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GENERALISED JORDAN-VON NEUMANN CONSTANTS AND UNIFORM NORMAL STRUCTURE

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We introduce a new geometric coefficient related to the Jordan-von Neumann constant. This leads to improved versions of known results and yields new ones on super-normal structure for Banach spaces.

1. INTRODUCTION

The notions of normal structure and uniform normal structure play an important role in metric fixed point theory (see Goebel and Kirk [**10**]). A number of Banach space properties have been shown to imply uniform normal structure. Some sufficient properties for a Banach space X to have uniform normal structure are:

(i) $J(X) < 3/2$ (see Gao and Lau [**6**]),
(ii) $R(X) > 0$ (see Gao [**5**]),
(iii) $C_{\mathrm{NJ}}(X) < 5/4$ (see Kato, Maligranda and Takahashi [**13**]), and
(iv) X is a u-space, a class of spaces that includes uniformly convex spaces and uniformly smooth spaces (see Gao and Lau [**6**]).

Recently, Kirk and Sims [**17**] introduced a new variant, ϕ-uniform normal structure, which lies strictly between normal structure and uniform normal structure.

In this paper we introduce a parameterised coefficient $C_{\mathrm{NJ}}(\cdot, X)$ generalising the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$. Utilising ultraproduct techniques, the coefficient $C_{\mathrm{NJ}}(\cdot, X)$ enables us to establish new sufficient conditions for a Banach space to have uniform normal structure. To achieve this, we first show that the coefficients $C_{\mathrm{NJ}}(\cdot, X)$ of the space X and $C_{\mathrm{NJ}}(\cdot, \widetilde{X})$ of its ultrapower $\widetilde{X}$ coincide. From this and some other new results, which also improve the number appearing in property (iii) from $5/4$ to $(3+\sqrt{5})/4$, we can apply the powerful ultraproduct technique to show that X has uniform normal structure whenever $C_{\mathrm{NJ}}(1, X) < 2$. An example of a Banach space X is given which has $C_{\mathrm{NJ}}(1, X) < 2$ and hence uniform normal structure, but for which neither (i) or (iii) apply. An exact determination of the coefficient $C_{\mathrm{NJ}}(\cdot, X)$ is obtained when X is a Hilbert space. More generally, a connection between $C_{\mathrm{NJ}}(\cdot, X)$ and the modulus of convexity δ_X is established. Finally, we investigate the constants $C_{\mathrm{NJ}}(\cdot, X)$ when X is a u-space. This leads to an alternative proof of (iv).

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2. Preliminaries

Throughout the paper we let X and X^* stand for a Banach space and its dual space, respectively. By a non-trivial Banach space X we shall mean that either X is a real space with dim X $\geqslant 2$, or a complex space with dim X $\geqslant 1$. We shall denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. For a sequence (x_n) in X, $x_n \xrightarrow{w} x$ stands for weak convergence to x. For $x \in X\backslash\{0\}$, let ∇_x denote the set of norm 1 supporting functionals at x. This is the subdifferential of the norm at the point x, which is nonempty by the Hanh-Banach Theorem.

We shall say that a nonempty weakly compact convex subset C of X has the *fixed point property* (*fpp* for short) if every nonexpansive mapping $T : C \to C$ has a fixed point (that is, there exists $x \in C$ such that $T(x) = x$). Recall that T is nonexpansive if $\|Tx - Ty\| \leqslant \|x - y\|$ for every $x, y \in C$. We shall say that X has the *fixed point property* (*fpp*) if every weakly compact convex subset of X has the *fpp*. Let A be a nonempty bounded set in X. The number $r(A) = \inf\{\sup_{y\in A} \|x - y\| : x \in A\}$ is called the *Chebyshev radius* of A. The number $\operatorname{diam} A = \sup_{x,y\in A} \|x - y\|$ is called the *diameter* of A. A Banach space X has *normal structure* if

$$r(A) < \operatorname{diam} A \tag{2.1}$$

for every bounded convex closed subset A of X with $\operatorname{diam} A > 0$. When (2.1) holds for every weakly compact convex subset A of X with $\operatorname{diam} A > 0$, we say X has *weak normal structure*. Normal structure and weak normal structure coincide if X is reflexive. A space X is said to have *uniform normal structure* if $\inf\big\{(\operatorname{diam} A)/\big(r(A)\big)\big\} > 1$, where the infimum is taken over all bounded convex closed subsets A of X with diam $A > 0$. Weak normal structure, as well as many other properties imply the fixed point property. Some relevant papers are Opial [**22**], Kirk [**16**], Sims [**24**], Garcia-Falset [**7**], and Gacia-Falset and Sims [**8**].

The *modulus of convexity of* X (see [**3, 4, 19, 20, 21**]) is the function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x - y\| \geqslant \varepsilon\right\}. \tag{2.2}$$

When X is non-trivial, we can deduce that

$$\begin{aligned}\delta_X(\varepsilon) &= \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x - y\| \geqslant \varepsilon\right\}\\ &= \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X, \|x - y\| = \varepsilon\right\}\\ &= \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in B_X, \|x - y\| = \varepsilon\right\}.\end{aligned}$$

If $\delta_X(1) > 0$, then X has uniform normal structure (see [**9**]).

The *modulus of smoothness* of X (see [**3, 4, 19, 20**]) is the function $\rho_X : [0,\infty) \to [0,\infty)$ defined by

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x, y \in S_X\right\} = \sup\left\{\frac{\tau\varepsilon}{2} - \delta_{X^*}(\varepsilon) : \varepsilon \in [0,2]\right\}. \tag{2.3}$$

A space X is called *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $0 < \varepsilon < 2$. It is called *uniformly smooth* if $\rho'_X(0) = \lim_{\tau\to 0}(\rho_X(\tau))/\tau = 0$. Uniformly convex spaces and uniformly smooth spaces are examples of u-spaces, where a space X is called a *u-space* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x, y \in S_X$,

$$\left\|\frac{x+y}{2}\right\| > 1-\delta \Rightarrow f(y) > 1-\varepsilon \text{ for all } f \in \nabla_x. \tag{2.4}$$

The notion of u-spaces was introduced by Lau [**18**]. Examples of uniformly convex spaces are the spaces $L^p(\Omega)$ where Ω is a measure space such that $L^p(\Omega)$ is at least two dimensional and $1 < p < \infty$.

A Banach space X is called *uniformly nonsquare* provided that there exists $\delta > 0$ such that if $x, y \in S_X$, then $\|x+y\|/2 \leqslant 1-\delta$ or $\|x-y\|/2 \leqslant 1-\delta$. Uniformly nonsquare spaces are superreflexive (see James [**11**]). Every u-space is uniformly nonsquare (see Lau [**18**]), hence, it is superreflexive.

The Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of a Banach space X is defined by

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ not both zero}\right\} = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x \in S_X, y \in B_X\right\}. \tag{2.5}$$

REMARK 2.1. We collect together some properties of the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ (see [**2, 12, 13, 14, 15, 25**]):

(1) $1 \leqslant C_{\mathrm{NJ}}(X) \leqslant 2$.
(2) X is a Hilbert space if and only if $C_{\mathrm{NJ}}(X) = 1$.
(3) $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(X^*)$.
(4) X is uniformly nonsquare if and only if $C_{\mathrm{NJ}}(X) < 2$ and this happens if and only if $\delta_X(\varepsilon) > 0$ for some $\varepsilon \in (0,2)$.
(5) If $C_{\mathrm{NJ}}(X) < 5/4$ then X, as well as its dual X^*, have uniform normal structure, and hence both X and X^* have the fixed point property.

One technique used in this paper is the "ultraproduct" technique. We refer to Askoy and Khamsi [1] and Sims [23] for a complete discussion on the topic. However, let us briefly recall the construction of an ultrapower of a Banach space X. As a first step we consider the space $l_\infty(X)$ consisting of all bounded sequences (x_n) of elements of X. The norm in $l_\infty(X)$ is given by the formula $\|(x_n)\| = \sup_{n\in\mathbb{N}} \|x_n\|$, where $\mathbb{N}$ is the set of positive integers. Now, let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$. The set $\mathcal{N} = \{(x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}$ is a closed linear subspace of $l_\infty(X)$. Here, $\lim_{\mathcal{U}}$ stands for the limit over the ultrafilter $\mathcal{U}$. The *ultrapower* $\widetilde{X}$ of X with respect to $\mathcal{U}$ is defined to be the quotient space $l_\infty(X)/\mathcal{N}$. By $\widetilde{x}$ we denote the equivalent class of $x = (x_n)$. From the definition of the quotient norm, we can derive the following canonical formula $\|\widetilde{x}\| = \lim_{\mathcal{U}} \|x_n\|$. Identifying an element $x \in X$ with the equivalence class of the constant sequence $(x, x, \dots)$, we can treat X as a subspace of $\widetilde{X}$. In what follow, we shall consider only non-trivial ultrafilters on the set of positive integers. Under this setting, the ultrapower $\widetilde{X}$ is finitely representable in X. Consequently, $\widetilde{X}$ inherits all finite-dimensional geometrical properties of X.

DEFINITION 2.2: Let $\mathcal{P}$ be a Banach space property. We say that a Banach space X has the property *super-$\mathcal{P}$* if every Banach space finitely representable in X has property $\mathcal{P}$.

THEOREM 2.3. (See [1, Theorem 3.5].) *Let X and Y be Banach spaces and suppose that Y is finitely representable in X. Then there is an ultrafilter $\mathcal{U}$ on the set $\mathbb{N}$ such that Y is isometrically isomorphic to a subspace of $\widetilde{X}$.*

We remark that when the property $\mathcal{P}$ is hereditary: that is, any subspace of a space with $\mathcal{P}$ also has $\mathcal{P}$, one has the following stronger conclusion.

COROLLARY 2.4. (See [1].) *Let $\mathcal{P}$ be a Banach space property which is inherited by subspaces. Then a Banach space X has super-$\mathcal{P}$ if and only if every ultrapower $\widetilde{X}$ of X has $\mathcal{P}$.*

THEOREM 2.5. (See [1].) *Let X be a Banach space. If X has super-normal structure, then X has uniform normal structure.*

3. Results

Let us begin with our generalisation of the Jordan-von Neumann constant. For $a \geqslant 0$ define,

$$C_{\mathrm{NJ}}(a, X) = \sup\left\{\frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} : x, y, z \in X \text{ not all zero and } \|y-z\| \leqslant a\|x\|\right\}$$

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$$= \sup\left\{\frac{\|x+y\|^2+\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2} : x,y,z \in B_X \text{ not all zero and } \|y-z\| \leqslant a\|x\|\right\}$$

$$= \sup\left\{\frac{\|x+y\|^2+\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2} : x,y,z \in B_X \text{ of which at least one belongs to } S_X \text{ and } \|y-z\| \leqslant a\|x\|\right\}.$$

REMARK 3.1.

(1) Obviously, $C_{\mathrm{NJ}}(0,X) = C_{\mathrm{NJ}}(X)$ (see (2.5)).
(2) $C_{\mathrm{NJ}}(a,X)$ is a nondecreasing function with respect to a.
(3) If $C_{\mathrm{NJ}}(a,X) < 2$, for some $a \geqslant 0$, then $C_{\mathrm{NJ}}(X) < 2$ and consequently X is uniformly nonsquare (see Remark 2.1(4)).
(4) $1 + \left(4a/4 + a^2\right) \leqslant C_{\mathrm{NJ}}(a,X) \leqslant 2$ for all $a \geqslant 0$ and $C_{\mathrm{NJ}}(a,X) = 2$ for all $a \geqslant 2$.

To see that (4) is true, we begin by proving the left inequality. For this, we take any $x \in S_X$ and put $y = (a/2)x = -z$. We then have $y - z = ax$ and so,

$$C_{\mathrm{NJ}}(a,X) \geqslant \frac{\|x+y\|^2+\|x-z\|^2}{2\|x\|^2+\|y\|^2+\|z\|^2} = \frac{(1+(a/2))^2\|x\|^2+(1+(a/2))^2\|x\|^2}{2\|x\|^2+2(a^2/4)\|x\|^2}$$
$$= \frac{2(1+(a/2))^2}{2(1+(a^2/4))} = \frac{4+4a+a^2}{4+a^2} = 1 + \frac{4a}{4+a^2}.$$

Next, we show that $C_{\mathrm{NJ}}(a,X) \leqslant 2$. By the triangle inequality, we have

$$\begin{aligned}\|x+y\|^2+\|x-z\|^2 &\leqslant \left(\|x\|^2+2\|x\|\|y\|+\|y\|^2\right)+\left(\|x\|^2+2\|x\|\|z\|+\|z\|^2\right)\\ &\leqslant \left(2\|x\|^2+2\|y\|^2\right)+\left(2\|x\|^2+2\|z\|^2\right)\\ &= 4\|x\|^2+2\|y\|^2+2\|z\|^2,\end{aligned}$$

from which it is clear that $C_{\mathrm{NJ}}(a,X) \leqslant 2$. Finally, we observe that the function $a \mapsto 1 + \left(4a/4+a^2\right)$ is strictly increasing on $[0,2]$ and attains its maximum of 2 at $a = 2$. It follows that $C_{\mathrm{NJ}}(a,X) = 2$ for all $a \geqslant 2$.

EXAMPLES 3.2. (1) ($l_\infty - l_1$ norm) Let $X = \mathbb{R}^2$ be equipped with the norm defined by

$$\|x\| = \begin{cases} \|x\|_\infty & \text{if } x_1x_2 \geqslant 0,\\ \|x\|_1 & \text{if } x_1x_2 \leqslant 0.\end{cases}$$

Take $x = (1,1), y = (0,1)$ and $z = (-1,0)$. Then we have $y - z = (1,1) = x$ and $\|x+y\| = \|(1,2)\|_\infty = 2$, $\|x-z\| = \|(2,1)\|_\infty = 2, \|z\| = 1$. So $2 = (4+4)/4$

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$= \left(\|x+y\|^2 + \|x-z\|^2\right)/\left(2\|x\|^2 + \|y\|^2 + \|z\|^2\right) \leqslant C_{\mathrm{NJ}}(1,X) \leqslant 2$. Hence $C_{\mathrm{NJ}}(1,X) = 2$. It is not difficult to see that $\delta_X(\varepsilon) = \max\{0, (\varepsilon-1)/2\}$ and so $\delta_X(1) = 0$. We shall shortly see (Remark 3.12(1)) that this implies $C_{\mathrm{NJ}}(0,X) \geqslant 5/4$, however, we do not know its exact value. This example shows that sometimes it is easy to compute $C_{\mathrm{NJ}}(a,X)$ at some point $a \in (0,2)$, but not at $a = 0$.

(2) Let $1 < p < 2$ and let the norm on $X = \mathbb{R}^2$ now be defined by

$$\|x\| = \begin{cases} \|x\|_1 & \text{if } x_1x_2 \geqslant 0, \\ \|x\|_p & \text{if } x_1x_2 \leqslant 0. \end{cases}$$

Under this norm, it can be shown that $\delta_X(1) = 0$, $C_{\mathrm{NJ}}(X) = 1 + 2^{2/p-2}$, $J(X) \geqslant 2^{1/p}$ and $C_{\mathrm{NJ}}(1,X) < 2$, where James' nonsquare constant $J(X)$ is defined by $J(X) = \sup\left\{\min\{\|x+y\|, \|x-y\} : x, y \in S_X\right\}$. The verification that $C_{\mathrm{NJ}}(1,X) < 2$ follows by an argument similar to that given later in the proof of Theorem 3.15. We shall shortly see that all spaces X with $C_{\mathrm{NJ}}(1,X) < 2$ have uniform normal structure (Corollary 3.7). This example also reveals that we may have $C_{\mathrm{NJ}}(X)$ close to 2 but still have uniform normal structure (also see the observation given later at the beginning of Remark 3.16).

These examples show that information on $C_{\mathrm{NJ}}(a,X)$ for general a proves to be useful. We note in passing that $C_{\mathrm{NJ}}\left(1, l_2(X)\right) < 2$ whenever $C_{\mathrm{NJ}}(1,X) < 2$, where $l_2(X)$ is the space of sequences (x_n) of elements of X for which the sequence of norms $\left(\|x_n\|\right)$ is in l_2, with the norm of (x_n) defined to be the l_2-norm of $\left(\|x_n\|\right)$.

We aim to show that the generalised Jordan-von Neumann constants $C_{\mathrm{NJ}}(a,X)$ of the space X and $C_{\mathrm{NJ}}\left(a, \widetilde{X}\right)$ of its ultrapower coincide. Before that we need to establish the continuity of the function $C_{\mathrm{NJ}}(\cdot, X)$.

Proposition 3.3. *$C_{\mathrm{NJ}}(\cdot,X)$ is a continuous function on $[0,\infty)$.*

Proof: We have already noted that $C_{\mathrm{NJ}}(\cdot,X)$ is nondecreasing, thus suppose that for some $a > 0$,

$$\sup_{b<a} C_{\mathrm{NJ}}(b,X) = \alpha < \beta < \gamma = \inf_{b>a} C_{\mathrm{NJ}}(b,X).$$

Choose $\gamma_n \downarrow a$ and $x_n, y_n, z_n \in B_X$ of which at least one belongs to S_X and such that $\|y_n - z_n\| = \gamma_n\|x_n\|$ and $g(x_n,y_n,z_n) \geqslant \beta$ for all $n \in \mathbb{N}$. Here $g(x,y,z) = \left(\|x+y\|^2 + \|x-z\|^2\right)/\left(2\|x\|^2 + \|y\|^2 + \|z\|^2\right)$. Choose $\eta_n \downarrow 1$ such that $\gamma_n/\eta_n < a$ for all n. Thus, $g(\eta_n x_n, y_n, z_n) = g\left(x_n, (y_n/\eta_n), (z_n/\eta_n)\right) \leqslant \alpha$ for all $n \in \mathbb{N}$. Take a subsequence (n') of (n) such that all the sequences

$$\|x_{n'} + y_{n'}\|, \|x_{n'} - z_{n'}\|, \|x_{n'}\|, \|y_{n'}\| \text{ and } \|z_{n'}\|$$

converge. As $\|x_n + w\| - (\eta_n - 1)\|x_n\| \leqslant \|\eta_n x_n + w\| \leqslant \|x_n + w\| + (\eta_n - 1)\|x_n\|$ for any $w \in X$ and $\eta_n \to 1$, we have $\lim_{n'} \|\eta_{n'}x_{n'} + y_{n'}\| = \lim_{n'} \|x_{n'} + y_{n'}\|$ and $\lim_{n'} \|\eta_{n'}x_{n'}$

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$- z_{n'}\| = \lim_{n'} \|x_{n'} - z_{n'}\|$. Consequently, $\beta - \alpha \leqslant g(x_{n'}, y_{n'}, z_{n'}) - g(\eta_{n'} x_{n'}, y_{n'}, z_{n'}) \to 0$, a contradiction. This finishes the proof when $a > 0$.

For $a = 0$, given $\varepsilon > 0$ we take a triple (x_n, y_n, z_n) in B_X^3 with at least one of x_n, y_n, z_n belonging to S_X, $\|y_n - z_n\| = \alpha_n \|x_n\|$, $\alpha_n \downarrow 0$, and

$$C_{\mathrm{NJ}}(0+, X) - \varepsilon := \inf_{a>0} C_{\mathrm{NJ}}(a, X) - \varepsilon < \lim_{n\to\infty} g(x_n, y_n, z_n).$$

Put $\varepsilon_n = 4\alpha_n + \alpha_n^2$ and $\gamma_n = \alpha_n \|x_n\| (\|y_n\| - \alpha_n \|x_n\|)$. Thus $\varepsilon_n, \gamma_n \to 0$. Passing through subsequences if necessary, we may assume that $\lim_{n\to\infty} (\|x_n\|^2 + \|y_n\|^2) = b$ exists. By the choice of (x_n, y_n, z_n) we see that $b \neq 0$. Next we observe that, for all large n,

$$\begin{aligned} g(x_n, y_n, z_n) &\leqslant \frac{\|x_n + y_n\|^2 + \|x_n - y_n\|^2 + \varepsilon_n}{2\|x_n\|^2 + 2\|y_n\|^2 - \gamma_n} \\ &\leqslant g(x_n, y_n, y_n) + \frac{\varepsilon_n + \gamma_n g(x_n, y_n, y_n)}{2\|x_n\|^2 + 2\|y_n\|^2 - \gamma_n} \\ &\leqslant C_{\mathrm{NJ}}(X) + \frac{\varepsilon_n + \gamma_n C_{\mathrm{NJ}}(X)}{2\|x_n\|^2 + 2\|y_n\|^2 - \gamma_n}. \end{aligned}$$

Thus $C_{\mathrm{NJ}}(0+, X) - \varepsilon < C_{\mathrm{NJ}}(X) \leqslant C_{\mathrm{NJ}}(0+, X)$ for all $\varepsilon > 0$. Therefore $C_{\mathrm{NJ}}(0+, X) = C_{\mathrm{NJ}}(X)$ which implies that $C_{\mathrm{NJ}}(\cdot, X)$ is continuous at 0. Hence the continuity of $C_{\mathrm{NJ}}(\cdot, X)$ is established. ∎

We are now ready to obtain an important tool.

Corollary 3.4. $C_{\mathrm{NJ}}(a, X) = C_{\mathrm{NJ}}(a, \widetilde{X})$.

Proof: Clearly, $C_{\mathrm{NJ}}(a, X) \leqslant C_{\mathrm{NJ}}(a, \widetilde{X})$. To show $C_{\mathrm{NJ}}(a, X) \geqslant C_{\mathrm{NJ}}(a, \widetilde{X})$, let $\delta > 0$, $\alpha \in [0, a]$ and suppose $\widetilde{x}, \widetilde{y}, \widetilde{z} \in \widetilde{X}$ not all of which are zero and for which $\|\widetilde{y} - \widetilde{z}\| = \alpha \|\widetilde{x}\|$. If $\widetilde{x} = 0$, then $g(\widetilde{x}, \widetilde{y}, \widetilde{z}) = 1 \leqslant C_{\mathrm{NJ}}(a, X)$. If $\widetilde{x} \neq 0$, choose $\varepsilon > 0$ such that $\varepsilon < \delta \|\widetilde{x}\|$. Since

$$c := \frac{\|\widetilde{x} + \widetilde{y}\|^2 + \|\widetilde{x} - \widetilde{z}\|^2}{2\|\widetilde{x}\|^2 + \|\widetilde{y}\|^2 + \|\widetilde{z}\|^2} = \lim_{\mathcal{U}} \frac{\|x_n + y_n\|^2 + \|x_n - z_n\|^2}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2} := \lim_{\mathcal{U}} c_n,$$

the set $\{n \in \mathbb{N} : |c_n - c| < \delta \text{ and } \|y_n - z_n\| \leqslant \alpha \|x_n\| + \varepsilon < (\alpha + \delta)\|x_n\|\}$ belongs to $\mathcal{U}$. In particular,

$$\begin{aligned} c &< g(x_n, y_n, z_n) + \delta \\ &\leqslant C_{\mathrm{NJ}}(a + \delta, X) + \delta \qquad \text{for some } n. \end{aligned}$$

The inequality $C_{\mathrm{NJ}}(a, \widetilde{X}) \leqslant C_{\mathrm{NJ}}(a, X)$ follows from the arbitrariness of δ and the continuity of $C_{\mathrm{NJ}}(\cdot, X)$. ∎

This result also follows from the fact that the parameterised Jordan-von Neumann constant is finitely determined.

The following Lemma is a modification of [**6**, Lemma 2.3].

Lemma 3.5. *Let X be a Banach space without weak normal structure, then for any $0 < \varepsilon < 1$ and each $1/2 < r \leqslant 1$, there exist $x_1 \in S_X$ and $x_2, x_3 \in rS_X$ satisfying*

(i) $x_2 - x_3 = ax_1$ *with* $|a - r| < \varepsilon$,

(ii) $\|x_1 - x_2\| > 1 - \varepsilon$, *and*

(iii) $\|x_1 + x_2\| > (1 + r) - \varepsilon, \|x_3 + (-x_1)\| > (3r - 1) - \varepsilon$.

Proof: Put $\eta = \min\{(\varepsilon/12r), 2 - (1/r)\}$, and let z_n be a sequence in S_X with $z_n \overset{w}{\to} 0$ and

$$1 - \eta < \|z_{n+1} - z\| < 1 + \eta$$

for sufficiently large n and for any $z \in \operatorname{co}\{z_k\}_{k=1}^n$. Take $n_0 \in \mathbb{N}$, $y \in \operatorname{co}\{z_n\}_{n=1}^{n_0}$ and a norm 1 supporting functional f of z_1 such that

$$\|y\| < \eta,\ \left|\langle f, z_{n_0}\rangle\right| < \eta,\ 1 - \eta < \|z_{n_0} - z_1\|, \left\|z_{n_0} - \frac{z_1}{2}\right\| < 1 + \eta,$$

and

$$\left\|\frac{z_1 - z_{n_0}}{\|z_1 - z_{n_0}\|} - z_{n_0}\right\| > 2 - 3\eta.$$

Put $x_1 = (z_1 - z_{n_0})/(\|z_1 - z_{n_0}\|)$, $x_2 = rz_1$ and $x_3 = rz_{n_0}$. We show that (i), (ii) and (iii) hold. We first note that $x_2 - x_3 = r(z_1 - z_{n_0}) = r\|z_1 - z_{n_0}\|x_1$. Observe that $1 - \eta < \|z_1 - z_{n_0}\| < 1 + \eta$, so $\left|r\|z_1 - z_{n_0}\| - r\right| < r\eta < \varepsilon$, hence (i) holds. Next, since $1/2 < r \leqslant 1$,

$$\left|r\left(1 + \|z_1 - z_{n_0}\|\right) - 1\right| = r\left(1 + \|z_1 - z_{n_0}\|\right) - 1 < r(2 + \eta) - 1 = (2r - 1) + r\eta.$$

This implies

$$\begin{aligned}
\|x_1 - x_2\| &= \left\|rx_1 + (1 - r)x_1 - r\|z_1 - z_{n_0}\|x_1 - rz_{n_0}\right\| \\
&\geqslant r\|x_1 - z_{n_0}\| - \left|1 - r - r\|z_1 - z_{n_0}\|\right| \\
&> r(2 - 3\eta) - (2r - 1) - r\eta \\
&= 2r - 3r\eta - 2r + 1 - r\eta \\
&> 1 - \varepsilon.
\end{aligned}$$

Thus (ii) follows.

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To verify (iii) we first note the estimate $\|rz_1 - rz_{n_0} - x_1\| = \Big\|(1-r)x_1 + r\big(x_1 - (z_1 - z_{n_0})\big)\Big\| \leqslant (1-r) + r\eta < (1-r) + r\eta$. Using this we have,

$$\begin{aligned}
\|x_1 - x_3\| &= \|x_1 - rz_{n_0}\| \\
&\geqslant \big\|rz_{n_0} - (rz_1 - rz_{n_0})\big\| - \|rz_1 - rz_{n_0} - x_1\| \\
&\geqslant 2r\Big\|z_{n_0} - \frac{z_1}{2}\Big\| - (1-r) - r\eta \\
&> 2r - 2r\eta - (1-r) - r\eta \\
&> (3r-1) - \varepsilon.
\end{aligned}$$

We now estimate $\|x_1 + x_2\|$. From the definition of f, we have

$$\begin{aligned}
\|x_1 + x_2\| \geqslant \langle f, x_1 + rz_1\rangle &= r + \langle f, x_1\rangle \\
&= r + \frac{\langle f, z_1\rangle - \langle f, z_{n_0}\rangle}{\|z_1 - z_{n_0}\|} \\
&> r + \frac{1-\eta}{1+\eta} \\
&= (r+1) - \frac{2\eta}{1+\eta} \\
&> (r+1) - \varepsilon.
\end{aligned}$$

The proof of the Lemma is now complete. ∎

We now obtain sufficient conditions for X to have uniform normal structure, the second of which improves [**13**, Corollary 4] which states that "A Banach space X with $C_{\mathrm{NJ}}(X) < 5/4$ has uniform normal structure."

Theorem 3.6. *Let X be a Banach space. If*

$$C_{\mathrm{NJ}}(r, X) < \frac{(1+r)^2 + (3r-1)^2}{2(1+r^2)}, \quad \text{for some } r \in \left(\frac{1}{2}, 1\right],$$

or

$$C_{\mathrm{NJ}}(0, X) < \frac{3+\sqrt{5}}{4},$$

then X has uniform normal structure.

Proof: It suffices to show that these conditions imply X has normal structure. As then, by Corollary 3.4, it follows that $\widetilde{X}$ also has normal structure, so X has super-normal structure, by Corollary 2.4, and hence X has uniform normal structure by Theorem 2.5.

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For the case $C_{\mathrm{NJ}}(r,X) < \left((1+r)^2+(3r-1)^2\right)/\left(2(1+r^2)\right)$ we first observe that from Remark 3.1(3), X is uniformly nonsquare and so in turn is reflexive. Thus, normal structure and weak normal structure coincide. It then suffices to prove that X has weak normal structure.

By the continuity of $C_{\mathrm{NJ}}(\cdot,X)$, $C_{\mathrm{NJ}}(r',X) < \left((1+r)^2+(3r-1)^2\right)/\left(2(1+r^2)\right)$ for some $r'>r$. Choose $m\in\mathbb{N}$ such that $r+(1/m)\leqslant r'$. Suppose X does not have weak normal structure. By Lemma 3.5 there exist $x_n\in S_X$ and $y_n,z_n\in rS_X$ such that, for each $n\in\mathbb{N}$,

$$y_n-z_n=\alpha_n x_n \text{ with } |\alpha_n-r|<\frac{1}{n+m},$$
$$\|x_n-y_n\|^2>\left(1-\frac{1}{n+m}\right)^2,\ \|x_n+y_n\|^2>\left(1+r-\frac{1}{n+m}\right)^2,$$

and

$$\|x_n-z_n\|^2>\left((3r-1)-\frac{1}{n+m}\right)^2.$$

Observe that $\|y_n-z_n\|=\alpha_n<r+(1/n+m)<r+(1/m)\leqslant r'$ and

$$\liminf_{n\to\infty}\|x_n+y_n\|^2\geqslant(1+r)^2 \text{ and } \liminf_{n\to\infty}\|x_n-z_n\|^2\geqslant(3r-1)^2.$$

Thus

$$\begin{aligned}\text{(3.1)}\qquad \frac{(1+r)^2+(3r-1)^2}{2(1+r^2)} &\leqslant \liminf_{n\to\infty}\frac{\|x_n+y_n\|^2+\|x_n-z_n\|^2}{2\|x_n\|^2+\|y_n\|^2+\|z_n\|^2}\\ &\leqslant C_{\mathrm{NJ}}(r',X)\\ &<\frac{(1+r)^2+(3r-1)^2}{2(1+r^2)}.\end{aligned}$$

This contradiction shows that X must have weak normal structure as desired.

For the case $C_{\mathrm{NJ}}(0,X) < (3+\sqrt{5})/4$, we first show that $C_{\mathrm{NJ}}(0,X) < \left((1+r)^2+1\right)/\left(2(1+r^2)\right)$ for any $r\in(1/2,1]$. The proof of this is the same as above except that here we consider the lower bound $\left(1-(1/m+n)\right)^2$ for $\|x_n-y_n\|^2$ instead of the one for $\|x_n-z_n\|^2$. Thus (3.1) becomes

$$\frac{(1+r)^2+1}{2(1+r^2)}\leqslant\liminf_{n\to\infty}\frac{\|x_n+y_n\|^2+\|x_n-y_n\|^2}{2(\|x_n\|^2+\|y_n\|^2)}\leqslant C_{\mathrm{NJ}}(0,X)<\frac{(1+r)^2+1}{2(1+r^2)}$$

which is impossible. The conclusion now follows by noting that $\left((1+r)^2+1\right)/\left(2(1+r^2)\right)$ achieves a maximum of $(3+\sqrt{5})/4$ at $r=(\sqrt{5}-1)/2\in(1/2,1]$. ∎

NOTE. The restriction $r \in (1/2, 1]$ in the first inequality of Theorem 3.6 reflects the fact that for $r \leqslant 1/2$ the right hand side is less than or equal to one. Indeed, from Remark 3.1(4) the first inequality in Theorem 3.6 is only possible if

$$\frac{(1+r)^2 + (3r-1)^2}{2(1+r^2)} \geqslant 1 + \frac{4r}{4+r^2},$$

that is, if $r \in (r_1, 1]$ where $r_1 \doteq 0.87$ is the real root of the polynomial $2x^3 - 3x^2 + 8x - 6$. Thus, Theorem 3.6 only gives us information near $r = 1$.

COROLLARY 3.7. *Let X be a Banach space. If $C_{\mathrm{NJ}}(1, X) < 2$, then X has uniform normal structure.*

PROOF: This follows immediately from Theorem 3.6 with $r = 1$. ∎

Utilising Corollary 3.7, Tasena **[26]** has shown "$C_{\mathrm{NJ}}(a, X) < (1+a)^2/(1+a^2)$ for some $a \in (0, 1]$ implies X has uniform normal structure". This improvement of Theorem 3.6 is quite strong since

$$\frac{(1+a)^2}{1+a^2} > \max\left(1 + \frac{4a}{4+a^2}, \frac{(1+a)^2 + (3a-1)^2}{2(1+a^2)}\right) \text{ for } a \in (0, 1).$$

We now consider the case when X is a Hilbert space, thereby extending Remark 2.1(2).

THEOREM 3.8. *Let H be a Hilbert space. Then*

$$C_{\mathrm{NJ}}(a, H) = 1 + \frac{4a}{4+a^2}$$

for all $a \in [0, 2]$.

PROOF: Let $a \in [0, 2]$ and $x, y, z \in H$ with $x \neq 0$ and $\|y - z\| = \alpha\|x\|$ for some $\alpha \in [0, a]$. Then

$$\begin{aligned}
\frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} &\leqslant \frac{2\|x\|^2 + \|y\|^2 + \|z\|^2 + 2\|x\|\|y-z\|}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \\
&\leqslant 1 + \frac{2\alpha\|x\|^2}{2\|x\|^2 + (\|y-z\|^2 + \|y+z\|^2)/2} \\
&\leqslant 1 + \frac{2\alpha\|x\|^2}{2\|x\|^2 + \|y-z\|^2/2} \\
&= 1 + \frac{4\alpha}{4+\alpha^2} \\
&\leqslant 1 + \frac{4a}{4+a^2}.
\end{aligned}$$

Thus, by Remark 3.1(4), $C_{\mathrm{NJ}}(a, H) = 1 + (4a)/(4+a^2)$. ∎

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QUESTION. Is X a Hilbert space if $C_{\mathrm{NJ}}(a,X) = 1+(4a)/(4+a^2)$ for some $a \in (0,2)$?

Theorem 3.8 and Corollary 3.7 give us the following

COROLLARY 3.9. *Every Hilbert space has uniform normal structure.*

We now give a connection between the constant $C_{\mathrm{NJ}}(\cdot,X)$ and the modulus of convexity $\delta_X(\cdot)$ (see (2.2)).

THEOREM 3.10. *Let X be a Banach space, $\varepsilon \in [0,2]$, and $\beta \geqslant 0$. If $C_{\mathrm{NJ}}(\beta,X) < \left(4+(\varepsilon-\beta)^2\right)/\left(3+(\beta+1)^2\right)$, then $\delta_X(\varepsilon) > 0$.*

PROOF: Suppose $\delta_X(\varepsilon) = 0$, then there exist $x_n, y_n \in S_X$ such that $\|x_n - y_n\| = \varepsilon$ for all $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} \|x_n + y_n\| = 2$. Put $z_n = y_n - \beta x_n$. Then, for each $n \in \mathbb{N}$, we have $y_n - z_n = \beta x_n$, $\|z_n\| = \|y_n - \beta x_n\| \leqslant 1+\beta$ and $\|x_n - z_n\| \geqslant \big|\|x_n - y_n\| - \|\beta x_n\|\big| = |\varepsilon - \beta|$. Thus

$$\frac{4+(\varepsilon-\beta)^2}{3+(\beta+1)^2} \leqslant \liminf_{n\to\infty} \frac{\|x_n+y_n\|^2 + \|x_n - z_n\|^2}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2} \leqslant C_{\mathrm{NJ}}(\beta,X) < \frac{4+(\varepsilon-\beta)^2}{3+(\beta+1)^2},$$

a contradiction. ∎

Note that Theorem 3.10 is applicable for all $\beta \in [0,\beta_1]$ where β_1 is the root of the equation

$$1 + \frac{4\beta}{4+\beta^2} = \frac{4+(\varepsilon-\beta)^2}{3+(1+\beta)^2}.$$

The above theorem immediately yields the following.

COROLLARY 3.11. *If, for $\varepsilon \in [0,2]$, $C_{\mathrm{NJ}}(0,X) < \left(4+\varepsilon^2\right)/4$, then $\delta_X(\varepsilon) > 0$. In particular, every Hilbert space is uniformly convex, that is, $\delta_X(\varepsilon) > 0$ for every $\varepsilon \in (0,2)$.*

REMARK 3.12.

(1) Corollary 3.11 shows that if $C_{\mathrm{NJ}}(X) < 5/4$, then $\delta_X(1) > 0$.
(2) $C_{\mathrm{NJ}}(0,X) < 2$ if and only if $C_{\mathrm{NJ}}(0,X) < \left(4+\varepsilon^2\right)/4$ for some $\varepsilon \in (0,2)$. Thus, this gives us a simpler proof of [**13**, Theorem 1] which states that "$C_{\mathrm{NJ}}(0,X) < 2$ if and only if X is uniformly nonsquare."
(3) Since $C_{\mathrm{NJ}}(0,X) = C_{\mathrm{NJ}}(0,X^*)$, the corresponding results in Theorem 3.6 and Corollary 3.11 hold for X^* as well.

QUESTION. Does the equality $C_{\mathrm{NJ}}(a,X) = C_{\mathrm{NJ}}(a,X^*)$ hold for $a \in (0,2]$?

COROLLARY 3.13. *If $C_{\mathrm{NJ}}(\cdot,X)$ is concave and $C_{\mathrm{NJ}}(a,X) < \left(3+\sqrt{5}+(5-\sqrt{5})a\right)/4$ for some $a \in [0,1]$, then X has uniform normal structure.*

PROOF: If $C_{\mathrm{NJ}}(1,X) < 2$, we are done by Corollary 3.7. Let $C_{\mathrm{NJ}}(1,X) = 2$ and suppose that X does not have uniform normal structure. Therefore $C_{\mathrm{NJ}}(0,X)$

$\geqslant (3+\sqrt{5})/4$ by Theorem 3.6. By the concavity of $C_{\mathrm{NJ}}(\cdot, X)$, we have for all a $\in [0,1]$,

$$C_{\mathrm{NJ}}(a,X) \geqslant (1-a)C_{\mathrm{NJ}}(0,X) + aC_{\mathrm{NJ}}(1,X) \geqslant \frac{3+\sqrt{5}+(5-\sqrt{5})a}{4},$$

a contradiction. ∎

QUESTION. Is Corollary 3.13 still valid if we drop the assuption of concavity?

REMARK 3.14. In the definition of a u-space (see (2.4)), we can replace x, y in S_X by $x, y \in B_X$. To see this, we first observe that, $\|x\| \geqslant \|x+y\| - \|y\|$. Thus,

$$\text{(3.2)} \qquad \text{if } x, y \in B_X \text{ and } \left\|\frac{x+y}{2}\right\| > 1-\delta \text{ for some } \delta > 0, \text{ then } \|x\| \geqslant 1-2\delta \text{ and } \|y\| \geqslant 1-2\delta.$$

From (3.2) if we put $x' = x/\|x\|$ and $y' = y/\|y\|$ we obtain

$$\text{(3.3)} \qquad \left\|\frac{x'+y'}{2}\right\| > 1-3\delta, \text{ whenever } \left\|\frac{x+y}{2}\right\| > 1-\delta.$$

Indeed, (3.3) follows from the fact that $\|x'-x\| < 2\delta$ and $\|y'-y\| < 2\delta$, together with the inequality

$$\|x'+y'\| \geqslant \|x+y\| - \|x'-x\| - \|y'-y\|.$$

Now, given any $\varepsilon > 0$, choose $\delta \in \big(0, (3\varepsilon)/4\big)$ so that for $x', y' \in S_X$,

$$\left\|\frac{x'+y'}{2}\right\| > 1-\delta \Rightarrow f(y') > 1-\frac{\varepsilon}{2} \text{ for all } f \in \nabla_{x'}.$$

Then, if $x, y \in B_X$, and $\|(x+y)/2\| > 1-(\delta/3)$, (3.3) implies that $\|(x'+y')/2\| > 1-\delta$ where $x' = x/\|x\|$ and $y' = y/\|y\|$. Note, by (3.2), that $\|y'-y\| < (2\delta)/3$. Fix $f \in \nabla_x = \nabla_{x'}$ and consider the inequalities

$$f(y) + \frac{\varepsilon}{2} > f(y) + \frac{2\delta}{3} \geqslant f(y) + \|y'-y\| \geqslant f(y) + f(y'-y) = f(y') > 1-\frac{\varepsilon}{2}.$$

Consequently, $f(y) > 1-\varepsilon$ as required.

THEOREM 3.15. *For $1 < p < \infty$, all $L^p(\Omega)$ spaces satisfy $C_{\mathrm{NJ}}\big(1, L^p(\Omega)\big) < 2$. Indeed, all u-spaces X have $C_{\mathrm{NJ}}(a,X) < 2$ for all $0 < a < 2$.*

PROOF: Suppose $C_{\mathrm{NJ}}(2-\delta, X) = 2$ for all sufficiently small $\delta > 0$. For one such δ choose $x_n, y_n, z_n \in B_X$ of which at least one belongs to S_X and such that

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$\|y_n - z_n\| \leqslant (2-\delta)\|x_n\|$ for each n and $g(x_n, y_n, z_n) \nearrow 2$. Consider

$$(3.4) \qquad \begin{aligned} g(x,y,z) &= \frac{\|x+y\|^2 + \|x-z\|^2}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \\ &\leqslant \frac{2\|x\|^2 + \|y\|^2 + \|z\|^2 + 2(\|x\|\|y\| + \|x\|\|z\|)}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \\ &= 1 + \frac{2(\|x\|\|y\| + \|x\|\|z\|)}{2\|x\|^2 + \|y\|^2 + \|z\|^2} \leqslant 2. \end{aligned}$$

This implies

$$\frac{2\|x_n\|\|y_n\| + 2\|x_n\|\|z_n\|}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2} \to 1$$

and then

$$\frac{(\|x_n\| - \|y_n\|)^2 + (\|x_n\| - \|z_n\|)^2}{2\|x_n\|^2 + \|y_n\|^2 + \|z_n\|^2} \to 0.$$

Since, for each n, one of x_n, y_n, z_n belongs to S_X, we must have $\|x_{n'}\|, \|y_{n'}\|, \|z_{n'}\| \to 1$ for some subsequence (n') of (n). From this, together with (3.4), one can conclude that

$$(3.5) \qquad \|x_{n'} + y_{n'}\|, \|x_{n'} - z_{n'}\| \to 2.$$

Take $f_{n'} \in \nabla_{x_{n'}}$ for each n. Since X is a u-space, we have, by (3.5) and (2.4), $f_{n'}(x_{n'} - y_{n'}) \to 0$ and $f_{n'}(x_{n'} + z_{n'}) \to 0$. Therefore,

$$\begin{aligned} 2\|x_{n'}\| = 2f_{n'}(x_{n'}) &= f_{n'}(x_{n'} - y_{n'}) + f_{n'}(x_{n'} + z_{n'}) + f_{n'}(y_{n'} - z_{n'}) \\ &\leqslant f_{n'}(x_{n'} - y_{n'}) + f_{n'}(x_{n'} + z_{n'}) + \|y_{n'} - z_{n'}\| \\ &\leqslant f_{n'}(x_{n'} - y_{n'}) + f_{n'}(x_{n'} + z_{n'}) + 2 - \delta. \end{aligned}$$

Thus, $2 \leqslant 2 - \delta$ a contradiction. ∎

REMARK 3.16.

(1) In [2], it is shown that $C_{\mathrm{NJ}}(L^p) = 2^{(2/t)-1}$, for $1 \leqslant p \leqslant \infty$, where $t = \min\{p,q\}$ and $(1/p) + (1/q) = 1$. Thus, while $C_{\mathrm{NJ}}(L^p)$ is close to 2 for p large, or near 1, Theorem 3.15 still applies and says that for $1 < p < \infty$, all L^p spaces have uniform normal structure.

(2) As a measure of uniform nonsquareness, we say X is ε*-inquadrate* (ε-InQ), for $0 \leqslant \varepsilon \leqslant 2$, if for any sequences $(x_n), (y_n)$ in B_X,

$$\|x_n + y_n\| \to 2 \text{ implies } \limsup_{n\to\infty} \|x_n - y_n\| \leqslant \varepsilon.$$

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In **[26]**, Tasena introduces ε-u-spaces and ε-u-smooth spaces and proves that "all ε-u-spaces have $C_{\rm NJ}(2-\delta, X) < 2$ for all $\delta > 2\varepsilon$". He also observes that $\varepsilon - InQ$ spaces are ε-u-spaces.

(3) A long standing open problem is whether $C_{\rm NJ}(0, X) < 2$ implies the fixed point property. It now appears that $C_{\rm NJ}(1, X) < 2$ implies uniform normal structure which in turn implies the fpp. Concerning this open problem, it is interesting to ask what is the smallest $a \in (0,1)$ for which the fpp follows whenever $C_{\rm NJ}(a, X) < 2$.

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