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VARIANCE MINIMIZATION – RELATIONSHIP BETWEEN COMPLETION-TIME VARIANCE AND WAITING-TIME VARIANCE

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Abstract

The completion-time variance (CTV) and the waiting-time variance (WTV) are two performance measures which are commonly used in optimization of single-machine scheduling systems. This paper shows that when the number of jobs is large the two measures are nearly equivalent in a probabilistic environment.

1. Introduction

The model of scheduling jobs to minimize their completion-time variance was initially formulated by Merten and Muller [14] in 1972, motivated by the file-organization problem in a computing system, where it is desirable to provide uniform response times to users' requests to retrieve data files. It is known that many other optimal scheduling problems espouse the same mathematical model. These include just-in-time (JIT) production, commercial service systems, scheduling of data transmissions from a satellite to an earth station, and any other situations where a uniform treatment of jobs is desirable. For example, in a JIT manufacturing setting, a critical task is to minimize the earliness as well as the tardiness of job completion times from their due date. Another example is in commercial service systems, where delays in providing services are the major source of complaints, while getting too many jobs done well ahead of their due time would lead to unnecessary high cost and wastage of resources. It can be shown that, if the due date involved is also a decision variable that the decision maker wishes to determine, and if in the problem large deviations of job completion times from the due date are highly undesirable, then the problem can be formulated by a variance minimization model. The reason is that the optimal due

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date should be equal to the mean completion time of the optimal job schedule for the variance minimization model. For a comprehensive review on these problems, see Baker and Scudder [2] and Cheng and Gupta [4].

CTV and WTV are two commonly used performance measures involved in variance minimization studies. See, for example, Bagchi [1], Cai [3], De, Ghosh, and Wells [5], Eilon and Chowdhury [6], Merten and Muller [14], Schrage [15] and Vani and Raghavachari [16]. Merten and Muller have shown (Theorem H in [14]) that, if a sequence $\lambda = \{i_1, i_2, \ldots, i_n\}$ minimizes CTV, then its antithetical sequence $\lambda' = \{i_n, i_{n-1}, \ldots, i_1\}$ minimizes WTV, and *vice versa*. Nevertheless, λ and λ' are two solutions which are totally different in the order that they sequence the jobs. So far it is still unclear whether or not a solution is good for one measure even if it is the best with respect to the other measure. In fact, such a question relates to a fundamental issue – the equivalence of performance measures – in machine-scheduling studies. There have been some significant results which have identified the equivalence between a number of performance measures (see Rinnooy Kan [11], French [7], and Gerchak and Magazine [8]). However, it appears that the equivalence relationship between CTV and WTV still remains as an open problem.

This paper will study this problem within the framework of probabilistic analysis. We shall show that the two measures are approximately equivalent when the problem instances are randomly distributed. Theoretically, one of our main results is that, in such a 'random' situation, the relative error between CTV and WTV tends towards zero 'in probability' at a rate of order n^{-1} for any sequence λ as the number of jobs, n, increases to infinity. Through applying Merten and Muller's Theorem H, we shall further show that this result implies that, for any given λ and its antithetical sequence λ', both $\frac{|CTV(\lambda')-CTV(\lambda)|}{CTV(\lambda)}$ and $\frac{|WTV(\lambda)-WTV(\lambda')|}{WTV(\lambda')}$ tend to zero in probability at the rate $O(n^{-1})$.

This finding is significant, because it indicates that the quality (as measured by performance measure CTV or WTV) of a pair of sequences λ and λ' is actually nearly the same. This equivalence, together with the well-known result established by Theorem H of Merten and Muller [14] that λ' is optimal to WTV iff λ is optimal to CTV, suggests that a sequence minimizing one criterion (WTV or CTV) will also approximately minimize the other criterion (CTV or WTV) in a probabilistic environment. Therefore in a two-criteria problem in which both CTV and WTV are to be minimized simultaneously, it suffices to find an optimal solution to one criterion only, as such a solution will be also an approximate solution to the other criterion.

2. Problem formulation

We are given a set $\mathcal{N} = \{1, 2, \ldots, n\}, n > 1$, of independent and simultaneously available jobs which are to be processed nonpreemptively on a single machine. Job i requires a positive integer *processing time* p_i and is assigned a positive integer *weight* u_i, $\forall i \in \mathcal{N}$. The problem is to find a sequence to process the jobs so that the variance of job-completion times or the variance of job waiting times is minimized. To be precise, let Π be the set of all permutations of the first n integers, and let $\lambda = \{i_1, i_2, \ldots, i_n\} \in \Pi$ be a sequence in which integer i_k being at the k-th position denotes that job i_k is the k-th to be processed. Then, the problem is:

$$\min_{\lambda \in \Pi} \left\{ CTV(\lambda) = \frac{1}{U} \sum_{i=1}^{n} u_i (C_i - \bar{C})^2 \right\}, \tag{2.1}$$

or:

$$\min_{\lambda \in \Pi} \left\{ WTV(\lambda) = \frac{1}{U} \sum_{i=1}^{n} u_i (W_i - \bar{W})^2 \right\}, \tag{2.2}$$

where

$C_i = \sum_{i_k \le i} p_{i_k}$: the completion time of job i under λ,

$W_i = \sum_{i_k \le i-1} p_{i_k}$: the waiting time of job i under λ,

$\bar{C} = \frac{1}{U} \sum_{i=1}^{n} u_i C_i$: the mean completion time,

$\bar{W} = \frac{1}{U} \sum_{i=1}^{n} u_i W_i$: the mean waiting time,

and

$U = \sum_{i=1}^{n} u_i$: the sum of weights.

It may be shown that inserted idle time between jobs on the machine can reduce neither CTV nor WTV. Thus we assume that the machine starts processing from time zero without idleness until all jobs have been completed.

We are concerned with the relationship, in an average sense, between CTV and WTV. For this purpose, we consider throughout this paper that all the problem parameters are drawn randomly and independently. The relative errors between CTV and WTV under a given sequence $\lambda \in \Pi$ are defined as

$$r_c(\lambda) = \frac{|CTV(\lambda) - WTV(\lambda)|}{CTV(\lambda)} \tag{2.3}$$

or

$$r_w(\lambda) = \frac{|CTV(\lambda) - WTV(\lambda)|}{WTV(\lambda)}. \tag{2.4}$$

We shall prove that for any sequence $\lambda \in \Pi$, as $n \to \infty$, $r_c(\lambda)$ and $r_w(\lambda)$ tend to zero 'in probability' in the sense that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathrm{P}\{r_c(\lambda) > \epsilon\} = 0 \tag{2.5}$$

and

$$\lim_{n\to\infty} \mathrm{P}\{r_w(\lambda) > \epsilon\} = 0. \tag{2.6}$$

Furthermore, the convergence rates of $r_c(\lambda)$ and $r_w(\lambda)$ are of order n^{-1} in the sense of (3.3) of the next section.

Before we carry out the proof, let us now note two important implications of the results given by (2.3) - (2.6) above.

First, (2.5) and (2.6) show that $CTV(\lambda)$ is asymptotically equivalent to $WTV(\lambda)$ for all λ. As a result, an optimal solution to CTV should be approximately a good solution to WTV (and *vice versa*). Here we use a relative error to describe the goodness of an approximate solution. This is a common practice in analyzing the sub-optimality of approximate solutions. Some well-known examples in the scheduling literature include Karp [12], Lawler [13], Rinnooy Kan [10], Ibarra and Kim [9], to name just a few.

Secondly, since Theorem H of Merten and Muller [14] established that $CTV(\lambda) = WTV(\lambda')$ and $CTV(\lambda') = WTV(\lambda)$, where λ' and λ are mutually antithetical, it is not hard to see that (2.3) and (2.5) imply

$$\lim_{n\to\infty} \mathrm{P}\left\{\frac{|CTV(\lambda') - CTV(\lambda)|}{CTV(\lambda)} > \epsilon\right\} = 0, \tag{2.7}$$

while (2.4) and (2.6) imply

$$\lim_{n\to\infty} \mathrm{P}\left\{\frac{|WTV(\lambda) - WTV(\lambda')|}{WTV(\lambda')} > \epsilon\right\} = 0. \tag{2.8}$$

Also, the convergence rates are of order n^{-1} as well.

Equations (2.7) and (2.8) indicate that any pair of sequences λ and λ' are nearly the same in terms of the quality as measured by CTV or WTV. Consequently, if a sequence λ is optimal for CTV (or WTV), then its antithetical sequence λ' is also approximately optimal for CTV (or WTV).

3. Probabilistic analysis

Assume that $p_1, \dots, p_n$ are drawn from a discrete uniform distribution over integer values $\{1, 2, \dots, a\}$, that is, p_i is equally likely to take each one of $1, 2, \dots, a$, so that

$$\mathrm{P}(p_i = k) = 1/a, \quad k = 1, 2, \dots, a. \tag{3.1}$$

Such a distribution will be denoted by $\mathrm{DU}[1, a]$. Similarly we assume that $u_1, \dots, u_n$ are independently drawn from a $\mathrm{DU}[1, b]$, and we further assume that $u_1, \dots, u_n$ are independent of $p_1, \dots, p_n$. Let p denote a random variable with a $\mathrm{DU}[1, a]$ distribution and u a random variable with a $\mathrm{DU}[1, b]$ distribution. It can be easily calculated that

$$\begin{aligned}
\mathrm{E}(p) &= \sum_{j=1}^{a} j\,\frac{1}{a} = \frac{a+1}{2}, \\
\mathrm{E}(p^2) &= \sum_{j=1}^{a} j^2\,\frac{1}{a} = \frac{(a+1)(2a+1)}{6}, \\
\mathrm{Var}(p) &= \mathrm{E}(p^2) - \mathrm{E}^2(p) = \frac{(a+1)(2a+1)}{6} - \frac{(a+1)^2}{4} = \frac{1}{12}(a^2-1).
\end{aligned}$$

In general, the k-th moment of p is given by $\mathrm{E}(p^k) = \sum_{j=1}^{a} j^k\,\frac{1}{a}$. Similarly, $\mathrm{E}(u) = \frac{b+1}{2}$, $\mathrm{E}(u^2) = \frac{(b+1)(2b+1)}{6}$, etc. To state our results, let us introduce the following definition.

DEFINITION. Let X_n be a sequence of random variables and α_n a sequence of numbers. If

$$\lim_{M\to\infty} \liminf_{n\to\infty} \mathrm{P}\left(\left|\frac{X_n}{\alpha_n}\right| \le M\right) = 1, \tag{3.2}$$

then we say that $\{X_n/\alpha_n\}$ is bounded 'in probability' and write $X_n = \mathrm{O}_p(\alpha_n)$.

We now present our main result.

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THEOREM 1. *If the processing times $p_1, \ldots, p_n$ and weights $u_1, \ldots, u_n$ satisfy the above assumptions, then for any sequence λ,*

$$r_c(\lambda) = O_p(1/n) \quad \textit{and} \quad r_w(\lambda) = O_p(1/n). \tag{3.3}$$

Note that (2.5) and (2.6) are immediate consequences of (3.3).

In order to prove this Theorem, let us first prove several lemmas. Without loss of generality we assume, in all our proofs, that $\lambda = \{1, \ldots, n\}$. This can always be achieved by relabeling $p_1, \ldots, p_n$ and $u_1, \ldots, u_n$ if necessary. We will prove the theorem for $r_c(\lambda)$ only as the proof for $r_w(\lambda)$ is very similar.

The following lemma establishes an upper bound on $r_c(\lambda)$ for any given p_i and u_i, $i = 1, 2, \ldots, n$.

LEMMA 1.

$$r_c(\lambda) \leq 2\left\{\frac{\sum_{i=1}^n u_i(p_i - \bar{p})^2}{\sum_{i=1}^n u_i(C_i - \bar{C})^2}\right\}^{\frac{1}{2}} + \frac{\sum_{i=1}^n u_i(p_i - \bar{p})^2}{\sum_{i=1}^n u_i(C_i - \bar{C})^2}, \tag{3.4}$$

where $\bar{p} = \frac{1}{U}\sum_{i=1}^n u_i p_i$.

PROOF. Note that $W_i = C_i - p_i$ and $\bar{W} = \bar{C} - \bar{p}$. Hence

$$\begin{aligned} WTV(\lambda) &= \frac{1}{U}\sum_{i=1}^n u_i(W_i - \bar{W})^2 = \frac{1}{U}\sum_{i=1}^n u_i(C_i - p_i - \bar{C} + \bar{p})^2 \\ &= \frac{1}{U}\sum_{i=1}^n u_i\left\{(C_i - \bar{C})^2 - 2(C_i - \bar{C})(p_i - \bar{p}) + (p_i - \bar{p})^2\right\} \\ &= CTV(\lambda) - \frac{2}{U}\sum_{i=1}^n u_i(C_i - \bar{C})(p_i - \bar{p}) + \frac{1}{U}\sum_{i=1}^n u_i(p_i - \bar{p})^2. \end{aligned}$$

Thus by the Cauchy-Schwarz inequality,

$$\begin{aligned} |WTV(\lambda) - CTV(\lambda)| &\leq \left|\frac{2}{U}\sum_{i=1}^n u_i(C_i - \bar{C})(p_i - \bar{p})\right| + \frac{1}{U}\sum_{i=1}^n u_i(p_i - \bar{p})^2 \\ &\leq \frac{2}{U}\left\{\sum_{i=1}^n u_i(C_i - \bar{C})^2 \sum_{i=1}^n u_i(p_i - \bar{p})^2\right\}^{\frac{1}{2}} + \frac{1}{U}\sum_{i=1}^n u_i(p_i - \bar{p})^2 \\ &= 2\left\{CTV(\lambda)\,\frac{1}{U}\sum_{i=1}^n u_i(p_i - \bar{p})^2\right\}^{\frac{1}{2}} + \frac{1}{U}\sum_{i=1}^n u_i(p_i - \bar{p})^2. \end{aligned}$$

After dividing the above inequality by $CTV(\lambda)$ and using (2.3), the proof is complete.

In the next two lemmas, we examine the limit of (3.4) as $n \to \infty$.

LEMMA 2.

$$\frac{1}{n}\sum_{i=1}^{n} u_i(p_i - \bar{p})^2 \to \mathrm{E}(u)\,\mathrm{Var}(p) = \frac{(a^2-1)(b+1)}{24} \tag{3.5}$$

in probability as $n \to \infty$.

PROOF. Applying the law of large numbers, we obtain

$$\frac{U}{n} = \frac{1}{n}\sum_{i=1}^{n} u_i \to \mathrm{E}(u),$$
$$\frac{1}{n}\sum_{i=1}^{n} u_i p_i \to \mathrm{E}(up) = \mathrm{E}(u)\,\mathrm{E}(p)$$

and

$$\frac{1}{n}\sum_{i=1}^{n} u_i p_i^2 \to \mathrm{E}(up^2) = \mathrm{E}(u)\,\mathrm{E}(p^2)$$

in probability. Hence

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} u_i(p_i - \bar{p})^2 &= \frac{1}{n}\left\{\sum_{i=1}^{n} u_i p_i^2 - \frac{1}{U}\left(\sum_{i=1}^{n} u_i p_i\right)^2\right\} \\
&= \frac{1}{n}\sum_{i=1}^{n} u_i p_i^2 - \frac{n}{U}\left(\frac{1}{n}\sum_{i=1}^{n} u_i p_i\right)^2 \\
&\to \mathrm{E}(u)\,\mathrm{E}(p^2) - \frac{1}{\mathrm{E}(u)}(\mathrm{E}(u)\,\mathrm{E}(p))^2 \\
&= \mathrm{E}(u)\big(\mathrm{E}(p^2) - \mathrm{E}^2(p)\big) = \mathrm{E}(u)\,\mathrm{Var}(p)
\end{aligned}$$

in probability, which proves the lemma.

LEMMA 3.

$$\frac{1}{n^3}\sum_{i=1}^{n} u_i(C_i - \bar{C})^2 \to \frac{1}{12}\mathrm{E}(u)\,\mathrm{E}^2(p) = \frac{(a+1)^2(b+1)}{96} \tag{3.6}$$

in probability as $n \to \infty$.

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PROOF. First rewrite

$$\sum_{i=1}^{n} u_i (C_i - \bar{C})^2 = \sum_{i=1}^{n} u_i C_i^2 - \frac{1}{U} \left(\sum_{i=1}^{n} u_i C_i \right)^2. \tag{3.7}$$

We then calculate

$$\mathrm{E}(C_i) = \mathrm{E}(p_1 + \cdots + p_i) = i\,\mathrm{E}(p),$$

$$\mathrm{E}\left(\sum_{i=1}^{n} u_i C_i \right) = \sum_{i=1}^{n} \mathrm{E}(u_i)\,\mathrm{E}(C_i) = \mathrm{E}(u) \sum_{i=1}^{n} i\,\mathrm{E}(p) = \frac{1}{2} n(n+1)\,\mathrm{E}(u)\,\mathrm{E}(p). \tag{3.8}$$

Thus

$$\mathrm{E}\left(\frac{1}{n^2} \sum_{i=1}^{n} u_i C_i \right) \to \frac{1}{2}\,\mathrm{E}(u)\,\mathrm{E}(p). \tag{3.9}$$

Furthermore,

$$\begin{aligned} E(C_i^2) &= \mathrm{E}\left[(p_1 + \cdots + p_i)^2\right] \\ &= \sum_{j=1}^{i} E(p_j^2) + \sum_{1 \le j \ne k \le i} E(p_j) E(p_k) \\ &= i\,\mathrm{E}(p^2) + i(i-1)\,\mathrm{E}^2(p) = i[\mathrm{E}(p^2) - \mathrm{E}^2(p)] + i^2\,\mathrm{E}^2(p) \\ &= i\,\mathrm{Var}(p) + i^2\,\mathrm{E}^2(p) \end{aligned} \tag{3.10}$$

and for $i < j$,

$$\begin{aligned} E(C_i C_j) &= \mathrm{E}\left[(p_1 + \cdots + p_i)(p_1 + \cdots + p_j)\right] \\ &= \sum_{k=1}^{i} E(p_k^2) + \sum_{k=1}^{i} \sum_{l=1, l \ne k}^{j} E(p_k) E(p_l) \\ &= i\,\mathrm{E}(p^2) + i(j-1)\,\mathrm{E}^2(p) = i\,\mathrm{Var}(p) + ij\,\mathrm{E}^2(p). \end{aligned} \tag{3.11}$$

Hence

$$\begin{aligned} \mathrm{E}\left(\sum_{i=1}^{n} u_i C_i^2 \right) &= \sum_{i=1}^{n} \mathrm{E}(u_i)\,\mathrm{E}(C_i^2) \\ &= \mathrm{E}(u) \sum_{i=1}^{n} [i\,\mathrm{Var}(p) + i^2\,\mathrm{E}^2(p)] \\ &= \mathrm{E}(u) \left[\frac{1}{2} n(n+1)\,\mathrm{Var}(p) + \frac{1}{6} n(n+1)(2n+1)\,\mathrm{E}^2(p) \right] \end{aligned}$$

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$$= \frac{1}{3}n^3\,\mathrm{E}(u)\,\mathrm{E}^2(p) + \mathrm{O}(n^2) \tag{3.12}$$

and

$$\begin{aligned}
\mathrm{E}\left[\left(\sum_{i=1}^{n} u_i C_i\right)^2\right] &= \mathrm{E}\left[\sum_{i=1}^{n} u_i^2 C_i^2 + 2\sum_{i<j} u_i u_j C_i C_j\right] \\
&= \sum_{i=1}^{n} \mathrm{E}(u^2)\,\mathrm{E}(C_i^2) + 2\sum_{i<j} \mathrm{E}^2(u)\,\mathrm{E}(C_i C_j) \\
&= \mathrm{E}(u^2)\sum_{i=1}^{n}[i\,\mathrm{Var}(p) + i^2\,\mathrm{E}^2(p)] + 2\,\mathrm{E}^2(u)\sum_{i<j}[i\,\mathrm{Var}(p) + ij\,\mathrm{E}^2(p)] \\
&= \mathrm{E}(u^2)\left[\frac{1}{2}n(n+1)\,\mathrm{Var}(p) + \frac{1}{6}n(n+1)(2n+1)\,\mathrm{E}^2(p)\right] \\
&\quad + \mathrm{E}^2(u)\left[\mathrm{O}(n^3)\,\mathrm{Var}(p) + \left(\frac{n^4}{4} + \mathrm{O}(n^3)\right)\mathrm{E}^2(p)\right] \\
&= \frac{n^4}{4}\,\mathrm{E}^2(u)\,\mathrm{E}^2(p) + \mathrm{O}(n^3).
\end{aligned} \tag{3.13}$$

It follows from (3.8) and (3.13) that

$$\begin{aligned}
\mathrm{Var}\left(\sum_{i=1}^{n} u_i C_i\right) &= \mathrm{E}\left[\left(\sum_{i=1}^{n} u_i C_i\right)^2\right] - \left[\mathrm{E}\left(\sum_{i=1}^{n} u_i C_i\right)\right]^2 \\
&= \frac{n^4}{4}\,\mathrm{E}^2(u)\,\mathrm{E}^2(p) + \mathrm{O}(n^3) - \left[\frac{n+1}{2}\,\mathrm{E}(u)\,\mathrm{E}(p)\right]^2 \\
&= \mathrm{O}(n^3),
\end{aligned}$$

which implies

$$\mathrm{Var}\left(\frac{1}{n^2}\sum_{i=1}^{n} u_i C_i\right) = \frac{1}{n^4}\mathrm{O}(n^3) \to 0.$$

This, together with (3.9) and Lemma A.1 as given in the Appendix, shows that

$$\frac{1}{n^2}\sum_{i=1}^{n} u_i C_i \to \frac{1}{2}\,\mathrm{E}(u)\,\mathrm{E}(p) \tag{3.14}$$

in probability.

In order to calculate the variance of $\sum u_i C_i^2$, we need the following higher moments of C_i:

$$E(C_i^3) = \mathrm{E}\left[(p_1 + \cdots + p_i)^3\right]$$

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$$= i\,\mathrm{E}(p^3) + 3i(i-1)\,\mathrm{E}(p^2)\,\mathrm{E}(p) + i(i-1)(i-2)\,\mathrm{E}^3(p), \tag{3.15}$$

$$\begin{aligned} E(C_i^4) &= \mathrm{E}\left[(p_1 + \cdots + p_i)^4\right] \\ &= i\,\mathrm{E}(p^4) + 4i(i-1)\,\mathrm{E}(p^3)\,\mathrm{E}(p) + 3i(i-1)\,\mathrm{E}(p^2)\,\mathrm{E}(p^2) \\ &\quad + 6i(i-1)(i-2)\,\mathrm{E}(p^2)\,\mathrm{E}^2(p) + i(i-1)(i-2)(i-3)\,\mathrm{E}^4(p) \end{aligned} \tag{3.16}$$

and for $i < j$,

$$\begin{aligned} E(C_i^2C_j^2) &= \mathrm{E}\left[C_i^2(C_i + p_{i+1} + \cdots + p_j)^2\right] \\ &= \mathrm{E}\left[C_i^4\right] + 2\,\mathrm{E}\left[C_i^3(p_{i+1} + \cdots + p_j)\right] + \mathrm{E}\left[C_i^2(p_{i+1} + \cdots + p_j)^2\right] \\ &= \mathrm{E}\left[C_i^4\right] + 2\,\mathrm{E}\left[C_i^3\right](j-i)\,\mathrm{E}(p) + \mathrm{E}\left[C_i^2\right]\mathrm{E}\left[(p_{i+1} + \cdots + p_j)^2\right]. \end{aligned} \tag{3.17}$$

It follows that

$$\begin{aligned} \sum_{i=1}^{n} \mathrm{E}(C_i^4) &= \mathrm{E}(p^4)\sum_{i=1}^{n} i + \left[4\,\mathrm{E}(p^3)\,\mathrm{E}(p) + 3\,\mathrm{E}(p^2)\,\mathrm{E}(p^2)\right]\sum_{i=1}^{n} i(i-1) \\ &\quad + 6\left[\mathrm{E}^2(p)\right]^2\sum_{i=1}^{n} i(i-1)(i-2) + \mathrm{E}^4(p)\sum_{i=1}^{n} i(i-1)(i-2)(i-3) \\ &= \frac{1}{5}n^5\,\mathrm{E}^4(p) + \mathrm{O}(n^4) \end{aligned} \tag{3.18}$$

and by (3.15)-(3.17) together with (3.10),

$$\begin{aligned} \sum_{i<j} \mathrm{E}(C_i^2C_j^2) &= \sum_{i<j}\left\{\mathrm{E}(C_i^4) + 2\,\mathrm{E}(p)\,\mathrm{E}(C_i^3)(j-i) + \mathrm{E}(C_i^2)\,\mathrm{E}\left[(p_{i+1} + \cdots + p_j)^2\right]\right\} \\ &= \mathrm{E}^4(p)\sum_{i<j} i^4 + 2\,\mathrm{E}(p)\,\mathrm{E}^3(p)\sum_{i<j}(j-i)i^3 \\ &\quad + \sum_{i<j}\left[i^2\,\mathrm{E}^2(p)(j-i)^2\,\mathrm{E}^2(p)\right] + \mathrm{O}(n^5) \\ &= \left(\frac{4!}{6!} + 2\frac{3!}{6!} + \frac{2!2!}{6!}\right)n^6\,\mathrm{E}^4(p) + \mathrm{O}(n^5) \\ &= \frac{1}{18}n^6\,\mathrm{E}^4(p) + \mathrm{O}(n^5), \end{aligned} \tag{3.19}$$

where the third equality used the result of Lemma A.2 in the Appendix. Combining (3.18)–(3.19) with (3.12) we obtain

$$\mathrm{Var}\left(\sum_{i=1}^{n} u_iC_i^2\right) = \mathrm{E}\left[\left(\sum_{i=1}^{n} u_iC_i^2\right)^2\right] - \left[\mathrm{E}\left(\sum_{i=1}^{n} u_iC_i^2\right)\right]^2$$

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$$
\begin{aligned}
&= \mathrm{E}\left(\sum_{i=1}^{n} u_i^2 C_i^4 + 2\sum_{i<j} u_i C_i^2 u_j C_j^2\right) - \left(\frac{1}{3}n^3\,\mathrm{E}(u)\,\mathrm{E}^2(p) + \mathrm{O}(n^2)\right)^2 \\
&= \mathrm{E}(u^2)\sum_{i=1}^{n}\mathrm{E}(C_i^4) + 2\,\mathrm{E}^2(u)\sum_{i<j}\mathrm{E}(C_i^2C_j^2) - \frac{1}{9}n^6\,\mathrm{E}^2(u)\,\mathrm{E}^4(p) + \mathrm{O}(n^5) \\
&= \mathrm{E}(u^2)\left[\frac{1}{5}n^5\,\mathrm{E}^4(p) + \mathrm{O}(n^4)\right] + 2\,\mathrm{E}^2(u)\left[\frac{1}{18}n^6\,\mathrm{E}^4(p) + \mathrm{O}(n^5)\right] \\
&\quad - \frac{1}{9}n^6\,\mathrm{E}^2(u)\,\mathrm{E}^4(p) + \mathrm{O}(n^5) \\
&= \mathrm{O}(n^5).
\end{aligned}
$$

Thus

$$
\mathrm{Var}\left(\frac{1}{n^3}\sum_{i=1}^{n} u_i C_i^2\right) = \frac{1}{n^6}\mathrm{O}(n^5) \to 0,
$$

which together with (3.12) and Lemma A.1 in the Appendix, yields

$$
\frac{1}{n^3}\sum_{i=1}^{n} u_i C_i^2 \to \frac{1}{3}\,\mathrm{E}(u)\,\mathrm{E}^2(p) \tag{3.20}
$$

in probability. Finally, a combination of (3.14) and (3.20) with (3.7) leads to

$$
\begin{aligned}
\frac{1}{n^3}\sum_{i=1}^{n} u_i (C_i - \bar{C})^2 &= \frac{1}{n^3}\sum_{i=1}^{n} u_i C_i^2 - \frac{n}{U}\left(\frac{1}{n^2}\sum_{i=1}^{n} u_i C_i\right)^2 \\
&\to \frac{1}{3}\,\mathrm{E}(u)\,\mathrm{E}^2(p) - \frac{1}{\mathrm{E}(u)}\left(\frac{1}{2}\,\mathrm{E}(u)\,\mathrm{E}(p)\right)^2 \\
&= \frac{1}{12}\,\mathrm{E}(u)\,\mathrm{E}^2(p)
\end{aligned}
$$

in probability. This completes the proof of (3.6).

Having established these lemmas, we can now prove Theorem 1.

Proof of Theorem 1. Combining Lemmas 2 and 3, we get

$$
n^2\frac{\sum_{i=1}^{n} u_i (p_i - \bar{p})^2}{\sum_{i=1}^{n} u_i (C_i - \bar{C})^2} \to \frac{96(a^2-1)(b+1)}{24(a+1)^2(b+1)} = \frac{4(a-1)}{a+1} \tag{3.21}
$$

in probability as $n \to \infty$. This shows that

$$
\frac{\sum_{i=1}^{n} u_i (p_i - \bar{p})^2}{\sum_{i=1}^{n} u_i (C_i - \bar{C})^2} = \mathrm{O}_p\left(\frac{1}{n^2}\right), \tag{3.22}
$$

which together with (3.4) implies $r_c(\lambda) = O_p\left(\frac{1}{n}\right)$. On the other hand, letting

$$e_c(\lambda) = \frac{CTV(\lambda) - WTV(\lambda)}{CTV(\lambda)},$$

we have

$$r_w(\lambda) = \frac{|CTV(\lambda) - WTV(\lambda)|}{CTV(\lambda)} \frac{CTV(\lambda)}{WTV(\lambda)} = \frac{r_c(\lambda)}{1 - e_c(\lambda)}. \tag{3.23}$$

Since $r_c(\lambda) = |e_c(\lambda)| = O_p\left(\frac{1}{n}\right) \to 0$ as $n \to \infty$, (3.23) gives $r_w(\lambda) = O_p\left(\frac{1}{n}\right)$ as required.

4. Concluding remarks

In this paper we carried out a probabilistic analysis of the relationship between the CTV measure and the WTV measure. We have shown that, when the processing times and weights are randomly drawn evenly over certain ranges, the two measures are nearly equivalent; for any sequence λ, the relative error $r_c(\lambda)$ or $r_w(\lambda)$ tends towards zero at the rate of n^{-1}. Hence if a sequence λ is optimal to one measure, it will be an approximate optimum to the other measure.

Morever, an interesting finding is that the range limits (that is, a and b) have little effects on the limiting upper bound of the relative error, especially when the range is wide. This can be seen from (3.21), which shows that the limit on the right side never exceeds 4 and is close to 4 when a, the range for the processing times, is large. The relative error, when n is large, is close to zero no matter what a or b are.

The assumption that the parameters involved in a problem under consideration follow a uniform distribution has been widely used in the literature of probabilistic analysis. While we have adopted a discrete uniform distribution - for being consistent with the common practice of taking integer-valued parameters in scheduling problems — the same results will hold for continuous uniform distributions as well. In fact our proofs are valid for any type of distribution, provided certain conditions on the moments of the distribution are met. But unlike the uniform distribution (whether discrete or continuous), the limiting upper bound of the relative error may be significantly influenced by the parameters of the distribution. This would make (3.3) less meaningful if such a bound is large compared with n, which may occur when the processing times are too skewed.

The theoretical findings provide us with a deep understanding of the problem, which will be of significance in decision making. At least a decision maker can now be formally assured that, if the problem to be dealt with is not ill-conditioned, in the sense that the problem instances are relatively evenly distributed, then an optimal decision under one measure will be also a good decision under the other measure. This is the objective pursued in equivalence studies (Gerchak and Magazine [8]).

Appendix

LEMMA A.1. *Let X_n be a sequence of random variables. If, as $n \to \infty$, $\mathrm{E}(X_n) \to A$ and $\mathrm{Var}(X_n) \to 0$, then $X_n \to A$ in probability.*

PROOF.

$$(X_n - A)^2 = [(X_n - \mathrm{E}(X_n)) + (\mathrm{E}(X_n) - A)]^2 \le 2\left[(X_n - \mathrm{E}(X_n))^2 + (\mathrm{E}(X_n) - A)^2\right].$$

Hence

$$\mathrm{E}\left[(X_n - A)^2\right] \le 2\,\mathrm{Var}(X_n) + 2(\mathrm{E}(X_n) - A)^2 \to 0,$$

which implies $X_n \to A$ in probability by Chebyshev's inequality.

LEMMA A.2. *For any nonnegative integers α and β,*

$$\sum_{1 \le i < j \le n} i^\alpha (j-i)^\beta = \frac{\alpha!\beta!}{(\alpha+\beta+2)!} n^{\alpha+\beta+2} + \mathrm{O}\left(n^{\alpha+\beta+1}\right)$$

as $n \to \infty$.

PROOF. Note that the sum is obviously a polynomial in n. Hence it suffices to show

$$\frac{1}{n^{\alpha+\beta+2}} \sum_{1 \le i < j \le n} i^\alpha (j-i)^\beta \to \frac{\alpha!\beta!}{(\alpha+\beta+2)!}$$

as $n \to \infty$. To prove this, we may use a simple integral approximation

$$\begin{aligned}
\frac{1}{n^{\alpha+\beta+2}} \sum_{1 \le i < j \le n} i^\alpha (j-i)^\beta &= \sum_{1 \le i < j \le n} \left(\frac{i}{n}\right)^\alpha \left(\frac{i}{n} - \frac{j}{n}\right)^\beta \frac{1}{n^2} \\
\to \int\!\!\int_{0<x<y<1} x^\alpha (y-x)^\beta dx dy &= \int_0^1 x^\alpha \left(\int_x^1 (y-x)^\beta dy\right) dx \\
&= \int_0^1 x^\alpha \frac{(1-x)^{\beta+1}}{\beta+1} dx = \frac{1}{\beta+1} \frac{\alpha!(\beta+1)!}{(\alpha+\beta+2)!} \\
&= \frac{\alpha!\beta!}{(\alpha+\beta+2)!}.
\end{aligned}$$

The lemma is thus proven.

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