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Rudin-Shapiro sequences on compact groups

Alessandro Figà-Talamanca and J.F. Price

The existence of a certain type of Rudin-Shapiro sequence of functions is shown for all infinite compact groups which are not Lie, thus extending a recent result of the authors to all infinite compact groups.

1. Introduction

The notation will follow exactly that of [1] and any notation or definition unexplained here can be found in [1]. Briefly, G will denote a (Hausdorff) compact group, Γ the set of equivalence classes of continuous irreducible unitary representations of G, and

$$f \sim \sum_{\gamma \in \Gamma} d(\gamma) \operatorname{tr}\left[\hat{f}\left(D_\gamma\right) D_\gamma(\cdot)\right]$$

the Fourier series of $f \in L^1(G)$, where: D_γ is a representative (assumed fixed in the sequel) of the class $\gamma \in \Gamma$; $d(\gamma)$ is the dimension of γ; tr denotes the usual trace; and

$$\hat{f}\left(D_\gamma\right) = \int_G f(x) D_\gamma\left(x^{-1}\right) d\lambda_G(x) .$$

Denote $\sup\{\|\hat{f}(D_\gamma)\| : \gamma \in \Gamma\}$ by $\|\hat{f}\|_\infty$. By a *Rudin-Shapiro sequence of type* t (a t-*RS*-sequence), where $2 < t \leq \infty$, we shall mean a sequence (h_n) of functions in $L^t(G)$ with the properties:

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$$(1) \qquad \begin{cases} \inf_n \|h_n\|_2 > 0 \ , \quad \sup_n \|h_n\|_t < \infty \ , \\ \lim_n \|\hat{h}_n\|_\infty = 0 \ . \end{cases}$$

The purpose of this note is to prove the following result which has already been shown in [1] for infinite compact Lie groups. (An immediate consequence of this extension is that the inclusions of 4.2 and of Theorem 4.4 of [1] remain strict for all infinite compact groups.)

THEOREM. *Let* G *be an infinite non-Lie compact group and let* $t \in (2, \infty)$. *Then there exist two* t*-RS-sequences* (h_n) *and* (h_n^*) *and a positive number* ρ *such that*

$$(2) \qquad h_n^* * h_n = h_n * h_n^* \quad \text{and} \quad \|h_n\|_2 = \|h_n^*\|_2 = 1 \ ,$$

$$(3) \qquad \rho^{1+1/p}\|\hat{h}_n\|_\infty^{2/p} \le \|h_n^* * h_n\|_p \le \|\hat{h}_n\|_\infty^{2/p} \ ,$$

for $n = 1, 2, \ldots$.

2. Proof of the theorem

We begin with several observations on the harmonic analysis of functions on factor groups of G . Suppose that G_0 , with dual Γ_0 , is a closed normal subgroup of G . Corollary (28.10) of [2] shows that there exists a (hypergroup) isomorphism φ from $A_0 \equiv A(\Gamma, G_0)$, the annihilator of G_0 in Γ , onto Γ_0 . Moreover, a representative $D_{\varphi(\gamma)}$ of each $\varphi(\gamma) \in \Gamma_0$ can always be chosen so that

$$D_{\varphi(\gamma)}(\bar{x}) = D_\gamma(x)$$

for all $x \in G$, where $\bar{x} = \pi(x)$, π being the canonical projection of G onto G/G_0 . In the sequel we will always assume that $D_{\varphi(\gamma)}$ is so chosen and we will often identify γ and $\varphi(\gamma)$.

Given $f \in L^1(G/G_0)$ with Fourier series

$$f(\bar{x}) \sim \sum_{\varphi(\gamma)\in\Gamma_0} d(\varphi(\gamma))\,\mathrm{tr}\big[\hat{f}(D_{\varphi(\gamma)})D_{\varphi(\gamma)}(\bar{x})\big] \ ,$$

it follows that $h = f \circ \pi$ has the Fourier series

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$$h(x) \sim \sum_{\gamma \in A_0} d(\gamma)\mathrm{tr}\left[\hat{f}\left(D_{\varphi(\gamma)}\right)D_\gamma(x)\right]$$

and so, by the uniqueness theorem for Fourier series, $\hat{h}\left(D_\gamma\right) = \hat{f}\left(D_{\varphi(\gamma)}\right)$ for $\gamma \in A_0$, and 0 otherwise. Thus

$$\|\hat{h}\|_\infty = \|\hat{f}\|_\infty \; ; \tag{4}$$

also it is routine that

$$\|h\|_{L^p(G)} = \|f\|_{L^p(G/G_0)} \; . \tag{5}$$

The method of proof of the theorem will be determined by whether or not G is 0-dimensional. A topological space is said to be 0-*dimensional* if it has an open basis consisting of sets which are both open and closed; Theorem (7.7) of [2] shows that a compact group is 0-dimensional if and only if it has a basis of neighbourhoods of the identity consisting of compact open normal subgroups.

(a) Suppose that G is not 0-dimensional (and not a Lie group). Then there exists a representation $\gamma_0 \in \Gamma$ such that $\Gamma_0 = \left[\{\gamma_0\}\right]$, the *smallest subset of* Γ *closed under conjugation and tensor products* followed by decomposition, is infinite [2, (28.19)]. Define G_0 to be the annihilator of Γ_0 in G. Then the dual of G/G_0 is (isomorphic to) Γ_0, so that G/G_0 is an infinite closed subgroup of a finite dimensional unitary group (use [2, (44.55)], noting that Γ_0 is finitely generated); hence G/G_0 is a compact Lie group.

By Lemma 3.1 (b) of [1] there exist t-*RS*-sequences $\left(f_n\right)$, $\left(f_n^*\right)$ on G/G_0 which have the properties described in the statement of the theorem. Define

$$h_n = f_n \circ \pi \; , \quad h_n^* = f_n^* \circ \pi \; ;$$

then $h_n^* * h_n = f_n^* * f_n \circ \pi$ and so formulae (4) and (5) above can be used to show immediately that the sequences $\left(h_n\right)$ and $\left(h_n^*\right)$ satisfy the theorem.

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(b) Suppose that G is infinite 0-dimensional (and consequently is not Lie). There exists a basis of neighbourhoods $\{G_\alpha : \alpha \in A\}$ of the identity in G consisting of open compact normal subgroups. For each $n = 1, 2, \ldots$ choose G_{α_n} from this basis such that

$$\lambda_G\left(G_{\alpha_n}\right) \leq 1/n .$$

Write $G_n = G_{\alpha_n}$ and define χ_n to be the characteristic function of G_n. The Fourier series of χ_n may be seen to have the form

$$\chi_n = \lambda_G\left(G_n\right) \sum_{\gamma \in A_n} d(\gamma)\operatorname{tr}\left[D_\gamma(.)\right] ,$$

where $A_n \equiv A\left(\Gamma, G_n\right)$ is finite. As in the proof of Theorem 3.1 of [1], we can choose a family $W_n = \left\{W_n(\gamma) : \gamma \in \Gamma\right\}$ of unitary operators such that $\lambda\left(G_n\right)^{-1/2}\chi_n^{W_n}$ and $\lambda\left(G_n\right)^{-1/2}\chi_n^{W_n^*}$ are t-RS-sequences, where

$$\chi_n^{W_n} = \lambda_G\left(G_n\right) \sum_{A_n} d(\gamma)\operatorname{tr}\left[W_n(\gamma)D_\gamma(.)\right]$$

and

$$\chi_n^{W_n^*} = \lambda_G\left(G_n\right) \sum_{A_n} d(\gamma)\operatorname{tr}\left[W_n(\gamma)^*D_\gamma(.)\right] .$$

Let us denote these two t-RS-sequences by $\left(h_n\right)$ and $\left(h_n^*\right)$ respectively.

First note that $\|h_n\|_2 = \|h_n^*\|_2 = 1$ and that $h_n^* * h_n$ and $h_n * h_n^*$ are both equal to χ_n. Thus (2) is satisfied, and moreover

$$\|h_n^* * h_n\|_p = \lambda_G\left(G_n\right)^{1/p} . \tag{6}$$

On the other hand,

$$\|\hat{h}_n\|_\infty = \lambda_G\left(G_n\right)^{-1/2}\left\|\left(\chi_n^{W_n}\right)^\wedge\right\|_\infty = \lambda_G\left(G_n\right)^{1/2} . \tag{7}$$

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Equations (6) and (7) together show the validity of (3) with $\rho = 1$, thus completing the proof.

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Istituto di Matematica,
Università di Genova,
Genova,
Italy;

Department of Mathematics,
Institute of Advanced Studies,
Australian National University,
Canberra, ACT.

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