Canad. Math. Bull. Vol. **37** (4), 1994 pp. 522–526

CARLEMAN APPROXIMATION BY ENTIRE FUNCTIONS ON THE UNION OF TWO TOTALLY REAL SUBSPACES OF C^n

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ABSTRACT. Let $L_1, L_2 \subset \mathbf{C}^n$ be two totally real subspaces of real dimension n, and such that $L_1 \cap L_2 = \{0\}$. We show that continuous functions on $L_1 \cup L_2$ allow Carleman approximation by entire functions if and only if $L_1 \cup L_2$ is polynomially convex. If the latter condition is satisfied, then a function $f: L_1 \cup L_2 \longrightarrow \mathbf{C}$ such that $f|_{L_i} \in C^k(L_i)$, $i = 1, 2$, allows Carleman approximation of order k by entire functions if and only if f satisfies the Cauchy-Riemann equations up to order k at the origin.

1. **Introduction.** A real subspace $L \subset \mathbf{C}^n$ is called *totally real* if it does not contain any complex line, *i.e.* if any real basis of L is linearly independent over $\mathbf{C}$. It follows that $\dim_{\mathbf{R}} L \leq n$ for any totally real subspace $L \subset \mathbf{C}^n$.

A compact subset $K \subset \mathbf{C}^n$ is called *polynomially convex* if for each point $z \in \mathbf{C}^n \setminus K$ there exists a polynomial f such that $|f(z)| > \|f\|_K = \sup\{|f(\zeta)| : \zeta \in K\}$. We extend this notion to closed subsets $M \subset \mathbf{C}^n$ and say that M is polynomially convex if there is some sequence $\{K_m\}$ of polynomially convex compact subsets $K_m \subset M$ satisfying the conditions $K_m \subset K_{m+1}$ for all m, $\bigcup K_m = M$, and for any compact subset $K \subset M$ there is some K_m containing K.

Let $L_1, L_2 \subset \mathbf{C}^n$ be two totally real subspaces of real dimension n, and such that $L_1 \cap L_2 = \{0\}$. The set $L_1 \cup L_2$ arose in a natural manner in [4], and it was observed that this set was not necessarily polynomially convex. This observation was attributed to Dan Burns. Note that $L_1 \cup L_2$ is not totally real at the origin.

In [5], necessary and sufficient conditions for $L_1 \cup L_2$ to be polynomially convex were given. After a linear change of coordinates, we may assume that $L_1 = \mathbf{R}^n$ and $L_2 = (A + iI)\mathbf{R}^n$, where A is a real $(n \times n)$-matrix and I is the identity matrix. Then L_2 is totally real iff i is not an eigenvalue of A, and $L_1 \cup L_2$ is polynomially convex iff A has no eigenvalues on the form ti with $t \in \mathbf{R}$, $|t| > 1$. It is possible to approximate continuous functions by polynomials uniformly on compact subsets of $L_1 \cup L_2$ precisely when $L_1 \cup L_2$ is polynomially convex, in which case all compact subsets of $L_1 \cup L_2$ are polynomially convex.

A more detailed description of the polynomially convex hull $\hat{K} = \{z \in \mathbf{C}^n : |f(z)| \leq \|f\|_K$ for all polynomials $f\}$ of a compact set $K \subset L_1 \cup L_2$, and of the algebra $P(K)$ of uniform limits of the restriction of polynomials to K, in the case where $L_1 \cup L_2$ is not polynomially convex has been obtained by Nils Øvrelid (unpublished).

Received by the editors June 3, 1993; revised May 26, 1994.
AMS subject classification: 32E30.

In [1] the question of Carleman approximation was considered, and it was shown that if $L_1 = \mathbf{R}^n$ and $L_2 = (A+iI)\mathbf{R}^n$, where A is a real $(n\times n)$-matrix which can be diagonalized over $\mathbf{R}$, then continuous functions on $L_1 \cup L_2$ allow Carleman approximation by entire functions. (The notion of Carleman approximation is explained in the formulation of the theorems below.) It is this result which we generalize, and we are able to show that it is possible to obtain Carleman approximation on $L_1 \cup L_2$ by entire functions if and only if $L_1 \cup L_2$ is polynomially convex (Theorem 1). We also give necessary and sufficient conditions for Carleman approximation in C^k-topology on $L_1 \cup L_2$ by entire functions (Theorem 2).

These results were part of the author's doctoral thesis, written at the University of Oslo under the direction of Nils Øvrelid. I would like to thank Nils Øvrelid for his support and valuable suggestions during this work.

2. **Main part.**

THEOREM 1. *Let $L_1, L_2 \subset \mathbf{C}^n$ be two totally real subspaces of real dimension n such that $L_1 \cap L_2 = \{0\}$, and assume that $L_1 \cup L_2$ is polynomially convex. Then continuous functions on $L_1 \cup L_2$ allow Carleman approximation by entire functions, i.e. for any $f \in C(L_1 \cup L_2)$ and any positive continuous function $\epsilon\colon L_1 \cup L_2 \longrightarrow \mathbf{R}$ there exists $h \in O(\mathbf{C}^n)$ such that $|h(z) - f(z)| < \epsilon(z)$ for all $z \in L_1 \cup L_2$.*

NOTE. Since the condition that $L_1 \cup L_2$ be polynomially convex is necessary for obtaining uniform approximation on compact sets in $L_1 \cup L_2$ by polynomials, it is clearly also necessary for obtaining Carleman approximation on $L_1 \cup L_2$ by entire functions. As we mentioned in the Introduction, Weinstock [5] has solved the problem of describing when $L_1 \cup L_2$ is polynomially convex.

Let $\phi_i\colon \mathbf{C}^n \longrightarrow \mathbf{C}^n$ be bijective $\mathbf{C}$-linear maps such that $\phi_i(L_i) = \mathbf{R}^n$, $i = 1, 2$, and define the differential operators

$$\partial_{(i),j} = (\phi_i^{-1})_*\Big(\frac{\partial}{\partial x_j}\Big)$$

on L_i for $i = 1, 2$ and $j = 1, \ldots, n$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex then we write

$$\partial_{(i)}^{\alpha} = \partial_{(i),1}^{\alpha_1} \cdots \partial_{(i),n}^{\alpha_n}.$$

THEOREM 2. *Let $L_1, L_2 \subset \mathbf{C}^n$ be as in Theorem 1. Let $f\colon L_1 \cup L_2 \longrightarrow \mathbf{C}$ be such that $f|_{L_i} \in C^k(L_i)$, $i = 1, 2$, where k is a positive integer, and assume that there exists some holomorphic polynomial g such that*

$$\partial_{(i)}^{\alpha} g(0) = \partial_{(i)}^{\alpha} f(0)$$

for $i = 1, 2$ and for all multiindices α with $|\alpha| \leq k$. Then f allows Carleman approximation of order k by entire functions, i.e. for any positive continuous function $\epsilon\colon L_1 \cup L_2 \longrightarrow \mathbf{R}$ there exists $h \in O(\mathbf{C}^n)$ such that

$$|\partial_{(i)}^{\alpha} h(z) - \partial_{(i)}^{\alpha} f(z)| < \epsilon(z)$$

for all $z \in L_i$, $i = 1, 2$, *and all multiindices* α *with* $|\alpha| \leq k$.

NOTE. Another formulation of the condition on f at the origin is that if $\tilde{f}$ is any smooth extension of f to a neighborhood of the origin, then $\bar{\partial}\tilde{f}$ will vanish to order $k-1$ at the origin. This is clearly a necessary condition if we want to obtain approximation in C^k-topology by holomorphic functions on any neighborhood of the origin in $L_1 \cup L_2$.

PROPOSITION. *Let* $L_1, L_2 \subset \mathbf{C}^n$ *be as in Theorem 1. Let* $R > 1$ *be given, and let* $K = \{z \in L_1 \cup L_2 : |z| \leq R\}$ *and* $K_i = \{z \in L_i : 1 \leq |z| \leq R\}$, $i = 1, 2$. *Then there exists a closed ball* $B = B(0, \delta)$ *centered at the origin such that for any* $f \in C(L_1 \cup L_2)$ *with compact support contained in* $K_1 \cup K_2$ *and any* $\epsilon > 0$ *there is an entire function* $h \in \mathcal{O}(\mathbf{C}^n)$ *satisfying* $|h - f|_K < \epsilon$ *and* $|h|_B < \epsilon$. *If in addition* $f|_{L_i} \in C^k(L_i)$, $i = 1, 2$, *then we can replace the conclusion* $|h - f|_K < \epsilon$ *by* $|\partial^\alpha_{(i)} h - \partial^\alpha_{(i)} f|_K < \epsilon$ *for all multiindices* α *with* $|\alpha| \leq k$, $i = 1, 2$.

PROOF. Choose $\gamma > 0$ so small that

$$d\big(\phi_i(K_i), \phi_i(L_j)\big) > 3\gamma$$

for $i, j = 1, 2$, $i \neq j$, and let

$$\Omega_0 = \{z \in \mathbf{C}^n : d(z, K) < \eta\},$$

where $\eta > 0$ is so small that

$$\phi_i(\Omega_0) \subset \Big\{z \in \mathbf{C}^n : d\big(z, \phi_i(K)\big) < \gamma\Big\},$$

$i = 1, 2$. Let

$$\tilde{U}_i = \Big\{z \in \mathbf{C}^n : d\big(z, \phi_i(K)\big) < \gamma \text{ and } d\big(z, \phi_i(K_i)\big) < 2\gamma\Big\},$$
$$\tilde{V}_i = \Big\{z \in \mathbf{C}^n : d\big(z, \phi_i(K)\big) < \gamma \text{ and } d\big(z, \phi_i(K_i)\big) > \frac{3}{2}\gamma\Big\},$$

and define

$$\tilde{h}_i^{(t)}(z) = (t\sqrt{\pi})^{-n} \int_{\mathbf{R}^n} e^{-(u-z)^2/t^2} f \circ \phi_i^{-1}(u)\, du.$$

Then $\tilde{h}_i^{(t)} \in \mathcal{O}(\mathbf{C}^n)$, and $\tilde{h}_i^{(t)} \longrightarrow f \circ \phi_i^{-1}$ uniformly on $\phi_i(K_i)$, $\tilde{h}_i^{(t)} \longrightarrow 0$ uniformly on $\tilde{W}_i = \tilde{U}_i \cap \tilde{V}_i$ as $t \longrightarrow 0^+$. If f is smooth of class C^k then the convergence is also of class C^k.

Since K is polynomially convex, we can choose a Stein neighborhood $\Omega \subset \Omega_0$ of K which is Runge in $\mathbf{C}^n$. Let $U_i = \Omega \cap \phi_i^{-1}(\tilde{U}_i)$, $V_i = \Omega \cap \phi_i^{-1}(\tilde{V}_i)$, and $W_i = U_i \cap V_i = \Omega \cap \phi_i^{-1}(\tilde{W}_i)$. Note that $\Omega = U_i \cup V_i$ for $i = 1, 2$, and that $\tilde{h}_i^{(t)} \circ \phi_i \longrightarrow 0$ on W_i. Hence, by solving a Cousin problem on Ω and using the open mapping theorem, we can find $\alpha_i^{(t)} \in \mathcal{O}(U_i)$, $\beta_i^{(t)} \in \mathcal{O}(V_i)$ such that $\alpha_i^{(t)} - \beta_i^{(t)} = \tilde{h}_i^{(t)} \circ \phi_i$ on W_i, $\alpha_i^{(t)} \longrightarrow 0$ on U_i, and $\beta_i^{(t)} \longrightarrow 0$ on V_i. Again, the convergence is of class C^k if f is of class C^k. (See the proof of Theorem 3.1 in [3], or the Proposition in Section 4 of [2].) Let $h_i^{(t)} = \tilde{h}_i^{(t)} \circ \phi_i - \alpha_i^{(t)}$ on U_i and $h_i^{(t)} = -\beta_i^{(t)}$ on V_i, and let $h^{(t)} = h_1^{(t)} + h_2^{(t)} \in \mathcal{O}(\Omega)$. Finally, choose $\delta > 0$ so small that $B = B(0,\delta) \subset V_i$ for $i = 1,2$, let $t > 0$ be sufficiently small, and let $h \in \mathcal{O}(\mathbf{C}^n)$ be a good approximation to $h^{(t)}$ on Ω (in C^k-topology if f is of class C^k). Then h has the desired properties. ■

NOTE. As a corollary, we get by a simple change of coordinates that if $f(z) = 0$ for $|z| \le r$, then f can be approximated arbitrarily well on $K = \{z \in L_1 \cup L_2 : |z| \le rR\}$ by $h \in \mathcal{O}(\mathbf{C}^n)$, where h is small on $B(0, r\delta)$.

PROOF OF THEOREM 1. Let $f \in C(L_1 \cup L_2)$ be given, and let $\epsilon : L_1 \cup L_2 \longrightarrow \mathbf{R}$ be a positive continuous function. Choose $R > 1$, and let δ be as in the Proposition above. Let $\{r_\nu\}$ be a strictly increasing sequence of positive real numbers such that $r_{\nu+2} = r_\nu R$ for all ν. For each ν, let $\tilde\rho_\nu : \mathbf{R} \longrightarrow \mathbf{R}$ be a smooth nondecreasing function such that $\tilde\rho_\nu(t) = 0$ for all $t \le r_\nu$ and $\tilde\rho_\nu(t) = 1$ for all $t \ge r_{\nu+1}$, and define $\rho_\nu(z) = \tilde\rho_\nu(|z|)$ for $z \in L_1 \cup L_2$. Let

$$K_\nu = \{z \in L_1 \cup L_2 : |z| \le r_\nu R\},$$

let $B_\nu = B(0, r_\nu\delta)$, and let $\epsilon_\nu = \min\{\epsilon(z) : z \in K_\nu\}$.

Choose $h_0 \in \mathcal{O}(\mathbf{C}^n)$ such that

$$|h_0 - f|_{K_0} < \frac{\epsilon_0}{2}.$$

Inductively, choose $h_\nu \in \mathcal{O}(\mathbf{C}^n)$ such that

$$\Big| h_\nu - \rho_\nu\Big(f - \sum^{\nu-1} h_j\Big)\Big|_{K_\nu} < \frac{\epsilon_\nu}{2^{\nu+1}}$$

and $|h_\nu|_{B_\nu} < \epsilon_\nu / 2^{\nu+1}$. Let $h = \sum h_\nu$. Since $B_\nu \nearrow \mathbf{C}^n$ it follows that the sum converges uniformly on compact sets in $\mathbf{C}^n$, and that $h \in \mathcal{O}(\mathbf{C}^n)$.

We claim that $|h(z) - f(z)| < \epsilon(z)$ for all $z \in L_1 \cup L_2$. Let $z \in L_1 \cup L_2$ be given, and choose ν_0 such that $r_{\nu_0} < |z| \le r_{\nu_0+1}$. Then

$$\begin{aligned} |h(z) - f(z)| &\le \sum_{\nu>\nu_0} |h_\nu(z)| + \Big| h_{\nu_0}(z) - \rho_{\nu_0}(z)\Big(f(z) - \sum_{\nu<\nu_0} h_\nu(z)\Big)\Big| \\ &\quad + \big(1 - \rho_{\nu_0}(z)\big)\Big| h_{\nu_0-1}(z) - \Big(f(z) - \sum_{\nu<\nu_0-1} h_\nu(z)\Big)\Big| \\ &< \sum_{\nu>\nu_0} \frac{\epsilon_\nu}{2^{\nu+1}} + \frac{\epsilon_{\nu_0}}{2^{\nu_0+1}} + \frac{\epsilon_{\nu_0-1}}{2^{\nu_0}} < \epsilon(z) \end{aligned}$$

since $z \in K_{\nu_0-1}$ and $\rho_{\nu_0-1}(z) = 1$. ■

PROOF OF THEOREM 2. By considering $f - g$ instead of f, it is clear that it suffices to consider the case where f vanishes to order k at the origin. There is a constant $C > 0$ such that for any $\rho > 0$ there is a smooth function ψ_ρ satisfying $0 \le \psi_\rho(z) \le 1$ for all $z \in \mathbf{C}^n$, $\psi_\rho \equiv 0$ on a neighborhood of the origin in $\mathbf{C}^n$, $\psi_\rho \equiv 1$ outside $B(0,\rho)$, and if $|\beta| \le k$ and $z \in L_i \cap B(0,\rho)$ then $|\partial^\beta_{(i)}\psi_\rho(z)| \le C|z|^{-|\beta|}$. Since $\partial^\gamma_{(i)} f(z) = o(|z|^{k-|\gamma|})$ when $|z| \longrightarrow 0$ with $z \in L_i$, it follows that $\psi_\rho f \longrightarrow f$ in C^k-topology as $\rho \longrightarrow 0$. Hence it suffices to prove the theorem in the case where $f \equiv 0$ on a neighborhood of the origin.

We now proceed as in the proof of Theorem 1. Choose $\{r_\nu\}$ such that $f(z) = 0$ for all $z \in L_1 \cup L_2$ with $|z| \leq r_0$. Define ρ_ν, K_ν, B_ν, and ϵ_ν as above. Let $C_\nu > 0$ be so large that

$$\sum_{|\beta| \leq k} |\partial^\beta_{(i)} \rho_\nu(z)| < C_\nu$$

for all $z \in L_i$, $i = 1, 2$, and choose $h_\nu \in \mathcal{O}(\mathbf{C}^n)$ such that

$$\left| \partial^\alpha_{(i)} \Big(h_\nu(z) - \rho_\nu(z) \Big(f(z) - \sum^{\nu-1} h_j(z) \Big) \Big) \right| < \frac{\epsilon_\nu}{2^{\nu+1} C_{\nu+1}}$$

for all $z \in L_i \cap K_\nu$, $i = 1, 2$, and all multiindices α with $|\alpha| \leq k$, and such that $|h_\nu|_{B_\nu} < \epsilon_\nu / 2^{\nu+1}$. Let $h = \sum h_\nu$, it then follows in the same manner as in the proof of Theorem 1 that $h \in \mathcal{O}(\mathbf{C}^n)$ and that

$$|\partial^\alpha_{(i)} h(z) - \partial^\alpha_{(i)} f(z)| < \epsilon(z)$$

for all $z \in L_i$, $i = 1, 2$, and all multiindices α with $|\alpha| \leq k$. ∎

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