(c) The Hayman–Wu theorem that (with the previous notation)

$$\operatorname{length}[\phi^{-1}(L \cap \Omega)] \leqslant \text{const.},$$

for any simply connected domain Ω and straight line L.

(d) The famous (or infamous) Brennan conjecture and the 'dandelion' construction.

In addition to all this, many other topics are considered: Bloch functions, BMO, extremal length, Schwarzian derivatives.

This book will surely be the standard reference for this topic for the foreseeable future. Indeed, one might almost say that it is the 'last word', except that such a carefully considered and well-presented book is bound to stimulate much further research in the subject.

The authors deserve our congratulations and our thanks.

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DOI:10.1017/S0013091507225015

Hirota, R. *The direct method in soliton theory* (translated from the Japanese and edited by A. Nagai, J. Nimmo and C. Gilson) (Cambridge University Press, 2004), xi + 200 pp., 0 521 83660 3 (hardback), £40.

The first recorded observation of a solitary wave was by John Scott Russell on the Union Canal near Edinburgh in 1834. (The solitary wave was recreated in 1995 on the canal's Scott Russell Aqueduct near Heriot-Watt University; a photograph of this event can be seen at www.ma.hw.ac.uk/solitons.) In 1895 Korteweg and de Vries showed that the solitary wave is described mathematically by a solution of the nonlinear evolution equation now known as the Korteweg–de Vries (KdV) equation. This equation, which describes shallow-water waves, is one of the simplest evolution equations in which the effects of nonlinearity and dispersion are balanced. With the advent of computers, nonlinear evolution equations could be solved numerically. In 1965 Zabusky and Kruskal carried out a numerical experiment on the KdV equation. They discovered that, when two or more solitary waves interact, each appeared to emerge from the nonlinear interaction intact and undistorted. They coined the word 'soliton' to denote this extraordinary type of solitary wave. In order to prove rigorously that solitons are not destroyed after their interaction, it was necessary to find exact solutions describing the interaction. This was achieved in 1967 by Gardner, Greene, Kruskal and Miura. They discovered the inverse scattering transform (IST) method that can be used to solve the initial-value problem for a restricted class of evolution equations. In the 1970s Hirota developed an ingenious method that is geared to finding multi-soliton solutions to evolution equations directly. Although the method is less general than the IST method in that it does not solve initial-value problems, it has the advantage of being applicable to a wider class of equations and is more straightforward. The method is now known as 'the direct method' or, outside Japan, 'Hirota's method'.

There are numerous books on various aspects of solitons. Although some describe Hirota's method, there has not been an introductory English-language book devoted to it. Apparently, when some of the western practitioners of the method found out that Hirota had written an introductory text in Japanese in 1992, they felt it should be translated into English. The result is this book. The translators admit that they have made minor changes to the original, but state that all such changes have the blessing of Professor Hirota.

In Chapter 1 the role of nonlinearity and dispersion in the formation of a solitary wave is discussed briefly. There follows a fascinating discussion of the motivation and chain of thought that led Hirota to the first key essential of the original direct method: the use of an appropriate dependent-variable transformation in order to convert a nonlinear evolution equation into a system of bilinear differential equations with respect to the new dependent variables. At first sight the bilinear equations look more complicated than the original nonlinear equation! However, the observation that the derivatives occur in special combinations leads to the second key step: the introduction of a binary differential operator, now known as the Hirota D-operator, the use of which collapses the bilinear equations into rather compact equations in terms of the Hirota derivatives. These ideas are illustrated by using three familiar evolution equations: the KdV equation, the modified KdV (mKdV) equation, and the nonlinear Schrödinger equation, and three examples of dependent-variable transformations: the rational, the logarithmic and the bi-logarithmic.

The third key step in the original formulation of the direct method is the observation that the bilinear equations may be solved by a perturbation method, the unusual feature being that, by virtue of the properties of the D-operator, it is possible to arrange for the perturbation expansions to truncate as finite series. This in turn leads to exact solutions of the original evolution equation. This procedure is illustrated by finding the two-soliton solution of the KdV equation.

Although there are several rather sophisticated developments of the original direct method, Hirota concentrates on the 'relatively straightforward' approach that regards bilinear equations arising from soliton equations as Pfaffian identities. In order to prepare the reader for this, Chapter 2 is a review of some relevant relationships between determinants and Pfaffians, useful determinantal and Pfaffian identities, and expansion, addition and derivative formulae for Pfaffians. The use of Maya and Young diagrams is introduced.

In Chapter 3 it is shown that there is a structure that is common to the bilinear form of several soliton equations that are regarded as fundamental in the mathematical sense, namely the KP, BKP, coupled KP, Toda lattice and Toda molecule equations. By regarding the bilinear dependent variable as a Wronskian determinant, a Grammian determinant or a Pfaffian, it is shown that each bilinear equation may be identified as a Pfaffian identity!

In Chapter 4 Bäcklund transformations are discussed. The bilinear form of the Bäcklund transformation associated with a soliton equation can be derived from the bilinear form of the equation by using appropriate 'exchange formulae' for the Hirota D-operator. The Bäcklund transformation is useful in soliton theory because it may be used to generate (i) the Lax pairs used in the IST method, (ii) new soliton equations, and (iii) Miura transformations. All these remarks are illustrated by considering the KdV equation. In this case the new equation is the mKdV equation and the Miura transformation expresses the KdV dependent variable in terms of the mKdV dependent variable. Subsequently, the KP, BKP and Toda equations are considered together with their 'modified' forms. By use of methods similar to those employed in Chapter 3, it is shown that the bilinear forms of the Bäcklund transformations are Pfaffian identities.

The brief afterword, in which there is a list of some topics not discussed in the book, together with the list of references, is a useful guide to possible further reading.

The translators/editors have done a magnificent job in making the text very clear to read. The only hint that this book is a translation is the occasional aside designated as 'snake's legs'; this is a literal translation for which there is no equivalent in mathematical texts in English! Although the book contains many involved formulae, Cambridge University Press have ensured that the printed pages appear uncluttered.

Hirota's stated aim is 'to inform the reader as briefly as possible about the beauty and conciseness of the mathematical rules underlying soliton equations'. Remarkably, he achieves this ambitious task while demanding of the reader no more than a knowledge of basic calculus and determinant theory.

This book gives a valuable insight into how Hirota discovered the direct method, and how it led to the use of Pfaffians in soliton theory. However, readers seeking practical examples of the solution process, the extraction of asymptotic states, the calculation of phase shifts, and so on should consult more appropriate textbooks or review papers.

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