then*

(i) $||u|| \leqq 2K||x||$

(ii) $||u + v|| \leqq 2K||x||$

(iii) $||w|| \leqq 2K||x||$

(iv) $||v|| \leqq 3K||x||$

PROOF. (i) For an absolutely continuous function p equal to 1 from $-\infty$ almost to μ, then decreasing to 0 and remaining 0 in $[\mu, \infty)$, we have

$$p(T)u = u, \quad p(T)v = p(T)w = 0, \quad \sup |p(\lambda)| = \operatorname{var} p(\lambda) = 1,$$

so that

$$||u|| = ||p(T)x|| \leqq K\|p\| \cdot ||x|| = 2K||x||.$$

(ii) Similar; p should equal 1 in $(-\infty, \mu]$ and decrease to 0 just to the right of μ.

(iii) Similar; p should be zero in $(-\infty, \mu]$ and increase to 1 just to the right of μ.

(iv) Similar; p should equal 1 at μ and decrease to 0 on either side of μ.

I must thank Dr. Ringrose for drawing my attention to the need for the following lemma, and for giving a proof of it. (In the complex case it can be avoided by using $(\lambda - \mu + i)^{-1}$ in place of $((\lambda - \mu)^2 + 1)^{-1}$ in the proof of Theorem A.)

LEMMA 2.7. *If* $(T - \mu I)^2 x = 0$ *then* $(T - \mu I)x = 0$.

PROOF. If $(T - \mu I)^2 x = 0$ then for any $k > 0$,

$$(I + k(T - \mu I)^2)x = x$$

so that

$$(I + k(T - \mu I)^2)^{-1}x = x.$$

Thus

$$\begin{aligned} ||(T - \mu I)x|| &= ||(T - \mu I)(I + k(T - \mu I)^2)^{-1}x|| \\ &\leqq K||x|| \cdot \|(\lambda - \mu)(1 + k(\lambda - \mu)^2)^{-1}\| \\ &\leqq K||x|| \cdot \tfrac{5}{2}k^{-\frac{1}{2}}. \end{aligned}$$

As k can be taken arbitrarily large, $(T - \mu I)x = 0$.

3. We will prove the following special case of Theorem *A*.

THEOREM B. *If T is a well-bounded linear operator in a Banach space* $\mathfrak{B}$, *and* μ *is real and not an eigenvalue of* T^*, *then there is a unique bounded projection* F_μ *such that*

(i) $F_\mu \cap\cap T$;
(ii) $\sigma(T; F_\mu\mathfrak{B}) \subseteqq (-\infty, \mu] \cap \sigma(T)$;
(iii) $\sigma(T; (I - F_\mu)\mathfrak{B}) \subseteqq [\mu, \infty) \cap \sigma(T)$.

REMARK. In (ii) or (iii) the difference of the two sides is at most the single point μ.

REMARK. The ergodic theorem (used as in Lemma 4.1) shows that μ will be an eigenvalue of T if and only if it is an eigenvalue of T^*.

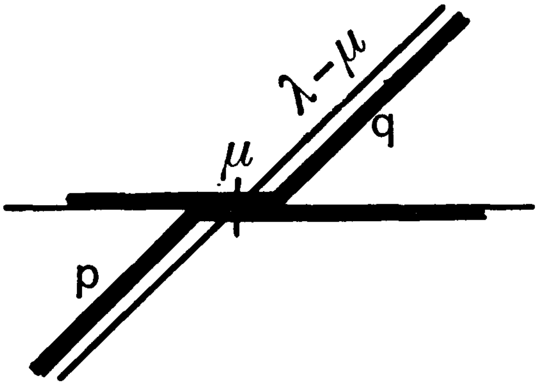
LEMMA 3.1. *Under the conditions of Theorem B, $\mathfrak{B}_\mu + \mathfrak{B}'_\mu$ is dense in $\mathfrak{B}$.*

PROOF. Suppose $y \perp \mathfrak{B}_\mu + \mathfrak{B}'_\mu$. Then for $p \in P$, $q \in Q$, $x \in \mathfrak{B}$,

$$(p(T^*)y, x) = (y, p(T)x) = 0$$
$$(q(T^*)y, x) = (y, q(T)x) = 0.$$

Thus $[p(T^*) + q(T^*)]y = 0$. Now choose $p \in P$, $q \in Q$ so that $\| p(\lambda) + q(\lambda) - (\lambda - \mu) \| < \varepsilon$.

Diagram 3



We obtain

$$||(T^* - \mu I)y|| < \varepsilon K||y||,$$

so that $T^*y = \mu y$. Thus $y = 0$, since μ is not an eigenvalue of T^*.

DEFINITION OF F_μ. If $x \in \mathfrak{B}_\mu + \mathfrak{B}'_\mu$ we can express x as $x = y + z$ with $y = p(T)u \in \mathfrak{B}_\mu$, $z = q(T)w \in \mathfrak{B}'_\mu$. By Lemma 2.4, y and z are uniquely determined, although $p \in P$ and $q \in Q$ are not unique. Define

$$F_\mu x = y.$$

Thus (if $s \in P$ is chosen so that $s(\lambda)p(\lambda) \equiv p(\lambda)$ and $\| s \| = 2$),

$$\begin{aligned} ||F_\mu x|| = ||p(T)u|| &= ||s(T)p(T)u + s(T)q(T)w|| \\ &= ||s(T)x|| \\ &\leqq 2K||x||. \end{aligned}$$

Similarly,

$$||(I - F_\mu)x|| = ||z|| \leqq 2K||x||.$$

Thus F_μ, defined as a bounded linear operator on a dense subspace of $\mathfrak{B}$, can be uniquely extended to the whole of $\mathfrak{B}$ by continuity. Clearly, *the range of* F_μ *is the closure of* $\mathfrak{B}_\mu$ *and the nullspace of* F_μ *is the closure of* $\mathfrak{B}'_\mu$.

We can now prove that F_μ has properties (i) to (iii) but its uniqueness will only be proved at the end of § 4.

PROOF OF (i). Let S be any bounded linear operator commuting with T. Then for any polynomials p, q (and hence for absolutely continuous functions p, q) we have

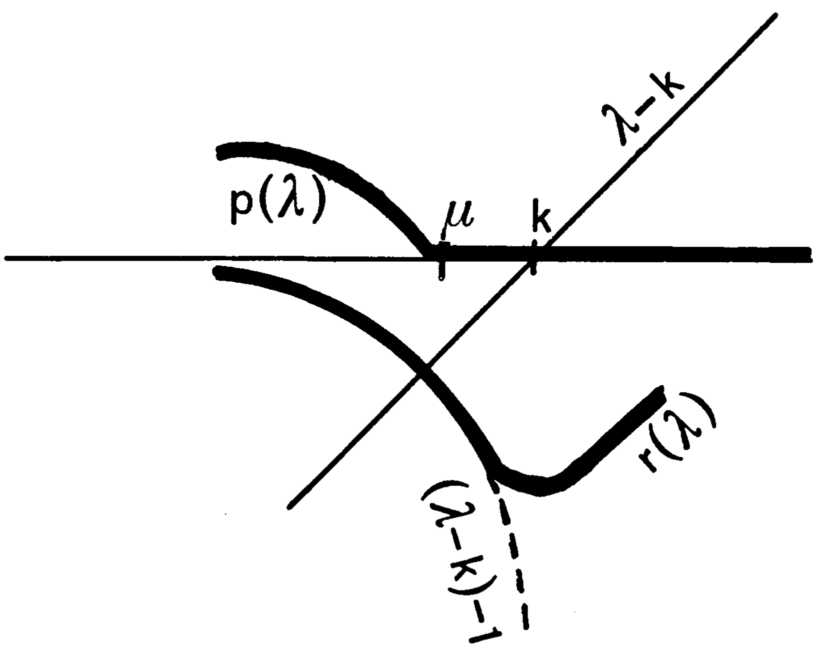
$$Sp(T)z \equiv p(T)Sz, \quad Sq(T)z \equiv q(T)Sz.$$

Thus $SF_\mu = F_\mu S$ on the dense subspace $\mathfrak{B}_\mu + \mathfrak{B}'_\mu$ and so, by continuity, S commutes with F_μ.

PROOF OF (ii). If $\kappa > \mu$, we can choose an absolutely continuous function $r(\lambda)$ such that

$$r(\lambda)(\lambda - \kappa)p(\lambda) \equiv p(\lambda) \qquad (p \in P).$$

Diagram 4



Thus for $x \in \mathfrak{B}_\mu$, $x = p(T)y$,

$$\begin{aligned} x = p(T)y &= r(T)(T - \kappa I)p(T)y \\ &= r(T)(T - \kappa I)x \\ &= (T - \kappa I)r(T)x. \end{aligned}$$

Thus $r(T)$ is the inverse of $T - \kappa I$ in $\mathfrak{B}_\mu$, and hence (both operators being bounded) in the closure of $\mathfrak{B}_\mu$, which is $F_\mu\mathfrak{B}$. Thus $\sigma(T; F_\mu\mathfrak{B})$ lies in $(-\infty, \mu]$ and it obviously lies in $\sigma(T)$.

PROOF OF (iii). Similar.

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4. Proof of Theorem A

CONSTRUCTION OF P_μ. Let $p(\lambda) = ((\lambda - \mu)^2 + 1)^{-1}$ so that

$$p(T) = ((T - \mu I)^2 + I)^{-1}.$$

By Lemma 2.7, the subspace $\mathfrak{B}_e$ of eigenvectors of μ for T is the subspace of eigenvectors of 1 for $p(T)$. Also

$$\|(p(\lambda))^n\| \leqq 3,$$

so that

$$\|(p(T))^n\| \leqq 3K \qquad (n \geqq 1).$$

By the ergodic theorem (4) the operator P_μ given by

$$P_\mu x = \lim q_n(T)x \qquad (x \in \mathfrak{B})$$

(where

$$q_n(\lambda) = \frac{1}{n}(1 + p(\lambda) + \cdots + p((\lambda))^{n-1}) \tag{4.1}$$

is a bounded projection onto $\mathfrak{B}_e$. Clearly P_μ commutes with all bounded operators which commute with T. *This proves* (i) *and* (ii).

LEMMA 4.1. *The restriction T_0 of T to $(I - P_\mu)\mathfrak{B}$ is well-bounded and has the additional property that μ is not an eigenvalue of T_0 or of the operator T_0^* in $((I - P_\mu)\mathfrak{B})^*$.*

PROOF. By the argument above,

$$P_{\mu 0} = \lim q_n(T_0)$$

projects onto the space of eigenvectors of T_0, i.e. onto the zero subspace of

$$\mathfrak{C} = (I - P_\mu)\mathfrak{B}.$$

Thus $P_{\mu 0} = 0$, so that

$$0 = (P_{\mu 0})^* = \lim q_n(T_0)^* = \lim q_n(T_0^*),$$

and the range of this projection, which is the eigenspace of μ for T_0^*, must be the zero subspace. This proves the lemma.

Thus in $\mathfrak{C}$ we can use Theorem B. *This proves* (iii) *to* (v). We now wish to show that it is indifferent whether we regard $\mathfrak{B}_\mu$ and $\mathfrak{B}'_\mu$ as subspaces of $\mathfrak{B}$ or as subspaces of $\mathfrak{C}$.

LEMMA 4.2.

$$\{p(T)x : x \in \mathfrak{B},\ p \in P\} = \{p(T)x : x \in \mathfrak{C},\ p \in P\} = \mathfrak{B}_\mu$$
$$\{q(T)x : x \in \mathfrak{B},\ q \in Q\} = \{q(T)x : x \in \mathfrak{C},\ q \in Q\} = \mathfrak{B}'_\mu$$

PROOF. This follows directly from Lemma 2.5.

Proof of (vi). We know now (Lemmas 3.1 and 4.1) that: *a dense set* $\mathfrak{B}_d$ *of vectors of* $\mathfrak{B}$ *can be written in the form*

$$x = u + v + w \quad (u \in \mathfrak{B}_\mu,\ v \in P_\mu \mathfrak{B},\ w \in \mathfrak{B}'_\mu). \tag{4.2}$$

For such an x, $||E_\mu x|| = ||u + v|| \leqq 2K||x||$, by Lemma 2.6. Thus $||E_\mu|| \leqq 2K$ and similar results hold for the other projections.

Proof of (vii). To show that $E_\nu G_\mu = E_\nu E_\mu$, I will show that $E_\nu P_\mu = 0$. Since the projections E_ν and P_μ commute, their product is a projection, which obviously commutes with T. To show that this projection is zero, it is enough to show that $\sigma = \sigma(T;\ E_\nu P_\mu \mathfrak{B})$ is the empty set. In fact, σ is a subset both of $\sigma(T;\ E_\nu \mathfrak{B})$ and of $\sigma(T;\ P_\mu \mathfrak{B})$. Thus σ is a subset of $(-\infty, \nu]\theta \cap \theta\{\mu\}$, which is the empty set.

To establish the equality of the projections $E_\nu G_\mu$ and E_ν, which commute with each other, it will be sufficient to show that they have the same range. Obviously, $E_\nu G_\mu \mathfrak{B} \subseteqq E_\nu \mathfrak{B}$ so it will be enough to show that $E_\nu G_\mu \mathfrak{B} \supseteqq E_\nu \mathfrak{B}$; and for this it is sufficient to show that

$$G_\mu \mathfrak{B} \supseteqq E_\nu \mathfrak{B}.$$

In fact, $P_\nu \mathfrak{B} + \mathfrak{B}_\nu$ is dense in $E_\nu \mathfrak{B}$ and $\mathfrak{B}_\nu \subseteqq \mathfrak{B}_\mu$ so it will be enough to show that $P_\nu \mathfrak{B} \subseteqq \mathfrak{B}_\mu$. Let $x \in P_\nu \mathfrak{B}$. Then $x = \lim q_n(T)x$ where $q_n(\lambda)$ is defined by (4.1) (with ν in place of μ). Choose an absolutely continuous function $r(\lambda)$ which equals 1 on some neighbourhood of ν and vanishes on some neighbourhood of $[\mu, \infty)$. Then $\| r(\lambda)q_n(\lambda) - q_n(\lambda) \| \to 0$ so that

$$x = \lim r(T)q_n(T)x = r(T) \lim q_n(T)x = r(T)x \in \mathfrak{B}_\mu.$$

Proof of (viii). For $x \in \mathfrak{B}_d$, we can write x in the form (4.2). By the definition of $\mathfrak{B}_\mu$, $u \in \mathfrak{B}_\nu$ for all ν sufficiently close to μ. Thus $E_\nu x = u = G_\mu x$. Since $||E_\nu|| < 2K$, $||G_\mu|| < 2K$ and $E_\nu x \to G_\mu x$ for x in the dense subset $\mathfrak{B}_d$, we have $E_\nu x \to G_\mu x$ for all $x \in \mathfrak{B}$.

Proof of (ix). Similar to (viii).

Proof of (x). Since $\alpha \notin J$, $\alpha \notin \sigma(T)$; thus $P_\alpha = 0$ so we have $\mathfrak{B} = \mathfrak{C}$, $E_\alpha = F_\alpha$. By (iv),

$$\sigma(T;\ E_\alpha \mathfrak{B}) = \phi \qquad (\alpha < a),$$

so that $E_\alpha \mathfrak{B} = \{0\}$, $E_\alpha = 0$ $(\alpha < a)$. Similarly, $I - E_\beta = 0$ if $\beta > b$. Thus the required results follow from (viii) and (ix).

Uniqueness of P_μ. Let P be a bounded projection onto the eigenspace of μ such that P commutes with T. Then P commutes with P_μ so that for all $x \in \mathfrak{B}$, $Px = P_\mu Px = PP_\mu x = P_\mu x$.

Uniqueness of F_μ. Let a bounded projection Π have the properties (iii), (iv) and (v) of F_μ. By Lemma 4.1 we need only consider the special

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case of Theorem B. Then $\mathfrak{B} = \mathfrak{C}$ so that Π and F_μ are operators in $\mathfrak{B}$, commuting with T and with each other. Thus $(I - \Pi)F_\mu$ is a projection and

$$\sigma(T; I-\Pi)F_\mu\mathfrak{B}) \subseteqq \sigma(T; (I-\Pi)\mathfrak{B}) \cap \sigma(T; F_\mu\mathfrak{B})$$
$$\subseteqq (-\infty, \mu] \cap [\mu, \infty) = \{\mu\}.$$

The Corollary to Theorem E below (which could be proved at this stage) shows that, in $(I-\Pi)F_\mu\mathfrak{B}$, T equals μI. Since μ has no eigenvectors this means that $(I-\Pi)F_\mu\mathfrak{B} = \{0\}$. Because F_μ and Π are projections, this implies

$$F_\mu\mathfrak{B} \subseteqq \Pi\mathfrak{B}.$$

Similarly we see that $F_\mu\mathfrak{B} \supseteqq \Pi\mathfrak{B}$. Thus $F_\mu\mathfrak{B} = \Pi\mathfrak{B}$ and similarly, $(I - F_\mu)\mathfrak{B} = (I-\Pi)\mathfrak{B}$. Thus $F_\mu = \Pi$.

This completes the proof of Theorems A and B.

5. The Scalar Operator $\mathbf{S} = \int \lambda d\mathbf{E}_\lambda$

I will write $E(\lambda)$ for E_λ and use the notation $\Delta E(\lambda_i)$ for $E(\lambda_{i+1}) - E(\lambda_i)$.

THEOREM C. *Let* $\{E(\lambda)\}_{-\infty<\lambda<\infty}$ *be a family of projections such that for all real* λ, μ, ν,

(vi)′ $\|E(\mu)\| \leqq K$

(vii)′ $E(\mu)E(\nu) = E(\min \mu, \nu)$

(ix) $\lim_{\nu\to\mu+0} E(\nu)x = E(\mu)x \qquad (x \in \mathfrak{B})$

(x) $E(\lambda) = 0 \ (\lambda < a); \qquad E(\lambda) = I \ (\lambda \geqq b)$.

Let p *be any continuously differentiable function. Choose a net* N *consisting of points* $(\lambda_i)_{1\leqq i\leqq n}$ *such that*

$$a - \theta = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b + \theta$$

(*where* θ *is some number* > 0). *Write* $\delta(N) = \max(|\lambda_0 - \lambda_1|, \cdots, |\lambda_{n-1} - \lambda_n|)$, *and* $S_N = \sum p(\lambda_i)\Delta E(\lambda_i)$.

Then (1) *as* $\delta(N) \to 0$, S_N *will converge strongly to an operator which will be written*

$$p(S) = \int p(\lambda)\, dE_\lambda.$$

In particular we write

$$S = \int \lambda dE_\lambda.$$

(2) *For this correspondence* $p(\lambda) \to p(S)$ *we have*

$$1 \to I$$
$$\lambda \to S$$
$$\alpha p(\lambda) + \beta q(\lambda) \to \alpha p(S) + \beta q(S)$$

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$$p(\lambda)q(\lambda) \to p(S)q(S) \tag{5.1}$$

$$\|p(S)\| \leqq |p(b)| + K \operatorname*{var}_{[a,b]} p(\lambda). \tag{5.2}$$

(3) *E_λ is the projection obtained by applying Theorem A to the well-bounded operator S.*

LEMMA. *Let $f(\lambda)$ be a function of a real variable λ taking values in a metric space. Let $f(\lambda)$ be continuous on the right at each point. Then $f(\lambda)$ has at most a countable set of discontinuities.*

PROOF. Define $d(\lambda)$, the discontinuity at λ, to be the upper limit, as x and y approach λ, of $\rho(f(x), f(y))$. Let S_n be the set of points where $d(\lambda) > 1/n$. To the right of any point of S_n there is an interval containing no point of S_n. Choose a rational number in this interval. This maps S_n one-one onto a subset of the rationals, showing that S_n is countable. Thus the set that concerns us, being $\cup_1^\infty S_n$, is countable.

PROOF OF THEOREM C (1). Consider some $x \in \mathfrak{B}$. By (ix) and the lemma, $E_\lambda x$ has a countable set of discontinuities. As $p'(\lambda)$ is continuous,

$$\int_{a-\theta}^{b+\theta} E(\lambda)\, x p'(\lambda)\, d\lambda \tag{5.3}$$

exists as a Riemann integral for any $\theta > 0$ (see (**13**), Theorem 1). Thus

$$\int_{a-\theta}^{b+\theta} p(\lambda)\, dE(\lambda)\, x \tag{5.4}$$

exists (in the sense stated in the theorem) and is equal to

$$[E(\lambda)p(\lambda)x]_{a-\theta}^{b+\theta} - \int_{a-\theta}^{b+\theta} p'(\lambda)\, E(\lambda)\, x\, d\lambda. \tag{5.5}$$

PROOF OF (2). By (x), (5.4) is independent of θ. (5.5) gives the inequality

$$\left\|\int p(\lambda)\, dE(\lambda)x\right\| \leqq |p(b)| \cdot \|E(b)\, x\| + |p(a)| \cdot \|E(a-0)\, x\| + \operatorname{lub} \|E(\lambda)\, x\| \int_a^b |p'(\lambda)|\, d\lambda$$

which, by (x) and (vi)′, gives (5.2).

For any net N,

$$[\sum p(\lambda_i)\,\Delta E(\lambda_i)][\sum q(\lambda_j)\,\Delta E(\lambda_j)] = \sum p(\lambda_i) q(\lambda_i)\,\Delta E(\lambda_i),$$

by (vii)′. Letting $\delta(N) \to 0$ we obtain (5.1).

PROOF OF (3). Since $E_\lambda \cap \sum p(\lambda_i)\Delta E(\lambda_i)$, we must have

$$E_\lambda \cap p(S).$$

Fix $x \in E_\mu \mathfrak{B}$ and $\theta > 0$. We have

$$x = E_\mu x = E_\lambda E_\mu x = E_\lambda x \qquad (\lambda \geqq \mu).$$

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Thus

$$\sum p(\lambda_i)\Delta E(\lambda_i)x = \sum_{\lambda_i<\mu} p(\lambda_i)\Delta E(\lambda_i)x,$$

since the remaining terms of the left-hand side are all zero. Thus

$$p(S)x = \int_{a-\theta}^{b+\theta} p(\lambda)\,dE_\lambda x = \int_{a-\theta}^{\mu+\theta} p(\lambda)\,dE_\lambda x. \tag{5.6}$$

We can now discuss the inverse of $(T-\nu I)$, regarded as an operator in $E_\mu\mathfrak{B}$. If $\nu>\mu$ or $\nu<a$ we choose $\theta>0$ so that $\nu>\mu+\theta$ or $\nu<a-\theta$. Then $(\lambda-\nu)^{-1}=r(\lambda)$ is a continuously differentiable function on $[a-\theta,\mu+\theta]=J'$, so that $r(S)$ can be defined by (5.6) as an operator in $E_\mu\mathfrak{B}$. The equation $r(\lambda)(\lambda-\nu)=1$ $(\lambda\in J')$ shows that

$$r(S)(S-\nu I)=(S-\nu I)r(S)=I,$$

by the argument of (2). Thus, *in* $E_\mu\mathfrak{B}$, *the spectrum of* S *is included in* $\sigma(S)\cap[a,\mu]$. Similarly, *in* $(I-E_\mu)\mathfrak{B}$ *the spectrum of* S *is included in* $\sigma(S)\cap[\mu,b]$.

As $\mathfrak{B}$ is reflexive, (vi)′, (vii)′ and Lorch's theorem (**5**) show that $E(\mu-0)$ exists (as a strong limit). For $x\in(E(\mu)-E(\mu-0))\mathfrak{B}$, the sum

$$\sum \lambda_i\Delta E(\lambda_i)x,$$

taken over a net N, reduces to the term with $\lambda_i<\mu\leqq\lambda_{i+1}$, which is

$$\lambda_i\Delta E(\lambda_i)x=\lambda_i(E(\mu)-E(\mu-0))x=\lambda_i x.$$

Upon allowing $\delta(N)\to 0$, we obtain $Sx=\mu x$. Thus $(E(\mu)-E(\mu-0))\mathfrak{B}$ consists of eigenvectors of μ. Conversely, if $Sx=\mu x$,

$$(SE(\mu-\theta)x)=E(\mu-\theta)(Sx)=\mu(E(\mu-\theta)x),$$

so that consideration of the spectrum of S in $E(\mu-\theta)\mathfrak{B}$ shows that

$$E(\mu-\theta)x=0 \qquad (\theta>0).$$

Similarly $E(\mu+\theta)x=x$ $(\theta>0)$.

Thus $x=E(\mu+0)x-E(\mu-0)x$

$$=E(\mu)x-E(\mu-0)x\in(E(\mu)-E(\mu-0))\mathfrak{B}.$$

Thus $(E(\mu)-E(\mu-0))$ is a projection, commuting with S, onto the eigenspace of μ. The uniqueness statements in Theorem A now show that E_λ is the projection which Theorem A describes (for S in place of T).

THEOREM D. *Let* T *be a well-bounded operator,* $\{E(\lambda)\}$ *the family of projections derived from* T *by Theorem* A, *and* S *the scalar operator derived from* $\{E(\lambda)\}$ *by Theorem* C. *Then*

(i) $S\cap\cap T$ *and*

(ii) $S-T$ *is a generalised nilpotent operator.*

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PROOF.

(i) $E(\lambda) \cap\cap T$. Thus $\sum \lambda_i \Delta E(\lambda_i) \cap\cap T$. Thus $S \cap\cap T$.

(ii) We have to show that $\sigma(S - T) = \{0\}$. We will show for each $\varepsilon > 0$ that $\sigma(S - T)$ lies inside the ε-neighbourhood of 0, i.e. that the spectral radius of $S - T$ is less than ε. Fix $\varepsilon > 0$. Choose a net N such that $\delta(N) < \varepsilon/2$. Then in $\Delta E(\lambda_i)\mathfrak{B}$, S and T each has its spectrum in $[\lambda_i, \lambda_{i+1}]$ so that $S - \lambda_i I$ and $T - \lambda_i I$ have spectra in $[0, \varepsilon/2]$. As these last two operators commute, the spectral radius of $S - T = (S - \lambda_i I) - (T - \lambda_i I)$ is at most ε ((**8**), § 149). Thus the spectrum of $S - T$ in $\mathfrak{B}$, being the union of the spectra of $S - T$ in the subspaces $\Delta E(\lambda_i)\mathfrak{B}$, lies in the ε-neighbourhood of 0.

It seems likely [1] that $S = T$, in the situation described in Theorem D. If $S - T$ is well-bounded (which is not obvious) the following theorem shows that S equals T. This equality can also be proved in some other special circumstances, for example if the space is finite-dimensional (by means of the corollary below) or if T has a complete set of eigenfunctions (for then $Sx = Tx$ for a dense set of x).

THEOREM E. *If T is well-bounded and generalised nilpotent, then $T = 0$.*

PROOF.

We can construct projections P_μ, G_μ, and F_μ, and subspaces $\mathfrak{B}_\mu$ and $\mathfrak{B}'_\mu$ $(-\infty < \mu < \infty)$, as in the proof of Theorem A. If $\mu < 0$, the spectrum of T in $\overline{\mathfrak{B}}_\mu = F_\mu(I - P_\mu)\mathfrak{B}$ is empty by Theorem A (iv). Thus $\mathfrak{B}_\mu = \{0\}$, so that

$$\mathfrak{B}_0 = \bigcup_{\mu<0} \mathfrak{B}_\mu = \{0\}.$$

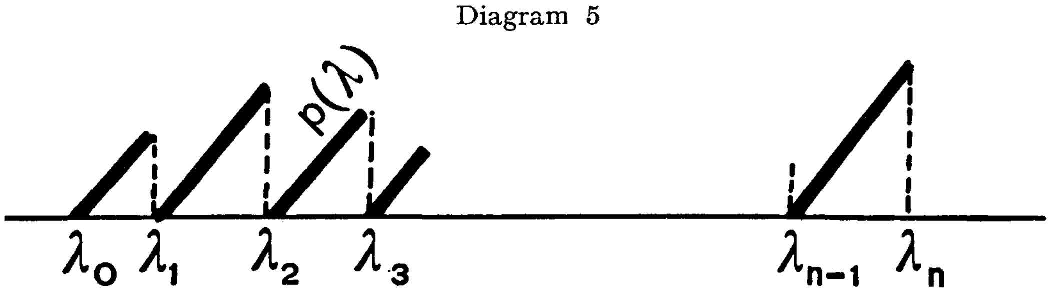
Similarly $\mathfrak{B}'_0 = \{0\}$. As $\mathfrak{B}_0 + \mathfrak{B}'_0$ is dense in $(I - P_0)\mathfrak{B}$, this means that $(I - P_0)\mathfrak{B} = \{0\}$, so that $P_0 = I$. Thus the nullspace of T is the whole of $\mathfrak{B}$.

COROLLARY. *If the spectrum of T consists of a single point μ, and T is well-bounded, then $T = \mu I$.*

PROOF. $T - \mu I$ satisfies the conditions of Theorem E.

My reasons (heuristic), for believing that $(S - T)^2 = 0$, are: $T - S$ is the limit, as $\sup\,(\lambda_{i+1} - \lambda_i) \to 0$, of operators $T - \sum \lambda_i \Delta E(\lambda_i)$. Such an operator corresponds (roughly) to the function $p(\lambda)$ of Diagram 5. Now $\|p(\lambda)\| \geqq |J|$,

Diagram 5



[1] See footnote, p. 3.

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however, fine the subdivision; but $\|(p(\lambda))^2\| \to 0$ as $\sup\ (\lambda_{i+1} - \lambda_i) \to 0$, so we expect that $(S - T)^2 = 0$. The fact that $\|p(\lambda)\|$ is approximately $|J|$ suggests the inequality (9) of the introduction.

6. The Spectral Theorem

In this section we will assume that $\mathfrak{B}$ is a Hilbert space and that T is a self-adjoint operator. It is well known that the bound of T is then equal to its spectral radius. The same theorem applied to $p(T)$, taken with the spectral mapping theorem, $p(\sigma(T)) = \sigma(p(T))$, shows that

$$||p(T)|| = \sup_{\lambda \in \sigma(T)} |p(\lambda)|$$

which is stronger than the statement that T is well-bounded.[2] We define the projections E_μ as in the proof of Theorem A. On inspection of the definition of E_μ it is easily seen that E_μ is self-adjoint. The argument of Theorem D (ii) shows that for a net N with $\delta(N) < \varepsilon$, the spectral radius of

$$T - \sum \lambda_i \Delta E(\lambda_i)$$

is less than ε so that, since this operator is self-adjoint, $||T - \sum \lambda_i \Delta E(\lambda_i)|| < \varepsilon$. Thus

$$T = \int \lambda \, dE_\lambda,$$

the right-hand side being the limit in operator norm of the corresponding Riemann sums.

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[2] Dr. Ringrose's results make the remaining lines of this proof superfluous.

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