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BERGMAN SPACES ON DISCONNECTED DOMAINS

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ABSTRACT. For a bounded region $G \subset \mathbb{C}$ and a compact set $K \subset G$, with area measure zero, we will characterize the invariant subspaces $\mathcal{M}$ (under $f \longrightarrow zf$) of the Bergman space $L^p_a(G \setminus K)$, $1 \leq p < \infty$, which contain $L^p_a(G)$ and with $\dim(\mathcal{M}/(z-\lambda)\mathcal{M}) = 1$ for all $\lambda \in G \setminus K$. When $G \setminus K$ is connected, we will see that $\dim(\mathcal{M}/(z-\lambda)\mathcal{M}) = 1$ for all $\lambda \in G \setminus K$ and thus in this case we will have a complete description of the invariant subspaces lying between $L^p_a(G)$ and $L^p_a(G \setminus K)$. When $p = \infty$, we will remark on the structure of the weak-star closed z-invariant subspaces between $H^\infty(G)$ and $H^\infty(G \setminus K)$. When $G \setminus K$ is not connected, we will show that in general the invariant subspaces between $L^p_a(G)$ and $L^p_a(G \setminus K)$ are fantastically complicated. As an application of these results, we will remark on the complexity of the invariant subspaces (under $f \longrightarrow \zeta f$) of certain Besov spaces on K. In particular, we shall see that in the harmonic Dirichlet space $B^{\frac{1}{2}}_2(\mathbb{T})$, there are invariant subspaces $\mathcal{F}$ such that the dimension of $\zeta\mathcal{F}$ in $\mathcal{F}$ is infinite.

1. Introduction. For a bounded open set $U \subset \mathbb{C}$ and $1 \leq p < \infty$, define the *Bergman space* $L^p_a(U)$ to be the space of functions $f \in L^p(U, dA)$ which are analytic on U. (Here dA is Lebesgue measure on $\mathbb{C}$.) It is well known that $L^p_a(U)$ is a closed subspace of $L^p(U) = L^p(U, dA)$ and that S on $L^p_a(U)$ defined by $(Sf)(z) = zf(z)$ is a continuous linear operator. A difficult and open problem in operator theory is to completely describe the subspaces $\mathcal{M}$ of $L^p_a(U)$ for which $S\mathcal{M} \subset \mathcal{M}$. We will call such subspaces *invariant subspaces*. In this paper we wish to continue an investigation begun in [19] and [21] of the invariant subspaces $\mathcal{M}$ with

$$L^p_a(G) \subset \mathcal{M} \subset L^p_a(G \setminus K),$$

where G is a bounded region in $\mathbb{C}$ and K is a compact subset of G with area measure zero. In particular, we focus our attention on the subspaces with the codimension 1 property. For an invariant subspace $\mathcal{M}$, the operator $(S-\lambda)|_{\mathcal{M}}$ is semi-Fredholm for all $\lambda \in G \setminus K$ and

$$-\operatorname{index}\big((S-\lambda)|_{\mathcal{M}}\big) = \dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big)$$

is constant on the components of $G \setminus K$ [16], Lemma 2.1. This constant is called the *codimension* of $\mathcal{M}$ on the component of $G \setminus K$. In this paper, we will characterize the

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invariant subspaces $\mathcal{M}$ with

$$L_a^p(G) \subset \mathcal{M} \subset L_a^p(G \setminus K) \tag{1.1}$$

$$\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = 1 \quad \forall \lambda \in G \setminus K. \tag{1.2}$$

It will turn out, Proposition 4.1, that for an invariant subspace $\mathcal{M}$ of the form (1.1), the condition (1.2) is equivalent to

$$L_\lambda \mathcal{M} = \mathcal{M} \quad \forall \lambda \in G \setminus K, \quad L_\lambda f = \frac{f - f(\lambda)}{z - \lambda}. \tag{1.3}$$

The operator L_λ is a continuous operator on $L_a^p(G \setminus K)$ and is a left inverse for $S - \lambda$.

NOTATION. Throughout this paper G will be a bounded region in $\mathbb{C}$, K will be a compact subset of G with area measure zero, and $\mathcal{M}$ will denote a closed invariant subspace of $L_a^p(G \setminus K)$ containing $L_a^p(G)$.

For $1 \leq p < 2$ we can describe our invariant subspaces $\mathcal{M}$ in terms of analytic continuation across parts of K.

THEOREM 1.1. *For $1 \leq p < 2$, the following are equivalent:*
1. $\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = 1 \; \forall \lambda \in G \setminus K$.
2. $L_\lambda \mathcal{M} = \mathcal{M} \; \forall \lambda \in G \setminus K$.
3. $\mathcal{M} = L_a^p(G \setminus E)$ *for some closed set* $E \subset K$.

For $p \geq 2$, not every $\mathcal{M}$ will be of the form $\mathcal{M} = L_a^p(G \setminus E)$, Section 6, but we still can describe $\mathcal{M}$ in terms of the q-capacity C_q (q is the conjugate index to p) associated with the Sobolev spaces W_1^q (see below). We say a set $E \subset \mathbb{C}$ is *quasi-closed* if given any $\varepsilon > 0$ there is an open set O with $C_q(O) < \varepsilon$ and $E \setminus O$ closed. Our main theorem for $p \geq 2$ is as follows:

THEOREM 1.2. *For $p \geq 2$, the following are equivalent:*
1. $\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = 1 \; \forall \lambda \in G \setminus K$.
2. $L_\lambda \mathcal{M} = \mathcal{M} \; \forall \lambda \in G \setminus K$.
3. *There is a quasi-closed set $E \subset K$ and a sequence of closed sets $F_1 \subset F_2 \subset \cdots \subset E$ with $C_q(F_n) \longrightarrow C_q(E)$ and*

$$\mathcal{M} = \mathcal{M}(E) \equiv \overline{\bigcup_n L_a^p(G \setminus F_n)}^{L^p}.$$

Moreover $\mathcal{M}(E)$ is independent of the choice of $\{F_n\}$ and if $E_1, E_2 \subset K$ are quasi-closed, then $\mathcal{M}(E_1) = \mathcal{M}(E_2)$ if and only if $C_q(E_1 \Delta E_2) = 0$.

We remark that Theorem 1.2 actually includes all the $1 < p < \infty$ since if $1 < p < 2$, then the conjugate index $q > 2$ and every non-empty set has positive C_q capacity [7], p. 157. Thus for $q > 2$, quasi-closed just means closed and hence $\mathcal{M}(E) = L_a^p(G \setminus E)$. It will turn out that if $G \setminus K$ is connected, then the condition (1.2) is automatic and we have the following:

COROLLARY 1.3. *If $G \setminus K$ is connected, then*
1. *for $1 \leq p < 2$, $\mathcal{M} = L^p_a(G \setminus E)$ for some closed $E \subset K$.*
2. *for $p \geq 2$, $\mathcal{M} = \mathcal{M}(E)$ for some quasi-closed $E \subset K$.*

When $G \setminus K$ is not connected, the condition (1.2) is not a vacuous one. Consider the following example:

EXAMPLE. Suppose $G \setminus K$ is not connected and let U be one of the bounded components of $\mathbb{C} \setminus K$. Consider the invariant subspace

$$\mathcal{M} = \chi_U L^p_a(U) + L^p_a(G). \tag{1.4}$$

One shows that $\mathcal{M}$ is closed in $L^p_a(G \setminus K)$ and that for $\lambda \in U$

$$\chi_U L^p_a(U) + L^p_a(G) = (z-\lambda)\chi_U L^p_a(U) + (z-\lambda)L^p_a(G) + \mathbb{C}\chi_U + \mathbb{C}.$$

Thus $\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = 2$.

In fact, relaxing (1.2) can produce even more pathological examples:

EXAMPLE. For $p = 2$ let G be a region which contains the closure of the unit disk $\mathbb{D}$ and consider the disconnected set $G \setminus \mathbb{T}$, where $\mathbb{T}$ is the unit circle. By [5], Corollary 6.9 and Proposition 5.4, given any $n \in \mathbb{N} \cup \{\infty\}$ there is an invariant subspace $\mathcal{N}_n$ of $L^2_a(\mathbb{D})$ with $\dim(\mathcal{N}_n/z\mathcal{N}_n) = n$. In fact, specific examples of this can be found in [10]. Consider the invariant subspace

$$\mathcal{M}_n = \chi_{\mathbb{D}}\mathcal{N}_n + L^2_a(G).$$

One shows that $\mathcal{M}_n$ is closed in $L^2_a(G \setminus \mathbb{T})$ and that

$$\dim(\mathcal{M}_n/z\mathcal{M}_n) = \dim(\mathcal{N}_n/z\mathcal{N}_n) + \dim\big(L^2_a(G)/zL^2_a(G)\big) = n+1,$$

making $\mathcal{M}_n$ difficult to understand. By imposing the condition (1.2), we avoid such pathologies as $\mathcal{M}_n$. In fact, this subspace $\mathcal{M}_n$ will be used to construct an invariant subspace $\mathcal{F}_{\mathcal{M}}$ (under multiplication by ζ) of the harmonic Dirichlet space $B^1_2(\mathbb{T})$ with $\dim(\mathcal{F}_{\mathcal{M}}/\zeta\mathcal{F}_{\mathcal{M}}) = n$, see Section 8.

The main tool used here will be to convert our Bergman space problem, via annihilators and the Cauchy transform, to an invariant subspace problem for the Sobolev space $W^{q,0}_1(G)$. Such invariant subspaces will be characterized in terms of their zero sets on K and for this we will use the fine properties of Sobolev functions and capacity.

Finally, we mention that as an application of these results, we will obtain information about polynomial and rational approximation, and characterize the Rat(K)-invariant subspaces of certain Besov classes of functions on K. This generalizes work of [17] for the harmonic Dirichlet space.

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2. Preliminaries.

2.1. *Sobolev spaces.* For $1 < p < \infty$ we identify the dual of $L^p(U) = L^p(U, dA)$ with $L^q(U)$ (q is the conjugate index to p) by the bi-linear pairing

$$\langle f, g\rangle = \int fg\, dA \quad f \in L^p(U),\ g \in L^q(U). \tag{2.1}$$

For a set $X \subset L^p(U)$ we let

$$X^\perp = \{g \in L^q(U) : \langle f, g\rangle = 0\ \forall f \in X\}$$

denote the annihilator of X and note that by the Hahn-Banach theorem $(X^\perp)^\perp$ is the closed linear span of X in $L^p(U)$. We let $\bar\partial f = \frac{1}{2}(\partial_x f + i\partial_y f)$ and $C_0^\infty(U)$ denote the infinitely differentiable functions with compact support in U.

For $1 < q < \infty$, define the *Sobolev space*

$$W_1^q = W_1^q(\mathbb{C}) = \{f \in L^q : \nabla f \in L^q\}$$
$$\|u\|_q = \Big(\int (|u|^2 + |\nabla u|^2)^{q/2}\, dA\Big)^{1/q}$$

and note that W_1^q is a separable, reflexive Banach space [1], Theorem 3.2 and Theorem 3.5. For a bounded domain $U \subset \mathbb{C}$, define $W_1^{q,0}(U)$ to be the closure of $C_0^\infty(U)$ in the W_1^q norm and note, by the Poincaré inequality [7], p. 154, we can equivalently norm $W_1^{q,0}(U)$ by

$$\|u\|_{q,0} = \Big(\int_U |\nabla u|^q\, dA\Big)^{1/q}.$$

If $q > 2$, the Sobolev imbedding theorem yields $W_1^{q,0}(U)$ is a Banach algebra of continuous functions [1], p. 115.

We now introduce the following Sobolev space which will be the key to much of our later approximations. We refer the reader to [21] for further discussion and proofs of the basic facts. Let

$$\mathcal{W} = \mathcal{W}(\mathbb{C}) = \{f \in L^\infty : \bar\partial f \in L^\infty\}$$
$$\|f\|_{\mathcal{W}} = \|f\|_\infty + \|\bar\partial f\|_\infty.$$

REMARK. We pause here for a moment to mention that $\mathcal{W}$ contains, but is not equal to $W^{1,\infty}(\mathbb{C}) = \{f \in L^\infty : \partial_x f, \partial_y f \in L^\infty\}$. In fact, if $f \in \mathcal{W}$, then $\partial_x f$ and $\partial_y f$ belong to BMO but are not always bounded [11].

One proves [21] that the functions in $\mathcal{W}$ have continuous representatives and thus for a bounded open set U we can define the (closed) subspace

$$\mathcal{W}_0(U) = \{f \in \mathcal{W} : f|_{\mathbb{C}\setminus U} = 0\}.$$

One can show that the norm $\|\bar\partial f\|_\infty$ is an equivalent norm on $\mathcal{W}_0(U)$ and that $\mathcal{W}_0(U)$ is a Banach algebra.

REMARK. $\mathcal{W}_0^{\prime}(U)$ is not the same as the closure of $C_0^\infty(U)$ in the $\mathcal{W}$ norm.

2.2. *Capacity.* Following [3] or [7], we let $1 < q < \infty$ and define the *q-capacity* C_q of a compact set F by

$$C_q(F) = \inf \|u\|_q,$$

where the infimum is taken over all real-valued functions $u \in C_0^\infty$ with $u \equiv 1$ on F. We extend this definition to arbitrary sets E by

$$C_q(E) = \sup\{C_q(F) : F \subset E, F \text{ compact}\}$$

and define the *exterior capacity* $C_q^*(E)$ of an arbitrary set E by

$$C_q^*(E) = \inf\{C_q(G) : G \supset E, G \text{ open}\}.$$

REMARK. C_q^* is equivalent to the q-Bessel capacity [9].

A set E is said to be *capacitable* if $C_q(E) = C_q^*(E)$. One notes [3] that C_q^* is a monotone, subadditive set function and that the Borel sets are capacitable. Recalling the definition of quasi-closed, one argues (using the fact that Borel sets are capacitable) that a quasi-closed set is capacitable, as is the difference of any two quasi-closed sets. We also say a property holds *quasi-everywhere* (abbreviation q.e.) if the set for which it fails has exterior capacity zero. For $q > 2$ every non-empty set has positive capacity [7], p. 151, hence quasi-closed and quasi-everywhere become closed and everywhere respectively.

Since functions in W_1^q, for $q \leq 2$, are not always continuous (or even bounded), we shall need the following definition: A complex-valued function f is *quasi-continuous* if for every $\varepsilon > 0$ there is an open set O with $C_q(O) < \varepsilon$ and $f|_{\mathbb{C}} \setminus O$ is continuous. One can show [3], Lemma 1, Theorem 2, that every $f \in W_1^q$ has a quasi-continuous representative and that any two quasi-continuous functions which agree a.e. dA must agree quasi-everywhere. In fact, one can find a formula for the quasi-continuous representative of a Sobolev function. For $f \in W_1^q$ we define

$$f^*(w) = \lim_{r \to 0} \frac{1}{\pi r^2} \int_{|z-w|<r} f(z)\, dA(z) \tag{2.2}$$

whenever this limit exists and notice by the Lebesgue differentiation theorem, $f = f^*$ a.e. By [7], p. 160, $f^*(w)$ is defined quasi-everywhere and moreover f^* is quasi-continuous. This next useful result of Bagby [3], Theorem 4, describes $W_1^{q,0}(U)$ in terms of zero sets.

PROPOSITION 2.1. *A function $f \in W_1^q$ belongs to $W_1^{q,0}(U)$ if and only if $f^* = 0$ q.e. off U.*

We again remark that for $q > 2$ and $f \in W_1^q$, the function f^* is defined everywhere and is continuous. Moreover $f \in W_1^q$ belongs to $W_1^{q,0}(U)$ if and only if $f^* \equiv 0$ on $\mathbb{C} \setminus U$.

3. **Correspondence.** We now relate our Bergman space problem to a certain Sobolev space problem via a technique of Havin [8]. We refer the reader to [8] and [19] for the details of this section and for further references. We begin with Havin's lemma.

LEMMA 3.1 (HAVIN). *Let U be a bounded open set and $1 < p < \infty$. Then $f \in L^q(U)$ satisfies*

$$\int_U uf\,dA = 0 \quad \forall u \in L^p_a(U)$$

if and only if there is an $F \in W^{q,0}_1(U)$ with $\bar\partial F = f$.

By the Calderon-Zygmund theory,

$$\|\bar\partial\phi\|_{L^q} \sim \|\nabla\phi\|_{L^q} \quad \forall \phi \in W^{q,0}_1(U) \tag{3.1}$$

and thus by Havin's lemma $\bar\partial\colon W^{q,0}_1(U) \longrightarrow L^p_a(U)^\perp$ is a continuous invertible operator with inverse given by the Cauchy transform

$$(\bar\partial^{-1}g)(w) = (Cg)(w) \equiv -\frac{1}{\pi}\int_U \frac{g(z)}{z-w}\,dA(z). \tag{3.2}$$

If R_z is multiplication by z on $L^p_a(U)^\perp$ (well defined and continuous by the bilinear pairing (2.1) and M_z is multiplication by z on $W^{q,0}_1(U)$ (also well defined and continuous) then, noticing that $\bar\partial(zf) = z\bar\partial f$ for all $f \in W^{q,0}_1(U)$, we have

$$R_z\bar\partial = \bar\partial M_z. \tag{3.3}$$

So if $L^p_a(G) \subset \mathcal{M} \subset L^p_a(G\setminus K)$ is invariant, then

$$L^p_a(G\setminus K)^\perp \subset \mathcal{M}^\perp \subset L^p_a(G)^\perp$$

is also z-invariant (by (2.1)) and applying the Cauchy transform C and (3.3) we obtain

$$W^{q,0}_1(G\setminus K) \subset C\mathcal{M}^\perp \subset W^{q,0}_1(G) \tag{3.4}$$

and $C\mathcal{M}^\perp$ is z-invariant.

If $p = 1$, one can use Weyl's lemma to prove $L^1_a(U)^\perp$ is the weak-star closure of $\bar\partial C^\infty_0(U)$, the transformation $\bar\partial\colon \mathcal{W}_0(U) \longrightarrow L^1_a(U)^\perp$ is invertible with inverse given by the Cauchy transform C, and $R_z\bar\partial = \bar\partial M_z$ [21]. (Note here that $\mathcal{W}_0(G)$ is endowed with the weak-star topology with the pairing $\int f\bar\partial g\,dA$, $f \in L^1$, $g \in \mathcal{W}$. Thus whenever we mention closure and density, we will be referring to the weak-star closure and weak-star density.) Thus, as before if $L^1_a(G) \subset \mathcal{M} \subset L^1_a(G\setminus K)$ is invariant, then $\mathcal{W}_0(G\setminus K) \subset C\mathcal{M}^\perp \subset \mathcal{W}_0(G)$ is z-invariant.

Thus for all $1 \le p < \infty$, our invariant subspaces $\mathcal{M}$ are in one-to-one correspondence with the z-invariant subspaces that lie between the Sobolev spaces $W^{q,0}_1(G\setminus K)$ and $W^{q,0}_1(G)$ (resp. $\mathcal{W}_0(G\setminus K)$ and $\mathcal{W}_0(G)$).

4. Codimension. For $\lambda \in G \setminus K$ we define $L_\lambda : L^p_a(G \setminus K) \to L^p_a(G \setminus K)$ by

$$L_\lambda f = \frac{f - f(\lambda)}{z - \lambda}.$$

One easily checks that L_λ is a continuous linear operator on $L^p_a(G \setminus K)$ with $L_\lambda(S - \lambda) = I$ and $L_\lambda \mathcal{M}$ is closed and invariant.

PROPOSITION 4.1. *For $\lambda \in G \setminus K$, $L_\lambda \mathcal{M} = \mathcal{M}$ if and only if $\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = 1$.*

PROOF. We first notice that since L_λ is a left inverse for $S_{z-\lambda}$, then $\mathcal{M} \subset L_\lambda \mathcal{M}$. Suppose that $L_\lambda \mathcal{M} = \mathcal{M}$. If $f \in \mathcal{M}$, then $(z - \lambda)L_\lambda f = f - f(\lambda) \in \mathcal{M}$ (since $1 \in \mathcal{M}$). So $\mathcal{M} = (z - \lambda)L_\lambda \mathcal{M} + \mathbb{C} = (z - \lambda)\mathcal{M} + \mathbb{C}$ and hence $\dim\big(\mathcal{M}/(z - \lambda)\mathcal{M}\big) = 1$.

If $\dim\big(\mathcal{M}/(z - \lambda)\mathcal{M}\big) = 1$, then since $1 \in \mathcal{M}$, every $f \in \mathcal{M}$ can be written as $f = (z - \lambda)g + f(\lambda)$ for some $g \in \mathcal{M}$. Hence $g = L_\lambda f \in \mathcal{M}$. ∎

THEOREM 4.2. *The following are equivalent:*
1. *$L_\lambda \mathcal{M} = \mathcal{M}$ for all $\lambda \in G \setminus K$*
2. *$\psi(C\mathcal{M}^\perp) \subset C\mathcal{M}^\perp$ for all $\psi \in \mathcal{W}$.*

PROOF. We first notice that if $g \in C\mathcal{M}^\perp \subset W^{q,0}_1(G)$ ($\mathcal{W}_0(G)$ respectively) then, by (3.1), $\psi g \in W^{q,0}_1(G)$ (resp. $\mathcal{W}_0(G)$) for all $\psi \in \mathcal{W}$. Also notice that if $\psi_0 \in \mathcal{W}$ with $\psi_0 = \psi$ on K, then $\psi_0 g - \psi g = 0$ q.e. on K and hence, by Proposition 2.1 and (3.4), $\psi_0 g - \psi g \in W^{q,0}_1(G \setminus K) \subset C\mathcal{M}^\perp$. Hence for $f \in \mathcal{M}$

$$\int f \bar\partial(\psi_0 g)\, dA = \int f \bar\partial(\psi g)\, dA$$

and so we can assume that ψ has compact support in the plane.

If $L_\lambda \mathcal{M} = \mathcal{M}$ for all $\lambda \in G \setminus K$, then for $f \in \mathcal{M}$ and $g \in C\mathcal{M}^\perp$

$$\int \frac{f(z) - f(\lambda)}{z - \lambda} \bar\partial g(z)\, dA(z) = 0 \quad \forall \lambda \in G \setminus K.$$

Thus

$$\int \frac{f(z)\bar\partial g(z)}{z - \lambda}\, dA(z) = f(\lambda) \int \frac{\bar\partial g(z)}{z - \lambda}\, dA(z) \quad \text{a.e. on } G.$$

Integrating both sides of this equation against $\bar\partial \psi \in L^\infty$ (which is possible since both sides belong to L^1_{loc}) and using Fubini's theorem, one obtains

$$\int f(z)\bar\partial g(z) \int \frac{\bar\partial \psi(\lambda)}{z - \lambda}\, dA(\lambda)\, dA(z) = \int f(\lambda)\bar\partial \psi(\lambda) \int \frac{\bar\partial g(z)}{z - \lambda}\, dA(z)\, dA(\lambda).$$

One rewrites the above (using the fact that ψ and g have compact support so $C(\bar\partial \psi) = \psi$ and $C(\bar\partial g) = g$) as

$$\int f \bar\partial(\psi g)\, dA = 0 \quad \forall \bar\partial g \in \mathcal{M}^\perp.$$

This implies $\psi g \in C\mathcal{M}^\perp$.

Conversely, suppose that $\psi(C\mathcal{M}^\perp) \subset C\mathcal{M}^\perp$ for all $\psi \in \mathcal{W}$. Then reverse the above argument to get that for all $f \in \mathcal{M}$ and $g \in C\mathcal{M}^\perp$

$$\int\Big(\int \frac{f(z)-f(\lambda)}{z-\lambda}\bar\partial g(z)\,dA(z)\Big)\bar\partial\,\psi(\lambda)\,dA(\lambda) = 0 \quad \forall\psi \in \mathcal{W}.$$

This will imply [1], p. 95,

$$\int \frac{f(z)-f(\lambda)}{z-\lambda}\bar\partial g(z)\,dA(z) = 0 \quad \text{a.e. on } G$$

and hence, by the Hahn-Banach theorem and the fact that $\mathcal{M} \subset L_\lambda\mathcal{M}$, we have $L_\lambda\mathcal{M} = \mathcal{M}$ for all $\lambda \in G\setminus K$. ∎

Let Rat(K) be the set of rational functions with poles off K and define the manifold

$$\mathcal{F} = \{g \in \mathcal{W}_0(G) : g|_K \in \mathrm{Rat}(K)\}.$$

COROLLARY 4.3. *If $\mathcal{F}(C\mathcal{M}^\perp) \subset C\mathcal{M}^\perp$, then $\mathcal{W}(C\mathcal{M}^\perp) \subset C\mathcal{M}^\perp$.*

PROOF. We will show that $L_\lambda\mathcal{M} \subset \mathcal{M}$ and apply Theorem 4.2. For $\lambda \in G\setminus K$, let $\phi \in \mathcal{W}_0(G)$ that is identically 1 near K and zero in a neighborhood of λ. For all $f \in \mathcal{M}$ and $\psi \in C\mathcal{M}^\perp$ we have (since $\psi(1-\phi) = 0$ on K and hence belongs to $W_1^{q,0}(G\setminus K) \subset C\mathcal{M}^\perp$, respectively $\mathcal{W}_0(G\setminus K) \subset C\mathcal{M}^\perp$)

$$\int \frac{f-f(\lambda)}{z-\lambda}\bar\partial\,\psi\,dA = \int \frac{f-f(\lambda)}{z-\lambda}\bar\partial\,(\psi\phi)\,dA = \int\big(f-f(\lambda)\big)\bar\partial\Big(\frac{\psi\phi}{z-\lambda}\Big)\,dA = 0$$

because $\psi\phi(z-\lambda)^{-1} \in \mathcal{F}(C\mathcal{M}^\perp)$ (note that $1 \in \mathcal{M}$). ∎

PROPOSITION 4.4. *The manifold $\mathcal{F} = \{g \in \mathcal{W}_0(G) : g|_K \in \mathrm{Rat}(K)\}$ is dense in both $\mathcal{W}_0(G)$ and $W_1^{q,0}(G)$.*

PROOF. Let $\bar{\mathcal{F}}$ denote the closure of $\mathcal{F}$ in $W_1^{q,0}(G)$. Then trivially

$$W_1^{q,0}(G\setminus K) \subset \mathcal{F} \subset \bar{\mathcal{F}} \subset W_1^{q,0}(G).$$

Since $\bar{\mathcal{F}}$ is z-invariant, it follows from Corollary 4.3 that $\mathcal{W}\bar{\mathcal{F}} \subset \bar{\mathcal{F}}$. Now let $\phi \in C_0^\infty(G)$ and identically 1 on K. Then $\phi \in W_1^{q,0}(G)$ and in fact $\phi \in \mathcal{F}$. Finally, if $\psi \in W_1^{q,0}(G)$, then $\phi\psi \in \bar{\mathcal{F}}$ and $(1-\phi)\psi \in W_1^{q,0}(G\setminus K) \subset \bar{\mathcal{F}}$, so $\psi = \phi\psi + (\psi - \phi\psi) \in \bar{\mathcal{F}}$. This shows that $\bar{\mathcal{F}}$ contains $W_1^{q,0}(G)$, which concludes the proof. A similar proof shows that $\mathcal{F}$ is weak-star dense in $\mathcal{W}_0(G)$. ∎

Let $\mathcal{P}$ be the set of analytic polynomials and define the manifold

$$\mathcal{F}_1 = \{g \in \mathcal{W}_0(G) : g|_K \in \mathcal{P}\}.$$

COROLLARY 4.5. *If $G\setminus K$ is connected, then the manifold $\mathcal{F}_1$ is dense in both $\mathcal{W}_0(G)$ and $W_1^{q,0}(G)$.*

PROOF. By Proposition 4.4, it suffices to show that $\overline{\mathcal{F}_1}$ contains $\mathcal{F}$. Indeed, if $g \in \mathcal{F}$ with $g|_K = r \in \mathrm{Rat}(K)$ then there exists a simply connected region Ω which does not contain the poles of r and $K \subset \Omega \subset G$. By Runge's theorem, r can be uniformly approximated by a sequence of polynomials $\{p_n\}$ in $\bar{\Omega}$. If $\phi \in C_0^\infty(G)$ vanishes off Ω and is identically 1 near K, then $\bar{\partial}(p_n\phi) = p_n\bar{\partial}\phi$ converges uniformly to $r\bar{\partial}\phi = \bar{\partial}(\phi r)$, that is $r\phi \in \overline{\mathcal{F}_1}$. Thus $g = r\phi + (g - r\phi) \in \overline{\mathcal{F}_1}$. ∎

COROLLARY 4.6. *If $G \setminus K$ is connected, then $\psi(C\mathcal{M}^\perp) \subset C\mathcal{M}^\perp$ for all $\psi \in \mathcal{W}$ and hence $L_\lambda\mathcal{M} = \mathcal{M}$ for all $\lambda \in G \setminus K$.*

PROOF. Let $g \in C\mathcal{M}^\perp$ and $\psi \in \mathcal{W}$. If $\psi_0 \in \mathcal{W}_0(G)$ with $\psi_0 = \psi$ on K, then $\psi_0 g - \psi g = 0$ q.e. on K and so by Proposition 2.1 $\psi_0 g - \psi g \in \mathcal{W}_0(G \setminus K) \subset C\mathcal{M}^\perp$. Thus

$$\mathrm{dist}(\psi g, C\mathcal{M}^\perp) = \mathrm{dist}(\psi_0 g, C\mathcal{M}^\perp).$$

By Corollary 4.5, given any $\varepsilon > 0$ there is a $\psi_\varepsilon \in \mathcal{W}_0(G)$ with $\psi_\varepsilon|_K = p \in \mathcal{P}$ and $\|\psi_\varepsilon - \psi_0\|_{\mathcal{W}} < \varepsilon$. Notice that $pg \in C\mathcal{M}^\perp$ and $pg - \psi_\varepsilon g = 0$ q.e. on K and so by Proposition 2.1 $pg - \psi_\varepsilon g \in W_1^{q,0}(G \setminus K) \subset C\mathcal{M}^\perp$. From this we obtain $\psi_\varepsilon g \in C\mathcal{M}^\perp$ and so

$$\mathrm{dist}(\psi_0 g, C\mathcal{M}^\perp) \le \|\psi_0 g - \psi_\varepsilon g\|_q \le C\varepsilon\|g\|_q.$$

Thus $\psi_0 g$ and hence ψg belongs to $C\mathcal{M}^\perp$. A similar proof works for $p = 1$. ∎

5. **Invariant subspaces: $1 \le p < 2$.** We now can prove our main theorem for $1 \le p < 2$.

THEOREM 5.1. *For $1 \le p < 2$, the following are equivalent:*
1. $\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = 1\ \forall\lambda \in G \setminus K$.
2. $L_\lambda\mathcal{M} = \mathcal{M}\ \forall\lambda \in G \setminus K$.
3. $\mathcal{M} = L_a^p(G \setminus E)$ *for some closed set* $E \subset K$.

PROOF. Notice that by our earlier work, we just need to prove (1) implies (3). By Theorem 4.2, $\psi(C\mathcal{M}^\perp) \subset C\mathcal{M}^\perp$ for all $\psi \in \mathcal{W}_0(G)$, hence $C\mathcal{M}^\perp$ is an ideal of the Banach algebra $W_1^{q,0}(G)$ (respectively $\mathcal{W}_0(G)$). Let

$$E = \{z \in K : g(z) = 0\ \forall g \in C\mathcal{M}^\perp\}$$

and proceed as in [19] [21] to show that $C\mathcal{M}^\perp = W_1^{q,0}(G \setminus E)$ (respectively $\mathcal{W}_0(G \setminus E)$) and thus $\mathcal{M} = L_a^p(G \setminus E)$. ∎

When $G \setminus K$ is connected we can apply the above along with Corollary 4.6 to get:

COROLLARY 5.2. *If $1 \le p < 2$ and $G \setminus K$ is connected, then $\mathcal{M} = L_a^p(G \setminus E)$ for some closed $E \subset K$.*

6. **Invariant subspaces:** $p \geq 2$. If $E \subset K$ is closed then $L^p_a(G \setminus E)$ is an invariant subspace containing $L^p_a(G)$ and is L_λ-invariant for all $\lambda \in G \setminus K$. For $1 \leq p < 2$ these are the only ones with this property, but for $p \geq 2$, there are others. For this reason we proceed with the following construction: For $p \geq 2$ (and hence $q \leq 2$) and a quasi-closed set $E \subset K$, we can find a sequence of closed sets $F_1 \subset F_2 \subset \cdots \subset E$ with $C_q(F_n) \longrightarrow C_q(E)$. Since $L^p_a(G \setminus F_n)$ increases with n, we can define the invariant subspace

$$\mathcal{M}(E) = \overline{\bigcup_n L^p_a(G \setminus F_n)}^{L^p}. \tag{6.1}$$

Notice that $\mathcal{M}(E)$ contains $L^p_a(G)$ and is L_λ-invariant for all $\lambda \in G \setminus K$ and that [19], Proposition 4.2:

PROPOSITION 6.1. *For $p \geq 2$, and quasi-closed sets $E, F \subset K$*
1. *$\mathcal{M}(E)$ is independent of the choice of $\{F_n\}$.*
2. *$\mathcal{M}(E) \subset \mathcal{M}(F) \Leftrightarrow C_q(E \setminus F) = 0$.*
3. *$\mathcal{M}(E) = \mathcal{M}(F) \Leftrightarrow C_q(E \Delta F) = 0$.*

Furthermore for $p \geq 2$ ($q \leq 2$) there are quasi-closed sets $E \subset K$ for which $\mathcal{M}(E)$ cannot be written as $L^p_a(G \setminus F)$ for any closed $F \subset K$ [19], Proposition 4.3. We record this example here for completeness and for further reference.

EXAMPLE. Fix $1 < q \leq 2$ and let G be a disk of radius 2 centered about the origin and $K = [0, 1]$. Let $B \subset [0, 1]$ be constructed in the same manner as the Cantor set except that the intervals removed (a_n, b_n) are such that $\sum_{n \geq 1} C_q(a_n, b_n) < C_q[0, 1]$. (This is justified since $C_q(a, b)^q \simeq (b - a)^{2-q}$ if $q < 2$ and $C_2(a, b)^2 \simeq \big(\log(2/(b - a))\big)^{-1}$ [25], and [13], p. 115, Proposition 6.) Set $E = [0, 1] \setminus B = \bigcup_{n \geq 1}(a_n, b_n)$ and notice that E is open and dense in $[0, 1]$ with $C_q(E) < C_q[0, 1]$. A straightforward argument shows that E is quasi-closed and $C_q(E \Delta F) > 0$ for any closed set F. Setting $\mathcal{M} = \mathcal{M}(E)$ and using Proposition 6.1, we are done.

One also notes that

$$W_q(E) \equiv C\big(\mathcal{M}(E)^\perp\big) = \bigcap_n W^{q,0}_1(G \setminus F_n) \tag{6.2}$$

and by Proposition 2.1, $f \in W^{q,0}_1(G)$ belongs to $W_q(E)$ if and only if $f^* = 0$ quasi-everywhere on E.

If $L_\lambda \mathcal{M} = \mathcal{M}$ for all $\lambda \in G \setminus K$, we can apply Theorem 4.2 to get $\psi C\mathcal{M}^\perp \subset C\mathcal{M}^\perp$ for every $\psi \in \mathcal{W}$. Using [19], one can show $C\mathcal{M}^\perp = W_q(E)$ and hence $\mathcal{M} = \mathcal{M}(E)$ for some quasi-closed $E \subset K$. For completeness, we outline the proof and refer the reader to [19] for the technical details. Let $f \in W^{q,0}_1(G)$ (assumed to be quasi-continuous), and define

$$[f] = \operatorname{span}\{\varphi f : \varphi \in \mathcal{W}\}.$$

If we define $Z_f = f^{-1}(0)$, we see (using the fact that $f^{-1}(F)$ is quasi-closed for closed F and quasi-continuous f) that Z_f is quasi-closed and, by [3], Lemma 1, $[f] \subset W_q(Z_f)$. These next two technical lemmas can be found in [19].

LEMMA 6.2. *If* $g, h \in W_1^{q,0}(G)$ *with* $|g(z)| \leq |h(z)|$ *a.e., then* $g \in [h]$.

LEMMA 6.3. *If* $f \in W_1^{q,0}(G)$ *is quasi-continuous, then* $[f] = W_q(Z_f)$.

Assuming these two facts, one can now show that $C\mathcal{M}^\perp = W_q(E)$, for some quasi-closed $E \subset K$.

COROLLARY 6.4. *There exists a quasi-continuous* $f \in W_1^{q,0}(G)$ *with*

$$C\mathcal{M}^\perp = [f] = W_q(Z_f).$$

PROOF. Since $C\mathcal{M}^\perp$ is separable, there is a sequence of non-zero quasi-continuous functions $\{f_n : n \geq 1\}$ in $W_1^{q,0}(G)$ with

$$C\mathcal{M}^\perp = \operatorname{span}\{[f_n] : n \geq 1\}.$$

By [7], p. 130, $|f_n| \in W_1^{q,0}(G)$, and by Lemma 6.2, $[|f_n|] = [f_n]$. Thus we may assume $f_n \geq 0$. For each $n \geq 1$, let $\varepsilon_n = \|f_n\|_q^{-1} 2^{-n}$ and define $f = \sum_n \varepsilon_n f_n \in W_1^{q,0}(G)$. Assuming f is quasi-continuous, we see that $Z_f = Z_{\cap Z_{f_n}}$ quasi-everywhere. (This will follow from the fact that if p_n is the n-th partial sum, then $C_q(f - p_n \geq \varepsilon) \leq \varepsilon^{-q} \|f - p_n\|_q$ [3], Theorem 2(i) and hence a subsequence of p_n will converge to f quasi-everywhere.) Thus

$$f \in \operatorname{span}\{[f_n] : n \geq 1\} = C\mathcal{M}^\perp \subset W_q(Z_f),$$

and hence, by Lemma 6.3, $[f] = C\mathcal{M}^\perp = W_q(Z_f)$. ∎

REMARK. We remark that there is a general result of Netrusov [14] which identifies the subspaces X of a Triebel-Lizorkin space $FL_{p,\theta}^l$ or a Besov space $BL_{p,\theta}^l$ in $\mathbb{R}^n$ with $\phi X \subset X$ for all smooth functions ϕ.

Before we prove our main theorem for $p \geq 2$, remark that since

$$W_1^{q,0}(G \setminus K) \subset C\mathcal{M}^\perp = W_q(Z_f),$$

then, by (6.2) and Proposition 6.1, $Z_f \subset K$ q.e.

THEOREM 6.5. *If* $p \geq 2$, *then the following are equivalent:*

1. $\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = 1\ \forall \lambda \in G \setminus K$.
2. $L_\lambda \mathcal{M} = \mathcal{M}\ \forall \lambda \in G \setminus K$.
3. $\mathcal{M} = \mathcal{M}(E)$ *for some quasi-closed set* $E \subset K$.

PROOF. By Corollary 6.4 and the above remark, there is quasi-closed set $E \subset K$ with $C\mathcal{M}^\perp = W_q(E)$ and hence $\mathcal{M} = \mathcal{M}(E)$. ∎

When $G \setminus K$ is connected, we can apply Corollary 4.6 to obtain:

COROLLARY 6.6. *If* $p \geq 2$ *and* $G \setminus K$ *is connected, then* $\mathcal{M} = \mathcal{M}(E)$ *for some quasi-closed set* $E \subset K$.

7. **Besov spaces** As an application of our results, we will characterize the Rat(K) invariant subspaces of certain Besov spaces on K which are generalizations of the well known Besov spaces on the unit circle [15]. This will be accomplished by creating a one-to-one correspondence with the invariant subspaces $\mathcal{M}$ of the Bergman space $L^p_a(G \setminus K)$ satisfying (1.1) and (1.2).

For $1 < q < \infty$ and $2 - q < \alpha < 2$ we follow [12], Chapter 2, and say a compact set K is an *α-set* if

$$\mathcal{H}_\alpha\big(B(z,r) \cap K\big) \sim r^\alpha, \quad \forall z \in K,\ \forall 0 < r < 1. \tag{7.1}$$

Here $\mathcal{H}_\alpha$ denotes α-dimensional Hausdorff measure [7]. In fact one checks [12], p. 33 that $B(z,r) \cap K$ has Hausdorff dimension α for all $z \in K$, $0 < r < 1$.

NOTATION. For the remainder of the paper, we fix $1 < q < \infty$ and $2 - q < \alpha < 2$ and assume that K is a compact α-set in G.

Define the *Besov space* $B^\alpha_q(K)$ as the space of functions $f \in L^q(K, d\mathcal{H}_\alpha)$ with norm

$$\|f\|_{B^\alpha_q} = \|f\|_{L^q(d\mathcal{H}_\alpha)} + \left(\int_K \int_K \frac{|f(z) - f(w)|^q}{|z - w|^{2\alpha - 2 + q}}\, d\mathcal{H}_\alpha(z)\, d\mathcal{H}_\alpha(w)\right)^{1/q}.$$

One can show [12], p. 213–214, that $B^\alpha_q(K)$ is a Banach space, $C^\infty|_K$ is dense in $B^\alpha_q(K)$, and if $q > 2$ then $B^\alpha_q(K)$ can be continuously embedded into $\mathrm{Lip}_{1-2/q}(K)$. Recalling that Rat(K) is the set of rational functions with poles off K, we say a subspace $\mathcal{F} \subset B^\alpha_q(K)$ is Rat(K)*-invariant* if $r\mathcal{F} \subset \mathcal{F}$ for all $r \in \mathcal{F}$; or equivalently $(\zeta - \lambda)\mathcal{F} = \mathcal{F}\ \forall \lambda \notin K$. We will characterize the Rat(K) invariant subspaces of $B^\alpha_q(K)$ by relating these subspaces to certain Bergman spaces.

If $L^p_a(G) \subset \mathcal{M} \subset L^p_a(G \setminus K)$ is invariant, then

$$L^p_a(G \setminus K)^\perp \subset \mathcal{M}^\perp \subset L^p_a(G)^\perp$$

is also invariant. Thus the invariant subspaces $\mathcal{M}$ are in one-to-one correspondence with the R-invariant subspaces of the quotient space $L^p_a(G)^\perp / L^p_a(G \setminus K)^\perp$, where R is the coset multiplication operator $R[g] = [zg]$. We now show that R is similar to M_ζ (multiplication by ζ) on $B^\alpha_q(K)$.

THEOREM 7.1. *The linear transformation*

$$J\colon L^p_a(G)^\perp / L^p_a(G \setminus K)^\perp \longrightarrow B^\alpha_q(K)$$

defined by

$$J[g](\zeta) = -\frac{1}{\pi} \int_G \frac{g(z)}{z - \zeta}\, dA(z)$$

is a continuous invertible operator with $JR = M_\zeta J$. Thus there is a one-to-one correspondence between the invariant subspaces $L^p_a(G) \subset \mathcal{M} \subset L^p_a(G \setminus K)$ and the lattice of ζ-invariant subspaces of $B^\alpha_q(K)$

PROOF. Recall from Section 3 that $C\colon L^p_a(U)^\perp \longrightarrow W^{q,0}_1(U)$ is continuous and invertible with $CR_z = M_zC$. The operator $C = \bar\partial^{-1}$ will induce the continuous invertible operator

$$\tilde C\colon L^p_a(G)^\perp / L^p_a(G\setminus K)^\perp \longrightarrow W^{q,0}_1(G)/W^{q,0}_1(G\setminus K), \quad C[g] = [Cg].$$

The operators R_z and M_z will induce the multiplication operators R and M on cosets of $L^p_a(G)^\perp / L^p_a(G\setminus K)^\perp$ and $W^{q,0}_1(G)/W^{q,0}_1(G\setminus K)$ respectively with

$$\tilde C R = M\tilde C.$$

By the trace theory in Sobolev spaces [12], p. 182, the operator

$$T\colon W^{q,0}_1(G) \longrightarrow B^\alpha_q(K), \quad Tf = f^*|_K$$

(recall the definition of f^* (2.2)) is a continuous surjective linear operator with $\ker(T) = W^{q,0}_1(G\setminus K)$, by Proposition 2.1. Thus

$$\tilde T\colon W^{q,0}_1(G)/W^{q,0}_1(G\setminus K) \longrightarrow B^\alpha_q(K), \quad \tilde T[h] = h^*|_K$$

is a continuous invertible operator.

If we define

$$J\colon L^p_a(G)^\perp / L^p_a(G\setminus K)^\perp \longrightarrow B^\alpha_q(K)$$

by $J = \tilde T \circ \tilde C$, we obtain

$$(J[g])(\zeta) = -\frac{1}{\pi}\int_G \frac{g(z)}{z-\zeta}\,dA(z)$$

(This is true since Cg is a quasi-continuous function [6], as is $(Cg)^*$, and $(Cg)^* = Cg$ a.e. (dA). We now apply [3], Theorem 2(iii) to get that $(Cg)^* = Cg$ quasi-everywhere.) It also follows that $JR = M_\zeta J$. ∎

NOTATION. Given an invariant subspace $L^p_a(G) \subset \mathcal{M} \subset L^p_a(G\setminus K)$, we let $\mathcal{F}_{\mathcal{M}}$ be the unique ζ-invariant subspace of $B^\alpha_q(K)$ that corresponds to $\mathcal{M}$ via J. One checks that

$$\mathcal{F}_{\mathcal{M}} = T(C\mathcal{M}^\perp). \tag{7.2}$$

If $\mathcal{F}$ is a ζ-invariant subspace of $B^\alpha_q(K)$, we let $\mathcal{M}_{\mathcal{F}}$ be the unique invariant subspace of $L^p_a(G\setminus K)$ containing $L^p_a(G)$ which corresponds to $\mathcal{F}$ via J. One checks that

$$\mathcal{M}_{\mathcal{F}} = \left(\bar\partial\{f \in W^{q,0}_1(G) : Tf \in \mathcal{F}\}\right)^\perp. \tag{7.3}$$

With this notation we notice that $\mathcal{M}_{\mathcal{F}_{\mathcal{M}}} = \mathcal{M}$.

PROPOSITION 7.2. *If $\lambda \in G\setminus K$ and $L^p_a(G) \subset \mathcal{M} \subset L^p_a(G\setminus K)$ is invariant, then*

$$L_\lambda \mathcal{M} = \mathcal{M}_{(\zeta-\lambda)\mathcal{F}_{\mathcal{M}}}.$$

PROOF. For any ζ-invariant $\mathcal{F} \subset B_q^\alpha(K)$ recall from (7.3) that

$$\mathcal{M}_{\mathcal{F}} = \left(\bar{\partial}\{g \in W_1^{q,0}(G) : Tg \in \mathcal{F}\}\right)^\perp.$$

Thus if $f \in \mathcal{M}$ and $Tg \in \mathcal{F}_{\mathcal{M}}$, then

$$\int L_\lambda f \bar{\partial}\left((z-\lambda)g\right) dA = \int (z-\lambda) L_\lambda f \bar{\partial} g \, dA = \int \left(f - f(\lambda)\right) \bar{\partial} g \, dA$$

which equals zero since $1 \in \mathcal{M}$. Thus $L_\lambda \mathcal{M} \subset \mathcal{M}_{(\zeta-\lambda)\mathcal{F}_{\mathcal{M}}}$.

If $f \in \mathcal{M}_{(\zeta-\lambda)\mathcal{F}_{\mathcal{M}}}$ and $Tg \in \mathcal{F}_{\mathcal{M}}$, then

$$0 = \int f \bar{\partial}\left((z-\lambda)g\right) dA = \int (z-\lambda) f \bar{\partial} g \, dA.$$

Thus $(z-\lambda)f \in \mathcal{M}_{\mathcal{F}_{\mathcal{M}}} = \mathcal{M}$ and hence $L_\lambda(z-\lambda)f = f \in L_\lambda \mathcal{M}$. ∎

Thus if $\lambda \in G \setminus K$, then the following are equivalent

1. $\dim\left(\mathcal{M}/(z-\lambda)\mathcal{M}\right) = 1$
2. $L_\lambda \mathcal{M} = \mathcal{M}$
3. $\mathcal{M} = \mathcal{M}_{\mathcal{F}_{\mathcal{M}}} = \mathcal{M}_{(\zeta-\lambda)\mathcal{F}_{\mathcal{M}}}$
4. $(\zeta - \lambda)\mathcal{F}_{\mathcal{M}} = \mathcal{F}_{\mathcal{M}}$.

We also mention that by the continuity of the surjective operator T along with Proposition 4.4 and Corollary 4.5, one has the following results about rational and polynomial approximation in the Besov spaces:

PROPOSITION 7.3. *1.* Rat(K) *is dense in* $B_q^\alpha(K)$.
2. If $\mathbb{C} \setminus K$ *is connected, then* $\mathcal{P}$ *is dense in* $B_q^\alpha(K)$.

Before we state our main theorem, we want to comment on capacity for the Besov spaces $B_q^\alpha(K)$. One can define a capacity associated with the Besov spaces $B_q^\alpha(K)$ as follows: For a compact set $F \subset K$, define the $B_{\alpha,q}$-capacity of F by

$$B_{\alpha,q}(F) = \inf \|f\|_{B_q^\alpha},$$

where the infimum is taken over all real-valued C_0^∞ functions (on all of $\mathbb{C}$) with $f \geq 1$ on F. Extend this definition to all sets $E \subset K$ and define an associated outer capacity $B^*_{\alpha,q}$. As with the C_q capacity, one defines the notions of capacitable, quasi-everywhere, quasi-closed, and quasi-continuous for the Besov capacity $B_{\alpha,q}$. Also notice that since the trace operator is continuous and surjective, then the capacities C_q and $B_{\alpha,q}$ are equivalent for sets $E \subset K$.

By the equivalence of the capacities C_q and $B_{\alpha,q}$ for subsets of K and quasi-continuity of $f^*, f \in W_1^{q,0}(G)$, along with the trace theorem, one has that every function in $B_q^\alpha(K)$ has a quasi-continuous representative. Moreover, one can prove in a very similar fashion to [3], Theorem 2, that if two quasi-continuous functions in $B_q^\alpha(K)$ agree $\mathcal{H}_\alpha$ a.e. then they agree quasi-everywhere. For $f \in B_q^\alpha(K)$, we let $\tilde{f}$ be any one of the quasi-continuous representatives of f.

THEOREM 7.4. *If* $\mathcal{F} \subset B_q^\alpha(K)$ *is* Rat(K)*-invariant, then there is a quasi-closed set* $E \subset K$ *with*

$$\mathcal{F} = \{f \in B_q^\alpha(K) : \tilde{f}|_E = 0 \quad q.e.\}.$$

PROOF. If $\mathcal{F}$ is Rat(K) invariant, then $L_\lambda \mathcal{M}_{\mathcal{F}} = \mathcal{M}_{\mathcal{F}}$ for all $\lambda \in G \setminus K$. Thus, by Theorem 1.2, $\mathcal{M}_{\mathcal{F}} = \mathcal{M}(E)$ for some quasi-closed $E \subset K$. But by (7.2) and (6.2),

$$\mathcal{F} = T\Big(C\big(\mathcal{M}(E)^{\perp}\big)\Big) = T\big(W_q(E)\big) = \{f \in B_q^\alpha(K) : f^*|_E = 0 \text{ q.e.}\}. \qquad \blacksquare$$

When $\mathbb{C} \setminus K$ is connected, then one can use the proof of Corollary 4.6 to show that if $\lambda \notin K$ and $f \in B_q^\alpha(K)$, then there is a sequence of polynomials $\{p_n\}$ with $p_n f \longrightarrow (z-\lambda)^{-1} f$ in $B_q^\alpha(K)$. Thus when $\mathbb{C} \setminus K$ is connected, every ζ-invariant subspace is Rat(K)-invariant and we have the following:

COROLLARY 7.5. *If $\mathbb{C} \setminus K$ is connected and $\mathcal{F} \subset B_q^\alpha(K)$ is ζ-invariant, there is a quasi-closed set $E \subset K$ with*

$$\mathcal{F} = \{f \in B_q^\alpha(K) : \tilde{f}|_E = 0 \ q.e.\}.$$

REMARK. We remark here that one could have computed the C^∞-invariant subspaces $\mathcal{F}$ of $B_q^\alpha(K)$, *i.e.* $\phi\mathcal{F} \subset \mathcal{F}$ for all $\phi \in C^\infty$ (here we mean C^∞ in a neighborhood of K), by using trace theory and the fact that the C^∞-invariant subspaces of W_1^q are known by Corollary 6.4. When one just looks at the rationally invariant subspaces $B_q^\alpha(K)$, it is not clear on first examination that rationally invariant implies C^∞-invariant. As it turns out though, it does.

We mention that we can employ the density of Rat(K) in $B_q^\alpha(K)$ to compute the commutant of M_ζ on $B_q^\alpha(K)$ and thus generalize [20], Theorem 1.1. We say a function ϕ is a *multiplier* for $B_q^\alpha(K)$ if $\phi f \in B_q^\alpha(K)$ for all $f \in B_q^\alpha(K)$. An application of the closed graph theorem shows that if ϕ is a multiplier, then M_ϕ (multiplication by ϕ) defines a continuous operator on $B_q^\alpha(K)$. For more information on multipliers of Sobolev and Besov spaces, see [13].

PROPOSITION 7.6. *Let B be an operator on $B_q^\alpha(K)$ with $BM_\zeta = M_\zeta B$. Then $B = M_\phi$ for some multiplier ϕ of $B_q^\alpha(K)$.*

PROOF. Since B commutes with M_ζ, then B commutes with M_r, where $r \in$ Rat(K). Let $h = B(1)$ and note that $B(r) = BM_r 1 = rh$, for all $r \in$ Rat(K). If $f \in B_q^\alpha(K)$, choose a sequence of $\{r_n\} \subset$ Rat(K) with $r_n \longrightarrow f$ in $B_q^\alpha(K)$. We assume (by passing to a subsequence if necessary) that $r_n \longrightarrow f$ $\mathcal{H}_\alpha$-a.e on K. Now $B(r_n) \longrightarrow B(f)$ in $B_q^\alpha(K)$ and $B(r_n) = hr_n$ converges to hf $\mathcal{H}_\alpha$-a.e on K, so $B(f) = hf$ $\mathcal{H}_\alpha$-a.e, *i.e.* h is a multiplier on $B_q^\alpha(K)$. $\blacksquare$

8. More on codimension. Recall that if $1 < p < \infty$, $L_a^p(G) \subset \mathcal{M} \subset L_a^p(G \setminus K)$ (K an α-set) is invariant, and $\lambda \in G \setminus K$, then $(S-\lambda)|_{\mathcal{M}}$ is a semi-Fredholm operator and

$$-\operatorname{index}\big((S-\lambda)|_{\mathcal{M}}\big) = \dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big)$$

is constant on the components of $G \setminus K$ and is called the codimension of $\mathcal{M}$ on the component of $G \setminus K$. In this section, we will prove an interesting relationship between the codimension of $\mathcal{M}$ and the codimension of $\mathcal{F}_{\mathcal{M}}$ which will help us understand the complexity of the invariant subspaces of both the Bergman space and the Besov space. Our result is the following:

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THEOREM 8.1. *For* $1 < p < \infty$, $\lambda \in G \setminus K$, *and an invariant subspace* $L^p_a(G) \subset \mathcal{M} \subset L^p_a(G \setminus K)$,

$$\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = 1 + \dim\big(\mathcal{F}_{\mathcal{M}}/(z-\lambda)\mathcal{F}_{\mathcal{M}}\big).$$

PROOF. We leave it to the reader to verify that

$$(z-\lambda)\mathcal{M} \subset (z-\lambda)L_\lambda\mathcal{M} \subset \mathcal{M} \subset L_\lambda\mathcal{M}. \tag{8.1}$$

Noticing that $(z-\lambda)L_\lambda\mathcal{M} = \{f \in \mathcal{M} : f(\lambda) = 0\}$, we see that (since $1 \in \mathcal{M}$) $\mathcal{M} = (z-\lambda)L_\lambda\mathcal{M} + \mathbb{C}$, and hence

$$\dim\big(\mathcal{M}/(z-\lambda)L_\lambda\mathcal{M}\big) = 1. \tag{8.2}$$

From basic linear algebra and using (8.1) we have

$$\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = \dim\big(\mathcal{M}/(z-\lambda)L_\lambda\mathcal{M}\big) + \dim\big((z-\lambda)L_\lambda\mathcal{M}/(z-\lambda)\mathcal{M}\big), \tag{8.3}$$

which by (8.2) becomes

$$\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = 1 + \dim\big((z-\lambda)L_\lambda\mathcal{M}/(z-\lambda)\mathcal{M}\big). \tag{8.4}$$

We now use (8.1) again along with the fact that $S - \lambda$ is bounded below to get

$$\dim(L_\lambda\mathcal{M}/\mathcal{M}) = \dim\big((z-\lambda)L_\lambda\mathcal{M}/(z-\lambda)\mathcal{M}\big). \tag{8.5}$$

Using (8.4) and (8.5) we have

$$\dim\big(\mathcal{M}/(z-\lambda)\mathcal{M}\big) = 1 + \dim(L_\lambda\mathcal{M}/\mathcal{M}). \tag{8.6}$$

From basic linear algebra and the fact that the Cauchy transform C is invertible, we obtain

$$\begin{aligned}\dim(L_\lambda\mathcal{M}/\mathcal{M}) &= \dim\big(\mathcal{M}^\perp/(L_\lambda\mathcal{M})^\perp\big) = \dim(C\mathcal{M}^\perp/C(L_\lambda\mathcal{M})^\perp) \\ &= \dim\left(\frac{C\mathcal{M}^\perp/W_1^{q,0}(G\setminus K)}{C(L_\lambda\mathcal{M})^\perp/W_1^{q,0}(G\setminus K)}\right).\end{aligned}$$

Now use (7.2) and Proposition 7.2 to see the above is equal to

$$\dim(\mathcal{F}_{\mathcal{M}}/(z-\lambda)\mathcal{F}_{\mathcal{M}}).$$

Combine this with (8.6) and we are done. ∎

REMARK. As a consequence of this theorem, we can make the following interesting observation about the codimension in the classical harmonic Dirichlet space $B^1_2(\mathbb{T})$ (see [17] for further details). Recall from the introduction that given $n \in \mathbb{N} \cup \{\infty\}$, the invariant subspace

$$\mathcal{M}_n = \chi_{\mathbb{D}}\mathcal{N}_n + L^2_a(G)$$

($\mathcal{N}_n$ is an invariant subspace of $L^2_a(\mathbb{D})$ with codimension n, see [10] for a specific example) has $\dim(\mathcal{M}_n/z\mathcal{M}_n) = n+1$. Thus by the above formula, the invariant subspace $\mathcal{F}_{\mathcal{M}_n} \subset B^1_2(\mathbb{T})$ has $\dim(\mathcal{F}_{\mathcal{M}_n}/\zeta\mathcal{F}_{\mathcal{M}_n}) = n$ (see [22] for an explicit example) which is in stark contrast to the analytic Dirichlet space $\{f \in B^1_2(\mathbb{T}) : \hat{f}(n) = 0 \ \forall n < 0\}$ where the codimension of a non-trivial invariant subspace is always one [18].

9. **Weak-star closed subspaces.** When the index $p = \infty$ and $G \setminus K$ is connected, one can ask about the weak-star closed invariant subspaces $\mathcal{A}$ with

$$H^\infty(G) \subset \mathcal{A} \subset H^\infty(G \setminus K). \tag{9.1}$$

Here $H^\infty(U)$ is the algebra of bounded analytic functions on a domain U. We refer the reader to [23] for a review of the basic facts about the weak-star topology on $H^\infty(U)$. We do not know a complete characterization of these subspaces but we do want to make a few remarks concerning the complexity of this problem.

If E is a closed subset of K, then $\mathcal{A} = H^\infty(G \setminus E)$ certainly satisfies (9.1), but these are not all of them. Consider the following example:

EXAMPLE. For $p = 2$, let $E \subset K = [0, 1]$ be the dense quasi-closed set (as in the example following Proposition 6.1) for which

$$\mathcal{M}(E) = \overline{\bigcup_n L^2_a(G \setminus F_n)}^{L^2} \neq L^2_a(G \setminus K). \tag{9.2}$$

We claim that

$$\mathcal{A} \equiv \overline{\bigcup_n H^\infty(G \setminus F_n)}^{*} \neq H^\infty(G \setminus K)$$

and since E is dense in K, then $\mathcal{A} \neq H^\infty(G \setminus F)$ for any closed $F \subset K$. Here we let $\bar{X}^*$ denote the weak-star closure of a set $X \subset H^\infty(G \setminus K)$.

Let

$$\mathcal{A}^0 = \bigcup_n H^\infty(G \setminus F_n)$$

and notice that $\mathcal{A}^0 \subset \mathcal{M}(E)$. For each countable ordinal α define $\mathcal{A}^\alpha$ to be the linear manifold of functions in $H^\infty(G \setminus K)$ which are weak-star limits of sequences of functions in

$$\bigcup_{\beta<\alpha} \mathcal{A}^\beta.$$

By using the basic fact that a sequence of $H^\infty(G \setminus K)$ functions $\{f_n\}$ converges to f weak-star if and only if

$$\sup_n \sup_{z \in G \setminus K} |f_n(z)| < \infty$$

and $f_n \longrightarrow f$ pointwise, we see that $\mathcal{A}^\alpha \subset \mathcal{M}(E)$ for each countable ordinal α. It is a well known fact [4], p. 213, that there is a least countable ordinal α' such that $\mathcal{A}^{\alpha'} = \mathcal{A}^{\alpha'+1}$ and moreover $\mathcal{A}^{\alpha'}$ is the weak-star closure of $\mathcal{A}^0$, that is $\mathcal{A}^{\alpha'} = \mathcal{A}$. Thus $\mathcal{A} \subset \mathcal{M}(E)$.

Suppose $\mathcal{A} = H^\infty(G \setminus K)$, then since $H^\infty(G \setminus K)$ is dense in $L^2_a(G \setminus K)$ [2], Lemma 4, we have $\mathcal{M}(E) = L^2_a(G \setminus K)$ which is a contradiction of (9.2).

We finish with these open questions:

QUESTION 1. Are all the weak-star closed invariant subspaces $\mathcal{A}$ of the form (9.1) of the form

$$\overline{\bigcup_n H^\infty(G \setminus F_n)}^*$$

for some increasing sequence of closed sets $\{F_n\}$?

This problem seems difficult since if one tries to put the problem in the context of some appropriate Sobolev space, the space will lie in L^1. In the proof of the $p \geq 2$ case one uses weak compactness in L^p [19], a luxury not afforded us in L^1. Moreover, the appropriate capacity here seems to be the analytic capacity since $H^\infty(G \setminus E) = H^\infty(G)$ if and only if E has analytic capacity zero. The capacities C_q used above are subadditive which allows us to define such concepts as quasi-closed and develop some useful properties of quasi-closed sets. It is an open question as to whether or not the analytic capacity is subadditive, making a useful notion of quasi-closed difficult to define.

QUESTION 2. In (9.1), is there a difference between the weak-star closed invariant subspaces and the weak-star closed subalgebras?

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