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Local–global principle for reduced norms over function fields of p-adic curves

R. Parimala, R. Preeti and V. Suresh

Abstract

Let K be a (non-archimedean) local field and let F be the function field of a curve over K. Let D be a central simple algebra over F of period n and $\lambda \in F^*$. We show that if n is coprime to the characteristic of the residue field of K and $D \cdot (\lambda) = 0$ in $H^3(F, \mu_n^{\otimes 2})$, then λ is a reduced norm from D. This leads to a Hasse principle for the group $\mathrm{SL}_1(D)$, namely, an element $\lambda \in F^*$ is a reduced norm from D if and only if it is a reduced norm locally at all discrete valuations of F.

1. Introduction

Let K be a p-adic field and F a function field in one variable over K. Let Ω_F be the set of all discrete valuations of F. Let G be a semi-simple simply connected linear algebraic group defined over F. It was conjectured in [CPS12] that the Hasse principle holds for principal homogeneous spaces under G over F with respect to Ω_F; i.e. if X is a principal homogeneous space under G over F with $X(F_\nu) \neq \emptyset$ for all $\nu \in \Omega_F$, then $X(F) \neq \emptyset$. If G is $\mathrm{SL}_1(D)$, where D is a central simple algebra over F of square-free index n, it follows from the injectivity of the Rost invariant [MS90] and a Hasse principle for $H^3(F, \mu_n^{\otimes 2})$ due to Kato [Kat86] that this conjecture holds. This conjecture has been settled for classical groups of type B_n, C_n and D_n [Hu14, Pre13]. It is also settled for groups of type ${}^2A_n$ with the assumption that $n+1$ is square-free [Hu14, Pre13].

The main aim of this paper is to prove that the conjecture holds for $\mathrm{SL}_1(D)$ for any central simple algebra D over F with period coprime to p. In fact we prove the following theorem (cf. Theorem 11.1).

Theorem 1.1. *Let K be a local field and F a function field in one variable over K. Let D be a central simple algebra over F of period coprime to the characteristic of the residue field of K and $\lambda \in F^*$. If $D \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then λ is a reduced norm from D.*

This, together with Kato's result on the Hasse principle for $H^3(F, \mu_n^{\otimes 2})$, gives the following theorem (cf. Corollary 11.2).

Theorem 1.2. *Let K be a local field and F a function field in one variable over K. Let Ω_F be the set of discrete valuations of F. Let D be a central simple algebra over F of period n coprime to the characteristic of the residue field of K and $\lambda \in F^*$. If λ is a reduced norm from $D \otimes F_\nu$ for all $\nu \in \Omega_F$, then λ is a reduced norm from D.*

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In fact we may restrict the set of discrete valuations to the set of divisorial discrete valuations of F; namely, those discrete valuations of F centered on a regular proper model of F over the ring of integers in K.

Here are the main steps in the proof. We reduce to the case where D is a division algebra of period ℓ^d with ℓ a prime not equal to p. Given a central division algebra D over F of period $n = \ell^d$ with $\ell \neq p$ and $\lambda \in F^*$ with $D \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, we construct an extension L of F of degree ℓ, and $\mu \in L^*$ such that $N_{L/F}(\mu) = \lambda$, $(D \otimes L) \cdot (\mu) = 0$ and the index of $D \otimes L$ is strictly smaller than the index of D. Then, by induction on the index of D, μ is a reduced norm from $D \otimes L$ and hence $N_{L/F}(\mu) = \lambda$ is a reduced norm from D.

Let $\mathscr{X}$ be a regular proper two-dimensional scheme over the ring of integers in K with function field F and X_0 the reduced special fiber of $\mathscr{X}$. By the patching techniques of Harbater, Hartmann and Krashen [HH10, HHK09], construction of such a pair (L, μ) is reduced to the construction of compatible pairs (L_x, μ_x) over F_x for all $x \in X_0$ (7.5), where for any $x \in X_0$, F_x is the field of fractions of the completion of the regular local ring at x on $\mathscr{X}$. We use local and global class field theory to construct such local pairs (L_x, μ_x). Our proof does not immediately extend to the more general situation where F is a function field in one variable over a complete discretely valued field with arbitrary residue field.

Here is a brief description of the organization of the paper. In § 3 we prove a few technical results concerning central simple algebras and reduced norms over global fields. These results are key to the later patching construction of the fields L_x and $\mu_x \in L_x$ with required properties.

In § 4 we prove the following local variant of Theorem 1.1.

Theorem 1.3. *Let F be a complete discrete valued field with residue field κ. Suppose that κ is a local field or a global field. Suppose further that if κ is a global field, then either n is odd or κ has no real places. Let D be a central simple algebra over F of period n. Suppose that n is coprime to* $\mathrm{char}(\kappa)$. *Let $\alpha \in H^2(F, \mu_n)$ be the class of D and $\lambda \in F^*$. If $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then λ is a reduced norm from D.*

Let A be a complete regular local ring of dimension 2 with residue field κ finite, field of fractions F and maximal ideal $m = (\pi, \delta)$. Let ℓ be a prime not equal to $\mathrm{char}(\kappa)$. Let D be a central simple algebra over F of period ℓ^n with $n \geqslant 1$ and α the class of D in $H^2(F, \mu_{\ell^n})$. Suppose that D is unramified on A, except possibly at π and δ. In § 5 we analyze the structure of D. We prove that the index of D is equal to the period of D. A similar analysis is done by Saltman [Sal97] with the additional assumption that F contains all the primitive ℓ^nth roots of unity, where ℓ^n is the period of D. Let $\lambda \in F^*$. Suppose that $\lambda = u\pi^r\delta^t$ for some unit $u \in A$ and $r, s \in \mathbb{Z}$ and $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_{\ell^n}^{\otimes 2})$. In § 6 we construct possible pairs (L, μ) with L/F of degree ℓ, $\mu \in L$ such that $N_{L/F}(\mu) = \lambda$, $\mathrm{ind}(D \otimes L) < \mathrm{ind}(D)$ and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_{\ell^n}^{\otimes 2})$.

Let K be a local field and F a function field of a curve over K. Let ℓ be a prime not equal to the characteristic of the residue field of K, D a central division algebra over F of period ℓ^n and α the class of D in $H^2(F, \mu_{\ell^n})$. Let $\lambda \in F^*$ with $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_{\ell^n}^{\otimes 2})$. Let $\mathscr{X}$ be a normal proper model of F over the ring of integers in K and X_0 its reduced special fiber. In § 7 we reduce the construction of (L, μ) to the construction of local (L_x, μ_x) for all $x \in X_0$ with some compatible conditions along the 'branches'.

Further, assume that $\mathscr{X}$ is regular and $\mathrm{ram}_{\mathscr{X}}(\alpha) \cup \mathrm{supp}_{\mathscr{X}}(\lambda) \cup X_0$ is a union of regular curves with normal crossings. In § 8, we group the components of X_0 into eight types depending on the valuation of λ, the index of D and the ramification type of D along those components. We call some nodal points of X_0 as special points depending on the type of components passing through

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the point. We also say that two components of X_0 are type 2 connected if there is a sequence of curves of type 2 connecting these two components. We prove that there is a regular proper model of F with no special points and no type 2 connection between certain types of components (Proposition 8.6).

Starting with a model constructed in Proposition 8.6, in § 9 we construct (L_P, μ_P) for all nodal points of X_0 (Proposition 9.8) with the required properties. In § 10, using the class field results of § 3, we construct (L_η, μ_η) for each of the components η of X_0 which are compatible with (L_P, μ_P) when P is in the component η.

Finally, in § 11, we prove the main results by piecing together all the constructions of §§ 7, 9 and 10.

2. Preliminaries

In this section we recall a few definitions and facts about Brauer groups, Galois cohomology groups, residue homomorphisms and unramified Galois cohomology groups. We refer the reader to [Col95] and [GS06].

Let K be a field and $n \geqslant 1$. Let ${}_n\mathrm{Br}(K)$ be the n-torsion subgroup of the Brauer group $\mathrm{Br}(K)$. Assume that n is coprime to the characteristic of K. Let μ_n be the group of nth roots of unity. For $d \geqslant 1$ and $m \geqslant 0$, let $H^d(K, \mu_n^{\otimes m})$ denote the dth Galois cohomology group of K with values in $\mu_n^{\otimes m}$. We have $H^1(K, \mu_n) \simeq K^*/K^{*n}$ and $H^2(K, \mu_n) \simeq {}_n\mathrm{Br}(K)$. For $a \in K^*$, let $(a)_n \in H^1(K, \mu_n)$ denote the image of the class of a in K^*/K^{*n}. When there is no ambiguity of n, we drop n and denote $(a)_n$ by (a). If K is a product of finitely many fields K_i, we denote $\prod H^d(K_i, \mu_n^{\otimes m})$ by $H^d(K, \mu_n^{\otimes m})$.

Let K_s be a separable closure of K. Then $H^1(K, \mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\mathrm{Gal}(K_s/K), \mathbb{Z}/n\mathbb{Z})$. Let $\chi : \mathrm{Gal}(K_s/K) \to \mathbb{Z}/n\mathbb{Z}$ be a continuous homomorphism and E the fixed field of $\ker(\chi)$. Then E/K is a cyclic extension of degree equal to the order of the image of χ. Hence the degree of E divides n. Let $\sigma \in \mathrm{Gal}(K_s/K)$ with $\chi(\sigma) = n/[E : K]$ modulo $n\mathbb{Z}$. Then χ is uniquely determined by the pair (E, σ). Thus every element of $H^1(K, \mathbb{Z}/n\mathbb{Z})$ is uniquely represented by a pair (E, σ), where E/K is a cyclic extension of degree t dividing n and σ a generator of $\mathrm{Gal}(E/K)$. Let $r \geqslant 1$. Then $(E, \sigma)^r \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ is represented by the pair (E', σ'), where E' is the fixed field of the subgroup of $\mathrm{Gal}(E/K)$ generated by $\sigma^{t/d}$, where $d = \gcd(t, r)$, and $\sigma' = \sigma^{r'}$, where $rr' + tt' = d$. In particular, if r is coprime to n, then $(E, \sigma)^r = (E, \sigma^{r'})$ with $rr' \equiv 1$ modulo t. Let $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ and $\chi : \mathrm{Gal}(K_s/K) \to \mathbb{Z}/n\mathbb{Z}$ be the associated homomorphism. Let L/K be a field extension. Then we have the restriction homomorphism $\mathrm{Gal}(L_s/L) \to \mathrm{Gal}(K_s/K)$. Let χ_L be the composition of χ with this restriction. Let E_L/L be the fixed field of $\ker(\chi_L)$ and σ_L be the corresponding generator of $\mathrm{Gal}(E_L/L)$. Then (E_L, σ_L) is the image of (E, σ) under the restriction map $H^1(K, \mathbb{Z}/n\mathbb{Z}) \to H^1(L, \mathbb{Z}/n\mathbb{Z})$. Further, $E \otimes_K L \simeq \prod E_L$.

Let $(E, \sigma) \in H^1(K, \mathbb{Z}/n\mathbb{Z})$ and $\lambda \in K^*$. Let $(E, \sigma, \lambda) = (E/K, \sigma, \lambda)$ denote the cyclic algebra over K,

$$(E, \sigma, \lambda) = E \oplus Ey \oplus \cdots \oplus Ey^{n-1},$$

with $y^n = \lambda$ and $ya = \sigma(a)y$. The cyclic algebra (E, σ, λ) is a central simple algebra and its index is the order of λ in $K^*/N_{E/K}(E^*)$ [Alb61, Theorem 18, p. 98]. The pair (E, σ) represents an element in $H^1(K, \mathbb{Z}/n\mathbb{Z})$ and the element $(E, \sigma) \cdot (\lambda) \in H^2(K, \mu_n)$ is represented by the central simple algebra (E, σ, λ). In particular, $(E, \sigma, \lambda) \otimes E$ is a matrix algebra and hence $\mathrm{ind}(E, \sigma, \lambda) \leqslant [E : K]$.

For $\lambda, \mu \in K^*$ we have [Alb61, p. 97]

$$(E, \sigma, \lambda) + (E, \sigma, \mu) = (E, \sigma, \lambda\mu) \in H^2(K, \mu_n).$$

In particular, $(E, \sigma, \lambda^{-1}) = -(E, \sigma, \lambda)$.

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Let (E,σ,λ) be a cyclic algebra over a field K and L/K be a field extension. Then $(E,\sigma,\lambda)\otimes L$ is Brauer equivalent to (E_L,σ_L,λ). In particular, if L is a finite extension of K and EL is the composite of E and L in an algebraic closure of K, then EL/L is cyclic with Galois group isomorphic to a subgroup of the Galois group of E/K and $(E,\sigma,\lambda)\otimes L$ is Brauer equivalent to (EL,σ_L,λ).

By an abuse of notation, when the role of σ is not important or is clear from the context, we denote (E,σ,λ) by (E,λ).

Lemma 2.1. *Let E/K be a cyclic extension of degree n, σ a generator of $\mathrm{Gal}(E/K)$ and $\lambda\in K^*$. Let m be a factor of n and $d=n/m$. Let M/K be the subextension of E/K with $[M:K]=m$. Then $(E,\lambda)\otimes K(\sqrt[d]{\lambda})=(M(\sqrt[d]{\lambda}),\sqrt[d]{\lambda})$.*

Proof. We have $(E,\sigma)^d=(M,\sigma)\in H^1(K,\mathbb{Z}/n\mathbb{Z})$ and hence

$$\begin{aligned}(E,\lambda)\otimes K(\sqrt[d]{\lambda}) &= (E(\sqrt[d]{\lambda}),\lambda)\\ &= (E(\sqrt[d]{\lambda}),(\sqrt[d]{\lambda})^d)\\ &= (\{E(\sqrt[d]{\lambda})\}^d,(\sqrt[d]{\lambda}))\\ &= (M(\sqrt[d]{\lambda}),\sqrt[d]{\lambda}).\end{aligned}$$ □

Lemma 2.2. *Let K be a complete discretely valued field and ℓ a prime. Let L/K be a cyclic field extension or the split extension of degree ℓ and $\mu\in L^*$. Then there exists $\theta\in L$ with $N_{L/K}(\theta)=1$ such that $L=K(\mu\theta)$ and θ is sufficiently close to 1.*

Proof. Since $[L:K]$ is a prime, if $\mu\notin K$, then $L=K(\mu)$. In this case $\theta=1$ has the required properties.

Suppose that $\mu\in K$. If $L=\prod K$, let $\theta_0\in K^*\backslash\{\pm1\}$ be sufficiently close to 1 and $\theta=(\theta_0,\theta_0^{-1},1,\ldots,1)$. Suppose that L is a field. Let σ be a generator of $\mathrm{Gal}(L/K)$. Suppose that $\mathrm{char}(K)\neq\ell$ contains a primitive ℓth root of unity. Since L/K is cyclic, we have $L=K(\sqrt[\ell]{a})$ for some $a\in K^*$. For any sufficiently large n, $\theta=(1+\pi^n\sqrt[\ell]{a})^{-1}\sigma(1+\pi^n\sqrt[\ell]{a})\in L$ has the required properties.

Suppose that $\mathrm{char}(K)=\ell$ or K contains no primitive ℓth root of unity. Since L/K is separable, we have $L=K(\alpha)$ for some $\alpha\in L^*$. Let $\theta=(1+\sigma(\pi^n\alpha))/(1+\pi^n\alpha)$. Then $\theta\neq1$ and $N_{L/K}(\theta)=1$. Suppose that $\theta\in K$. Then $\theta^\ell=N_{L/K}(\theta)=1$ and hence $\theta=1$, leading to a contradiction. Hence $\theta\notin K$. Therefore for sufficiently large n, θ has the required properties. □

Lemma 2.3. *Let K be a field and E/K be a finite extension of degree coprime to $\mathrm{char}(K)$. Let L/K be a subextension of E/K and $e=[E:L]$. Suppose L/K is Galois and $E=L(\sqrt[e]{\pi})$ for some $\pi\in L^*$. Then E/K is Galois if and only if E contains a primitive eth root of unity and, for every $\tau\in\mathrm{Gal}(L/K)$, $\tau(\pi)\in E^{*e}$.*

Proof. Suppose that E/K is Galois. Let $f(X)=X^e-\pi\in L[X]$. Since $[E:L]=e$ and $E=L(\sqrt[e]{\pi})$, $f(X)$ is irreducible in $L[X]$. Since $f(X)$ has one root in E and E/L is Galois, $f(X)$ has all the roots in E. Hence E contains a primitive eth root of unity. Let $\tau\in\mathrm{Gal}(L/K)$. Then τ can be extended to an automorphism $\tilde{\tau}$ of E. We have $\tau(\pi)=\tilde{\tau}(\pi)=(\tilde{\tau}(\sqrt[e]{\pi}))^e\in E^{*e}$.

Conversely, suppose that E contains a primitive eth root of unity and $\tau(\pi)\in E^{*e}$ for every $\tau\in\mathrm{Gal}(L/K)$. Let

$$g(X)=\prod_{\tau\in\mathrm{Gal}(L/K)}(X^e-\tau(\pi)).$$

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Then $g(X) \in K[X]$ and $g(X)$ splits completely in E. Since e is coprime to $\mathrm{char}(K)$, the splitting field E_0 of $g(X)$ over K is Galois. Since L/K is Galois and E is the composite of L and E_0, E/K is Galois. □

The following lemma is well known.

Lemma 2.4. *Let K be a complete discretely valued field with residue field κ and $\pi \in K$ a parameter. Let e be a natural number coprime to the characteristic of κ. If L/K is a totally ramified extension of degree e, then $L = K(\sqrt[e]{v\pi})$ for some $v \in K$ which is a unit in the valuation ring of K. Further, if e is a power of a prime ℓ, $\theta \in K^*$, $\theta \notin \pm K^{*\ell}$ and $-\theta$ is a norm from L, then $L = K(\sqrt[e]{\theta})$.*

Proof. Since K is a complete discretely valued field, there is a unique extension of the valuation ν on K to a valuation ν_L on L. Since L/K is totally ramified extension of degree e and e is coprime to $\mathrm{char}(\kappa)$, the residue field of L is κ and $\nu_L(\pi) = e$. Let $\pi_L \in L$ with $\nu_L(\pi_L) = 1$. Then $\pi = w\pi_L^e$ for some $w \in L$ with $\nu_L(w) = 0$. Since the residue field of L is same as the residue field of K, there exists $v \in K$ with $\nu(v) = 0$ and the image of v^{-1} is the same as the image of w in the residue field κ. Since L is complete and e is coprime to $\mathrm{char}(\kappa)$, by Hensel's lemma, there exists $u \in L$ such that $w = v^{-1}u^e$. Thus $\pi = w\pi_L^e = v^{-1}u^e\pi_L^e = v^{-1}(u\pi_L)^e$. In particular, $v\pi \in L^{*e}$ and hence $L = K(\sqrt[e]{v\pi})$.

Suppose that $\theta \in K^*$, $\theta \notin \pm K^{*\ell}$ and $-\theta$ is a norm from L. Let $\mu \in L$ with $N_{L/K}(\mu) = -\theta$. Since $L = K(\sqrt[e]{v\pi})$ with $v \in K$ a unit in the valuation ring of K and $\pi \in K$ a parameter, $\sqrt[e]{v\pi} \in L$ is a parameter at the valuation of L. Write $\mu = w_0(\sqrt[e]{v\pi})^s$ for some $w_0 \in L$ a unit at the valuation of L and $s \in \mathbb{Z}$. As above, we have $w_0 = v_1u_1^e$ for some $v_1 \in K$ and $u_1 \in L$. Since $v_1 \in K$, we have

$$\begin{aligned} -\theta = N_{L/K}(\mu) &= N_{L/K}(w_0(\sqrt[e]{v\pi})^s) \\ &= N_{L/K}(v_1u_1^e(\sqrt[e]{v\pi})^s) \\ &= v_1^eN_{L/K}(u_1)^e(-1)^{(e+1)s}(v\pi)^s \\ &= a^e(-1)^s(v\pi)^s, \end{aligned}$$

where $a = v_1N_{L/K}(u_1)(-1)^s$. Hence $\theta = (-1)^{s+1}(v\pi)^s \in K^*/K^{*e}$. Since $\theta \notin \pm K^{*\ell}$ and e is a power of ℓ, s is coprime to ℓ. In particular, $(-1)^{s+1} \in K^e$ and hence $K(\sqrt[e]{\theta}) = K(\sqrt[e]{(v\pi)^s}) = K(\sqrt[e]{v\pi}) = L$. □

Throughout this paper by a local field we mean a non-archimedean local field.

Lemma 2.5. *Let k be a local field and ℓ a prime not equal to the characteristic of the residue field of k. Let L_0/k be an extension of degree ℓ and $\theta_0 \in k^*$. If $\theta_0 \notin \pm k^{*\ell}$ and $-\theta_0$ is a norm from L_0, then $L_0 = k(\sqrt[\ell]{\theta_0})$.*

Proof. Suppose that L_0/k is ramified. Since $\theta_0 \notin \pm k^{*\ell}$, by Lemma 2.4, $L_0 = k(\sqrt[\ell]{\theta_0})$.

Suppose that L_0/k is unramified. Let π be a parameter in k and write $\theta_0 = u\pi^r$ with u a unit in the valuation ring of k. Since θ_0 is a norm from L_0, ℓ divides r and $k(\sqrt[\ell]{\theta_0}) = k(\sqrt[\ell]{u})$ is an unramified extension of k of degree ℓ. Since k is a local field, there is only one unramified field extension of k of degree ℓ and hence $L_0 = k(\sqrt[\ell]{\theta_0})$. □

Lemma 2.6. *Suppose K is a complete discretely valued field with residue field κ a local field. Let ℓ be a prime not equal to the characteristic of the residue field of κ. Let L/K be a field extension of degree ℓ and $\theta \in K^*$. If $\theta \notin \pm K^{*\ell}$ and $-\theta$ is a norm from L, then $L \simeq K(\sqrt[\ell]{\theta})$.*

Proof. If L/K is a ramified extension, then by Lemma 2.4, $L \simeq K(\sqrt[\ell]{\theta})$. Suppose that L/K is an unramified extension. Since $-\theta$ is a norm from L/K, the valuation of θ is divisible by ℓ. Thus, without loss of generality, we assume that θ has valuation zero. Let L_0 be the residue field of L and $\overline{\theta}$ be the image of θ in κ. Then L_0/κ is a field extension of degree ℓ and $-\overline{\theta}$ is a norm from L_0. Since $\theta \notin \pm K^{*\ell}$, $\overline{\theta} \notin \pm\kappa^{\ell}$. Since κ is a local field, $L_0 \simeq \kappa(\sqrt[\ell]{\overline{\theta}})$ (Lemma 2.5) and hence $L \simeq K(\sqrt[\ell]{\theta})$. □

For $L = \prod_1^\ell K$, let σ be the automorphism of L given by $\sigma(a_1, \dots, a_\ell) = (a_2, \dots, a_\ell, a_1)$. Set $\mathrm{Gal}(L/K) = \{\sigma^i \mid 0 \leqslant i \leqslant \ell - 1\}$. Then any σ^i, $1 \leqslant i \leqslant \ell - 1$, is called a generator of $\mathrm{Gal}(L/K)$.

Lemma 2.7. *Let K be a field and ℓ a prime not equal to the characteristic of K. Let L be a cyclic extension of K or the split extension of degree ℓ and σ a generator of the Galois group of L/K. Suppose that there exists an integer $t \geqslant 1$ such that K does not contain a primitive ℓ^tth root of unity. Let $\mu \in L$ with $N_{L/K}(\mu) = 1$ and $m \geqslant t$. If $\mu \in L^{*\ell^{2m}}$, then there exists $b \in L^*$ such that $\mu = b^{-\ell^m}\sigma(b^{\ell^m})$.*

Proof. Suppose $L = \prod K$ and $\mu \in L^{*\ell^s}$ for some $s \geqslant 1$ with $N_{L/K}(\mu) = 1$. Then $\mu = (\theta_1^{\ell^s}, \dots, \theta_\ell^{\ell^s}) \in L$ with $\theta_1^{\ell^s} \cdots \theta_\ell^{\ell^s} = 1$. Without loss of generality we assume that σ is given by $\sigma(a_1, \dots, a_\ell) = (a_2, \dots, a_\ell, a_1)$. Let $b = (1, b_1, \dots, b_{\ell-1}) \in L^*$, where $b_i = \theta_1 \cdots \theta_i$. Then $\mu = b^{-\ell^s}\sigma(b^{\ell^s})$.

Suppose L/K is a cyclic field extension. Write $\mu = \mu_0^{\ell^{2m}}$ for some $\mu_0 \in L$. Let $\mu_1 = \mu_0^{\ell^m}$. Then $\mu = \mu_1^{\ell^m}$. Let $\theta_0 = N_{L/K}(\mu_0)$ and $\theta_1 = N_{L/K}(\mu_1)$. Then $\theta_1 = \theta_0^{\ell^m}$. Since $N_{L/K}(\mu) = 1$, we have $\theta_1^{\ell^m} = N_{L/K}(\mu_1^{\ell^m}) = 1$. If $\theta_1 \neq 1$, then K contains a primitive ℓ^mth root of unity. Since $m \geqslant t$ and K has no primitive ℓ^tth root of unity, $\theta_1 = 1$. Hence $N_{L/K}(\mu_1) = 1$ and by Hilbert's Theorem 90, $\mu_1 = b^{-1}\sigma(b)$ for some $b \in L$. Thus $\mu = \mu_1^{\ell^m} = b^{-\ell^m}\sigma(b^{\ell^m})$. □

We end this section with the following well-known fact.

Lemma 2.8. *Let k be a local field and ℓ a prime not equal to* $\mathrm{char}(\kappa)$*. If $\theta \in k^*$, then there exist a field extension L/k of degree ℓ and $\mu \in L^*$ such that $N_{L/k}(\mu) = \theta$.*

Proof. Let ν be the discrete valuation on k and $\theta \in k^*$. Without loss of generality we assume that $0 \leqslant \nu(\theta) < \ell$. Suppose $\nu(\theta) > 0$. Let $L = k(\sqrt[\ell]{-\theta})$ and $\mu = -\sqrt[\ell]{-\theta} \in L$. Then $N_{L/k}(\mu) = \theta$. Suppose $\nu(\theta) = 0$. Then let L/k be the unramified extension of degree ℓ. Then θ is a norm from L (cf. [Ser79, p. 82, Proposition 3 and Remark 1]). □

3. Global fields

In this section we prove a few technical results concerning Brauer groups of global fields and reduced norms. We begin with the following lemma.

Lemma 3.1. *Let k be a global field, ℓ a prime not equal to* $\mathrm{char}(k)$, *$n, d \geqslant 2$ and $r \geqslant 1$ be integers. Let E_0 be a cyclic extension of k, σ_0 a generator of the Galois group of E_0/k and $\theta_0 \in k^*$. Let $\beta \in H^2(k, \mu_{\ell^n})$ be such that $r\ell\beta = (E_0, \sigma_0, \theta_0) \in H^2(k, \mu_{\ell^n})$. Let S be a finite set of places of k containing all the places of k with $\beta \otimes k_\nu \neq 0$. For each $\nu \in S$, let L_ν/k_ν be a cyclic field extension of degree ℓ or L_ν be the split extension of k_ν of degree ℓ and $\mu_\nu \in L_\nu^*$. Suppose that:*

(1) *$N_{L_\nu/k_\nu}(\mu_\nu) = \theta_0$;*
(2) *$r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu)$;*

(3) $\mathrm{ind}(\beta \otimes E_0 \otimes L_\nu) < d$;

(4) k *contains a primitive* ℓ*th root of unity.*

Then there exist a field extension L_0/k *of degree* ℓ *and* $\mu_0 \in L_0$ *such that:*

(1) $N_{L_0/k}(\mu_0) = \theta_0$;

(2) $r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0)$;

(3) $\mathrm{ind}(\beta \otimes E_0 \otimes L_0) < d$;

(4) $L_0 \otimes k_\nu \simeq L_\nu$ *for all* $\nu \in S$;

(5) μ_0 *is close to* μ_ν *for all* $\nu \in S$.

Proof. Let Ω_k be the set of all places of k and

$$S' = S \cup \{\nu \in \Omega_k \mid \theta_0 \text{ is not a unit at } \nu \text{ or } E_0/k \text{ is ramified at } \nu\}.$$

Let $\nu \in S' \backslash S$. Then $\beta \otimes k_\nu = 0$. Since k contains a primitive ℓth root of unity, there exists a cyclic field extension L_ν of k_ν of degree ℓ such that $\theta_0 \in N(L_\nu^*)$ (cf. the proof of Lemma 2.8). Let $\mu_\nu \in L_\nu$ with $N_{L_\nu/k_\nu}(\mu_\nu) = \theta_0$. Since $\beta \otimes k_\nu = 0$, $\mathrm{ind}(\beta \otimes E_0 \otimes L_\nu) = 1 < d$. Since the corestriction map $\mathrm{cor} : H^2(L_\nu, \mu_{\ell^n}) \to H^2(k_\nu, \mu_{\ell^n})$ is injective (cf. [Lor08, Theorem 10, p. 237]) and $\mathrm{cor}(E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) = (E_0 \otimes k_\nu, \sigma_0 \otimes 1, \theta_0) = r\ell\beta \otimes k_\nu = 0$, $(E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) = 0 = r\beta \otimes L_\nu$. Thus, if necessary, by enlarging S, we assume that S contains all those places ν of k with either θ_0 not a unit at ν or E_0/k ramified at ν and that there is at least one $\nu \in S$ such that L_ν is a field extension of k_ν of degree ℓ.

Let $\nu \in S$. By Lemma 2.2, there exists $\theta_\nu \in L_\nu$ such that $N_{L_\nu/k_\nu}(\theta_\nu) = 1$, $L_\nu = k_\nu(\theta_\nu\mu_\nu)$ and θ_ν is sufficiently close to 1. In particular, $\theta_\nu \in L_\nu^{\ell n}$ and hence $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu) = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu\theta_\nu)$. Thus, replacing μ_ν by $\mu_\nu\theta_\nu$, we assume that $L_\nu = k_\nu(\mu_\nu)$. Let $f_\nu(X) = X^\ell + b_{\ell-1,\nu}X^{\ell-1} + \cdots + b_{1,\nu}X + (-1)^\ell\theta_0 \in k_\nu[X]$ be the minimal polynomial of μ_ν over k_ν.

By Chebotarev's density theorem [FJ08, Theorem 6.3.1], there exists $\nu_0 \in \Omega_k \backslash S$ such that $E_0 \otimes k_{\nu_0}$ is the split extension of k_{ν_0}. By the strong approximation theorem [CF67, p. 67], choose $b_j \in k$, $1 \leqslant j \leqslant \ell - 1$, such that each b_j is sufficiently close to $b_{j,\nu}$ for all $\nu \in S$ and each b_j is an integer at all $\nu \notin S \cup \{\nu_0\}$. Let $L_0 = k[X]/(X^\ell + b_{\ell-1}X^{\ell-1} + \cdots + b_1X + (-1)^\ell\theta_0)$ and $\mu_0 \in L_0$ be the image of X. We now show that L_0 and μ_0 have the required properties.

Since each b_j is sufficiently close to $b_{j,\nu}$ at each $\nu \in S$, it follows from Krasner's lemma that there exists an isomorphism $L_0 \otimes k_\nu \simeq L_\nu$ with the image of $\mu_0 \otimes 1$ in L_ν close to μ_ν for all $\nu \in S$ (cf. [Ser79, ch. II, § 2]). Since L_ν is a field extension of k_ν of degree ℓ for at least one $\nu \in S$, L_0 is a field extension of degree ℓ over k. Since $X^\ell + b_{\ell-1}X^{\ell-1} + \cdots + (-1)^\ell\theta_0$ is the minimal polynomial of μ_0, we have $N(\mu_0) = \theta_0$.

To show that $\mathrm{ind}(\beta \otimes E_0 \otimes L_0) < d$ and $r\beta = (E_0, \sigma_0, \mu_0) \in H^2(L_0, \mu_{\ell^n})$, by the Hasse–Brauer–Noether theorem (cf. [CF67, p. 187]), it is enough to show that for every place w of L_0, $\mathrm{ind}(\beta \otimes E_0 \otimes L_w) < d$ and $r\beta \otimes L_w = (E_0, \sigma_0, \mu_0) \otimes L_w \in H^2(L_w, \mu_{\ell^n})$.

Let w be a place of L_0 and ν a place of k lying below w. Suppose that $\nu \in S$. Then $L_0 \otimes k_\nu \simeq L_\nu$. By the assumption on L_ν, we have $\mathrm{ind}(\beta \otimes E_0 \otimes k_\nu) < d$. Since μ_ν is close to μ_0, we have $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0, \mu_\nu) = (E_0 \otimes L \otimes k_\nu, \sigma_0, \mu_0)$.

Suppose that $\nu \notin S$ and $\nu \neq \nu_0$. Then θ_0 is a unit at ν, E_0/k is unramified at ν and $\beta \otimes k_\nu = 0$. Since each b_j is an integer at ν and μ_0 is a root of the polynomial $X^\ell + b_{\ell-1}X^{\ell-1} + \cdots + b_1X + (-1)^\ell\theta_0$, μ_0 is an integer at w. Since θ_0 is a unit at ν, μ_0 is a unit at w. In particular,

$(E_0 \otimes L_w, \sigma_0, \mu_0) = 0 = r\beta \otimes L_w$. If $\nu = \nu_0$, then by the choice of ν_0, $\beta \otimes k_\nu = 0$, $E_0 \otimes k_\nu$ is the split extension of k_ν and hence $(E_0, \sigma_0, \mu_0) \otimes L_w = 0 = r\beta \otimes L_w$. □

Corollary 3.2. *Let k be a global field, ℓ a prime not equal to* $\mathrm{char}(k)$*, n and $r \geqslant 1$ integers. Suppose that either $\ell \neq 2$ or κ has no real place. Let $\theta_0 \in k^*$ and $\beta \in H^2(k, \mu_{\ell^n})$. Suppose that $r\ell\beta = 0 \in H^2(k, \mu_{\ell^n})$ and $\beta \neq 0$. Then there exist a field extension L_0/k of degree ℓ and $\mu_0 \in L_0$ such that $N_{L_0/k}(\mu_0) = \theta_0$, $r\beta \otimes L_0 = 0$ and* $\mathrm{ind}(\beta \otimes L_0) < \mathrm{ind}(\beta)$.

Proof. Let S be a finite set of places of k containing all the places ν with $\beta \otimes k_\nu \neq 0$. Let $\nu \in S$. Let L_ν/k_ν be a field extension of degree ℓ and $\mu_\nu \in L_\nu$ be such that $N_{L_\nu/k_\nu}(\mu_v) = \theta_0$ (cf. Lemma 2.8).

Since L_ν/k_ν is a field extension of degree ℓ, ℓ divides $\mathrm{ind}(\beta)$ and k_ν is a local field, we have $\mathrm{ind}(\beta \otimes L_\nu) < \mathrm{ind}(\beta)$ [CF67, p. 131]. Since $r\ell\beta = 0$ and L_ν/k_ν is a field extension of degree ℓ, $r\beta \otimes L_\nu = 0$. Let $E_0 = k$. Then, by Lemma 3.1, there exist a field extension L_0/k of degree ℓ and $\mu \in L_0$ with required properties. □

We use the following notation for the rest of this section:

- k a global field with no real places and $\theta_0 \in k^*$;
- ℓ a prime not equal to $\mathrm{char}(k)$;
- k contains a primitive ℓth root of unity;
- E_0/k a cyclic extension of degree a power of ℓ and σ_0 a generator of $\mathrm{Gal}(E_0/k)$;
- $n \geqslant 1$;
- $\beta \in H^2(k, \mu_{\ell^n})$ with $r\ell\beta = (E_0, \sigma_0, \theta_0)$ for some $r \geqslant 1$.

Lemma 3.3. *Suppose that $r\beta \otimes E_0 \neq 0$. If ν is a place of k and L_ν/k_ν a field extension of degree ℓ such that $\theta_0 \in N_{L_\nu/k_\nu}(L_\nu^*)$, then* $\mathrm{ind}(\beta \otimes E_0 \otimes L_\nu) < \mathrm{ind}(\beta \otimes E_0)$.

Proof. Write $r\ell = m\ell^d$ with m coprime to ℓ. Then $d \geqslant 1$. Since $m\ell^d\beta = r\ell\beta = (E_0, \sigma_0, \theta_0)$, we have $m\ell^d\beta \otimes E_0 = 0$. Since m is coprime to ℓ and the period of β is a power of ℓ, it follows that $\ell^d\beta \otimes E_0 = 0$. Since $r\beta \otimes E_0 \neq 0$, $\ell^{d-1}\beta \otimes E_0 \neq 0$ and $\mathrm{per}(\beta \otimes E_0) = \ell^d$.

Let ν be a place of k and L_ν/k_ν a field extension of degree ℓ such that $\theta_0 \in N_{L_\nu/k_\nu}(L_\nu^*)$. Suppose that L_ν is not contained in $E_0 \otimes k_\nu$. Then $[E_0 \otimes L_\nu : E_0 \otimes k_\nu] = \ell$ and hence $\mathrm{ind}(\beta \otimes E_0 \otimes L_\nu) < \mathrm{ind}(\beta \otimes E_0)$ [CF67, p. 131]. Suppose that L_ν is contained in $E_0 \otimes k_\nu$. Then $E_0 \otimes L_\nu = \prod E_i$ with each E_i a cyclic field extension of k_ν. Since E_0/k is a Galois extension, $E_i \simeq E_j$ for all i and j and $m\ell^d\beta \otimes k_\nu = (E_0, \sigma_0, \theta_0) \otimes k_\nu = (E_i, \sigma_i, \theta_0)$ for all i, for suitable generators σ_i of $\mathrm{Gal}(E_i/k_\nu)$. Since L_ν is a field and contained in $E_0 \otimes k_\nu$, L_ν is contained in E_i for all i. Since θ_0 is a norm from L_ν, $\theta_0^{[E_i:k_\nu]/\ell} \in N_{E_i/k_\nu}(E_i^*)$. Since the period of $(E_i, \sigma_i, \theta_0)$ is equal to the order of the class of θ_0 in the group $k_\nu^*/N_{E_i/k_\nu}(E_i^*)$ [Alb61, p. 75], $\mathrm{per}(E_i, \sigma_i, \theta_0) \leqslant [E_i : k_\nu]/\ell < [E_i : k_\nu]$.

Suppose that $\mathrm{per}(\beta \otimes k_\nu) \leqslant [E_i : k_\nu]$. Since k_ν is a local field, $\mathrm{per}(\beta \otimes E_i) = 1$. Thus $\mathrm{per}(\beta \otimes E_0 \otimes k_\nu) = \mathrm{per}(\beta \otimes E_i) = 1 < \ell^d = \mathrm{per}(\beta \otimes E_0)$.

Suppose that $\mathrm{per}(\beta \otimes k_\nu) > [E_i : k_\nu]$. Since $m\ell^d\beta \otimes k_\nu = (E_i, \sigma_i, \theta_0)$ and m is coprime to ℓ, we have $\mathrm{per}(\beta \otimes k_\nu) \leqslant \ell^d\, \mathrm{per}(E_i, \sigma_i, \theta_0)$. Since k_ν is a local field,

$$\mathrm{per}(\beta \otimes E_0 \otimes k_\nu) = \mathrm{per}(\beta \otimes E_i) = \frac{\mathrm{per}(\beta \otimes k_\nu)}{[E_i : k_\nu]} \leqslant \frac{\ell^d\, \mathrm{per}(E_i, \sigma_i, \theta_0)}{[E_i : k_\nu]} < \ell^d = \mathrm{per}(\beta \otimes E_0).$$

Since k_ν is a local field, period equals index and hence the lemma follows. □

Proposition 3.4. *Suppose that $r\beta \otimes E_0 \neq 0$. Then there exist a field extension L_0/k of degree ℓ and $\mu_0 \in L_0$ such that:*

(1) $N_{L_0/k}(\mu_0) = \theta_0$;
(2) $\operatorname{ind}(\beta \otimes E_0 \otimes L_0) < \operatorname{ind}(\beta \otimes E_0)$;
(3) $r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0)$.

Proof. Let S be the finite set of places of k consisting of all those places ν with $\beta \otimes k_\nu \neq 0$. Let $\nu \in S$. By Lemma 2.8, we have a field extension L_ν/k_ν of degree ℓ and $\mu_\nu \in L_\nu$ such that $N_{L_\nu/k_\nu}(\mu_\nu) = \theta_0$ and, by Lemma 3.3, $\operatorname{ind}(\beta \otimes E_0 \otimes L_\nu) < \operatorname{ind}(\beta \otimes E_0)$. Since $\operatorname{cor}_{L_\nu/k_\nu}(r\beta \otimes L_\nu) = r\ell\beta = (E_0 \otimes k_\nu, \sigma_0, \theta_0) = \operatorname{cor}_{L_\nu/k_\nu}(E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu)$ and the corestriction map here is injective (cf. [Lor08, Theorem 10, p. 237]), we have $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, \mu_\nu)$.

By Lemma 3.1, we have the required L_0 and μ_0. □

Proposition 3.5. *Suppose that $r\beta \otimes E_0 = 0$ and $E_0 \neq k$. Let L_0 be the unique subfield of E_0 of degree ℓ over k. Then there exists $\mu_0 \in L_0$ such that:*

(1) $N_{L_0/k}(\mu_0) = \theta_0$;
(2) $r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0)$.

Proof. Since $r\beta \otimes E_0 = 0$ and E_0/k is a cyclic extension, we have $r\beta = (E_0, \sigma_0, \mu')$ for some $\mu' \in k$. We have $(E_0, \sigma_0, \mu'\,\ell) = \ell r\beta = (E_0, \sigma_0, \theta_0)$. Thus $\theta_0 = N_{E_0/k}(y)\mu'\,\ell$. Let $\mu_0 = N_{E_0/L_0}(y)\mu' \in L_0$. Since $L_0 \subset E_0$, we have $r\beta \otimes L_0 = (E_0/L_0, \sigma_0^\ell, \mu') = (E_0/L_0, \sigma_0^\ell, N_{E_0/L_0}(y)\mu') = (E_0/L_0, \sigma_0^\ell, \mu_0)$ (cf. § 2) and

$$N_{L_0/k}(\mu_0) = N_{L_0/k}(N_{E_0/L_0}(y))\mu'\,\ell = \theta_0. \qquad \square$$

Corollary 3.6. *Suppose that $r\beta \otimes E_0 = 0$ and $E_0 \neq k$. Let L_0 be the unique subfield of E_0 of degree ℓ over k. Let S be a finite set of places of k. Suppose that for each $\nu \in S$ there exists $\mu_\nu \in L_0 \otimes k_\nu$ such that:*

- $N_{L_0\otimes k_\nu/k_\nu}(\mu_\nu) = \theta_0$;
- $r\beta \otimes L_0 \otimes k_\nu = (E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, \mu_\nu)$.

Then there exists $\mu \in L_0$ such that:

(1) $N_{L_0/k}(\mu) = \theta_0$;
(2) $r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu)$;
(3) *μ is close to μ_ν for all $\nu \in S$.*

Proof. By Proposition 3.5, there exists $\mu_0 \in L_0$ such that:

- $N_{L_0/k}(\mu_0) = \theta_0$;
- $r\beta \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, \mu_0)$.

Let $\nu \in S$. Then we have:

- $N_{L_0/k}(\mu_0) = \theta_0 = N_{L_0\otimes k_\nu/k_\nu}(\mu_\nu)$;
- $(E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, \mu_0) = (E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, \mu_\nu)$.

Let $b_\nu = \mu_0\mu_\nu^{-1} \in L_0 \otimes k_\nu$. Then $N_{L_0\otimes k_\nu/k_\nu}(b_\nu) = 1$ and $(E_0 \otimes L_0 \otimes k_\nu, \sigma_0 \otimes 1, b_\nu) = 0$. Thus, there exists $a_\nu \in E_0 \otimes L_0 \otimes k_\nu$ with $N_{E_0\otimes L_0\otimes k_\nu/L_0\otimes k_\nu}(a_\nu) = b_\nu$. We have $N_{E_0\otimes L_0\otimes k_\nu/k_\nu}(a_\nu) = N_{L_0\otimes k_\nu/k_\nu}(b_\nu) = 1$. Since E_0/k is a cyclic extension with σ_0 a generator of $\operatorname{Gal}(E_0/k)$, for each $\nu \in S$, there exists $c_\nu \in E_0 \otimes L_0 \otimes k_\nu$ such that $a_\nu = c_\nu^{-1}(\sigma_0 \otimes 1)(c_\nu)$. By weak approximation, there exists $c \in E_0 \otimes L_0$ such that c is close to c_ν for all $\nu \in S$. Let $a = c^{-1}(\sigma \otimes 1)(c) \in E_0 \otimes L_0$ and $\mu = \mu_0 N_{E_0\otimes L_0/L_0}(a) \in L_0$. Then μ has all the required properties. □

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4. Complete discretely valued fields

Let F be a field with a discrete valuation ν, valuation ring R and residue field κ. Suppose that n is coprime to the characteristic of κ. For any $d \geqslant 1$, we have the residue map $\partial_F : H^d(F, \mu_n^{\otimes i}) \to H^{d-1}(\kappa, \mu_n^{\otimes i-1})$. We also denote ∂_F by ∂. An element α in $H^d(F, \mu_n^{\otimes i})$ is called *unramified* at ν or R if $\partial(\alpha) = 0$. The subgroup of all unramified elements is denoted by $H^d_{nr}(F/R, \mu_n^{\otimes i})$ or simply $H^d_{nr}(F, \mu_n^{\otimes i})$. Suppose that F is complete with respect to ν. Then we have an isomorphism $H^d(\kappa, \mu_n^{\otimes i}) \xrightarrow{\sim} H^d_{nr}(F, \mu_n^{\otimes i})$ and the composition $H^d(\kappa, \mu_n^{\otimes i}) \xrightarrow{\sim} H^d_{nr}(F, \mu_n^{\otimes i}) \hookrightarrow H^d(F, \mu_n^{\otimes i})$ is denoted by ι_κ or simply ι.

Let F be a complete discretely valued field with residue field κ, ν the discrete valuation on F and $\pi \in F^*$ a parameter. Suppose that n is coprime to the characteristic of κ. Let $\partial : H^2(F, \mu_n) \to H^1(\kappa, \mathbb{Z}/n\mathbb{Z})$ be the residue homomorphism. Let E/F be a cyclic unramified extension of degree n with residue field E_0 and σ a generator of $\mathrm{Gal}(E/F)$ with $\sigma_0 \in \mathrm{Gal}(E_0/\kappa)$ induced by σ. Then (E, σ, π) is a division algebra over F of degree n. For any $\lambda \in F^*$, we have

$$\partial(E, \sigma, \lambda) = (E_0, \sigma_0)^{\nu(\lambda)}.$$

For $\lambda, \mu \in F^*$, we have

$$\partial((E, \sigma, \lambda) \cdot (\mu)) = (E_0, \sigma_0) \cdot ((-1)^{\nu(\lambda)\nu(\mu)}\theta),$$

where θ is the image of $\lambda^{\nu(\mu)}/\mu^{\nu(\lambda)}$ in the residue field.

Suppose E_0 is a cyclic extension of κ of degree n. Then there is a unique unramified cyclic extension E of F of degree n with residue field E_0. Let σ_0 be a generator of $\mathrm{Gal}(E_0/\kappa)$ and $\sigma \in \mathrm{Gal}(E/F)$ be the lift of σ_0. Then σ is a generator of $\mathrm{Gal}(E/F)$. We call the pair (E, σ) the *lift of* (E_0, σ_0).

We use the following notation throughout this section:

- (F, ν) a complete discretely valued field;
- κ the residue field of F;
- $\pi \in F^*$ a parameter at ν;
- $n \geqslant 2$ an integer coprime to $\mathrm{char}(\kappa)$;
- D a central simple algebra over F of period n;
- $\alpha \in H^2(F, \mu_n)$ the class representing D.

Let $\lambda \in F^*$. In this section we analyze the condition $\alpha \cdot (\lambda) = 0$ and we use this analysis in the proof of our main result (§ 10). We also prove that if κ is either a local field or a global field and $\alpha \cdot (\lambda) = 0$ in $H^3(F, \mu_n^{\otimes 2})$, then λ is a reduced norm from D.

Let E_0 be the cyclic extension of κ and $\sigma_0 \in \mathrm{Gal}(E_0/\kappa)$ be such that $\partial(\alpha) = (E_0, \sigma_0)$. Let (E, σ) be the lift of (E_0, σ_0). The pair (E, σ) or E is called the *lift of the residue* of α. The following lemma is well known.

Lemma 4.1. *Let $\alpha \in H^2(F, \mu_n)$, (E, σ) the lift of the residue of α. Then $\alpha = \alpha' + (E, \sigma, \pi)$ for some $\alpha' \in H^2_{nr}(F, \mu_n)$. Further, $\alpha' \otimes E = \alpha \otimes E$ is independent of the choice of π.*

Proof. Since $\partial(E, \sigma, \pi) = \partial(\alpha)$, $\alpha' = \alpha - (E, \sigma, \pi) \in H^2_{nr}(F, \mu_n)$ and $\alpha = \alpha' + (E, \sigma, \pi)$. □

Lemma 4.2. *Let $\alpha \in H^2(F, \mu_n)$. If $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1, then* $\mathrm{ind}(\alpha) = \mathrm{ind}(\alpha' \otimes E)[E : F] = \mathrm{ind}(\alpha \otimes E)[E : F]$.

Proof. Cf. [FS39, Proposition 1(3)] and [JW90, 5.15]. □

Lemma 4.3. *Let E be the lift of the residue of α. Suppose there exists a totally ramified extension M/F which splits α. Then $\alpha \otimes E = 0$.*

Proof. Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1. Since $\alpha' \otimes E = \alpha \otimes E$, we have $\alpha' \otimes E \otimes M = 0$. Since $E \otimes M/E$ is totally ramified, the residue field of $E \otimes M$ is the same as the residue field of E. Since $\alpha' \otimes E \otimes M = 0$ and $\alpha' \otimes E$ is unramified, it follows from [Ser03, 7.9 and 8.4] that $\alpha' \otimes E = 0$ and hence $\alpha \otimes E = 0$. □

For an element $\zeta \in H^m(F, A)$ for any abelian group A, let $\mathrm{per}(\zeta)$ denote the order of ζ in the group $H^m(F, A)$.

Lemma 4.4. *Let $\alpha \in H^2(F, \mu_n)$ and (E, σ) be the lift of the residue of α. If $\alpha \otimes E = 0$, then $\alpha = (E, \sigma, u\pi)$ for some $u \in F^*$ which is a unit at the discrete valuation, and $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$.*

Proof. We have $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1. Since $\alpha' \otimes E = \alpha \otimes E = 0$, we have $\alpha' = (E, \sigma, u)$ for $u \in F^*$. Since E/F and α' are unramified at the discrete valuation of F, u is a unit at the discrete valuation of F. We have $\alpha = (E, \sigma, u) + (E, \sigma, \pi) = (E, \sigma, u\pi)$. Since E/F is an unramified extension and $u\pi$ is a parameter, $(E, \sigma, u\pi)$ is a division algebra and its period is $[E : F]$. In particular, $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)$. □

Theorem 4.5. *Let F be a complete discretely valued field with residue field κ. Suppose that κ is a local field. Let ℓ be a prime not equal to the characteristic of κ, $n = \ell^d$ and $\alpha \in H^2(F, \mu_n)$. Then $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$.*

Proof. Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1. Then E is an unramified cyclic extension of F with $\partial(\alpha) = (E_0, \sigma_0)$ and α' is unramified at the discrete valuation of F. Let $\overline{\alpha}'$ be the image of α' in $H^2(\kappa, \mu_n)$.

Suppose that $\mathrm{per}(\partial(\alpha)) = \mathrm{per}(\alpha)$. Then $\mathrm{per}(\partial(\alpha)) = [E_0 : \kappa]$. Since F is complete and E/F is an unramified extension, we have $[E_0 : \kappa] = [E : F]$. Thus,

$$\begin{aligned} 0 &= \mathrm{per}(\alpha)\alpha \\ &= \mathrm{per}(\alpha)(\alpha' + (E, \sigma, \pi)) \\ &= \mathrm{per}(\alpha)\alpha' + \mathrm{per}(\alpha)(E, \sigma, \pi) \\ &= \mathrm{per}(\alpha)\alpha' + [E : F](E, \sigma, \pi) \\ &= \mathrm{per}(\alpha)\alpha'. \end{aligned}$$

In particular, $\mathrm{per}(\alpha')$ divides $\mathrm{per}(\alpha) = [E_0, \kappa] = [E : F]$. Since κ is a local field, $\overline{\alpha}' \otimes E_0$ is zero [CF67, p. 131] and hence $\alpha' \otimes E$ is zero. By Lemma 4.4, we have $\alpha = (E, \sigma, \theta\pi)$ for some $\theta \in F$ which is a unit in the valuation ring. In particular, α is cyclic and $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha) = [E : F]$.

Suppose that $\mathrm{per}(\partial(\alpha)) \neq \mathrm{per}(\alpha)$. Then $\mathrm{per}(\partial(\alpha)) < \mathrm{per}(\alpha)$. Since $\mathrm{per}(\partial(\alpha)) = \mathrm{per}(E, \sigma, \pi)$, we have $\mathrm{per}(\alpha) = \mathrm{per}(\alpha')$. Since κ is a local field, $\mathrm{per}(\overline{\alpha}') = \mathrm{ind}(\overline{\alpha}')$. Since $\mathrm{per}(\overline{\alpha}') = \mathrm{per}(\alpha')$ and $\mathrm{per}(\partial(\alpha)) = [E_0 : \kappa]$, we have $[E_0 : \kappa] < \mathrm{per}(\overline{\alpha}')$. Since κ is a local field,

$$\mathrm{ind}(\overline{\alpha}' \otimes E_0) = \frac{\mathrm{per}(\overline{\alpha}')}{[E_0 : \kappa]}.$$

Since E is a complete discretely valued field with residue field E_0 and α' is unramified at the discrete valuation of E, we have $\mathrm{ind}(\alpha' \otimes E) = \mathrm{ind}(\overline{\alpha}' \otimes E_0)$. Thus, we have

$$\begin{aligned} \mathrm{ind}(\alpha) &= \mathrm{ind}(\alpha' \otimes E)[E : F] \quad \text{(by Lemma 4.2)} \\ &= \mathrm{ind}(\overline{\alpha}' \otimes E_0)[E_0 : \kappa] \\ &= \frac{\mathrm{per}(\overline{\alpha}')}{[E_0 : \kappa]}[E_0 : \kappa] \\ &= \mathrm{per}(\overline{\alpha}') = \mathrm{per}(\alpha). \end{aligned}$$ □

Proposition 4.6 [Kat79, Corollary 2, p. 331]. *Suppose that κ is a local field. If L/F is a finite field extension, then the corestriction homomorphism $H^3(L,\mu_n^{\otimes 2}) \to H^3(F,\mu_n^{\otimes 2})$ is bijective.*

Proof. Let κ' be the residue field of L. Since κ and κ' are local fields, $H^3(\kappa,\mu_n^{\otimes 2}) = H^3(\kappa',\mu_n^{\otimes 2}) = 0$ [Ser97, p. 86]. Since F and L are complete discretely valued fields, the residue homomorphisms $H^3(F,\mu_n^{\otimes 2}) \xrightarrow{\partial_F} H^2(\kappa,\mu_n)$ and $H^3(L,\mu_n^{\otimes 2}) \xrightarrow{\partial_L} H^2(\kappa',\mu_n)$ are isomorphisms (cf. [Ser03, 7.9]). The proposition follows from the commutative diagram

$$\begin{array}{ccc} H^3(L,\mu_n^2) & \xrightarrow{\partial_L} & H^2(\kappa',\mu_n) \\ \downarrow & & \downarrow \\ H^3(F,\mu_n^{\otimes 2}) & \xrightarrow{\partial_F} & H^2(\kappa,\mu_n) \end{array}$$

where the vertical arrows are the corestriction maps [Ser03, 8.6]. □

Lemma 4.7. *Let ℓ be a prime not equal to $\mathrm{char}(\kappa)$ and $n = \ell^d$ for some $d \geqslant 1$. Let $\alpha \in H^2(F,\mu_n)$ and $\lambda \in F^*$. Write $\lambda = \theta\pi^r$ for some $\theta,\pi \in F$ with $\nu(\theta) = 0$ and $\nu(\pi) = 1$. Let (E,σ) be the lift of the residue of α and $\alpha = \alpha' + (E,\sigma,\pi)$ as in Lemma 4.1. Then*

$$\partial(\alpha\cdot(-\lambda)) = 0 \iff r\alpha' = (E,\sigma,(-1)^{r+1}\theta) \iff r\alpha = (E,\sigma,(-1)^{r+1}\lambda).$$

In particular, if $\partial(\alpha\cdot(-\lambda)) = 0$ and $r = \nu(\lambda)$ is coprime to ℓ, then $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\lambda})) < \mathrm{ind}(\alpha)$ and $\partial_{F(\sqrt[\ell]{\lambda})}(\alpha\cdot(-\sqrt[\ell]{\lambda})) = 0$.

Proof. Since $r\alpha = r\alpha' + (E,\sigma,\pi^r)$ and $\lambda = \theta\pi^r$, $r\alpha = (E,\sigma,(-1)^{r+1}\lambda)$ if and only if $r\alpha' = (E,\sigma,(-1)^{r+1}\theta)$.

We have

$$\partial(\alpha\cdot(-\lambda)) = \partial((\alpha' + (E,\sigma,\pi))\cdot(-\theta\pi^r)) = r\overline{\alpha}' + (E_0,\sigma_0,(-1)^{r+1}\overline{\theta}^{-1}),$$

where $\partial(\alpha) = (E_0,\sigma_0)$.

Thus $\partial(\alpha\cdot(-\lambda)) = 0$ if and only if $r\overline{\alpha}' + (E_0,\sigma_0,(-1)^{r+1}\overline{\theta}^{-1}) = 0$ if and only if $r\overline{\alpha}' = (E_0,\sigma_0,(-1)^{r+1}\overline{\theta})$ if and only if $r\alpha' = (E,\sigma,(-1)^{r+1}\theta)$ (F being complete).

Suppose $r = \nu(\lambda)$ is coprime to ℓ and $\partial(\alpha\cdot(-\lambda)) = 0$. Clearly $(-1)^{r+1}$ is an ℓ^dth power in F. Thus, we have $r\alpha = (E,\sigma,(-1)^{r+1}\lambda) = (E,\sigma,\lambda)$. Since r is coprime to ℓ, we have

$$\mathrm{ind}(\alpha) = \mathrm{ind}(r\alpha) = \mathrm{ind}(E,\sigma,\lambda) = [E:F]$$

and

$$\begin{aligned}\mathrm{ind}(\alpha\otimes F(\sqrt[\ell]{\lambda})) &= \mathrm{ind}(r\alpha\otimes F(\sqrt[\ell]{\lambda})) = \mathrm{ind}(E(\sqrt[\ell]{\lambda}),\sigma,\lambda) \\ &= [E(\sqrt[\ell]{\lambda}) : F(\sqrt[\ell]{\lambda})]/\ell < [E:F] = \mathrm{ind}(\alpha).\end{aligned}$$

Further, $\partial_{F(\sqrt[\ell]{\lambda})}(r\alpha\cdot(-\sqrt[\ell]{\lambda})) = \partial_{F(\sqrt[\ell]{\lambda})}((E,\sigma,\lambda)\cdot(-\sqrt[\ell]{\lambda})) = (E_0,\sigma_0)\cdot((-1)^{r^2\ell+r\ell})$. If ℓ is even, then $(-1)^{r^2\ell+r\ell} = 1$. If ℓ is odd, then n is odd and -1 is an nth power. Thus, in either case, $(E_0,\sigma_0)\cdot((-1)^{r^2\ell+r\ell}) = 0 \in H^2(\kappa,\mu_n)$. Since r is coprime to ℓ, $\partial_{F(\sqrt[\ell]{\lambda})}(\alpha\cdot(-\sqrt[\ell]{\lambda})) = 0$. □

Lemma 4.8. *Let $n \geqslant 2$ be coprime to $\mathrm{char}(\kappa)$ and ℓ a prime which divides n. Let $\alpha \in H^2(F,\mu_n)$, $\lambda = \theta\pi^{\ell r} \in F^*$ with θ a unit in the valuation ring of F, π a parameter and $\alpha = \alpha' + (E,\sigma,\pi)$ be as in Lemma 4.1. Let L_0/κ be an extension of degree ℓ and $\mu_0 \in L_0$. Suppose that:*

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- $N_{L_0/\kappa}(\mu_0) = -\overline{\theta}$;
- $r\overline{\alpha}' \otimes L_0 = (E_0 \otimes L_0, \sigma_0 \otimes 1, (-1)^r \mu_0)$.

Let L/F be the unramified extension of degree ℓ with residue field L_0. Then, there exists $\mu \in L$ such that:

- *μ a unit in the valuation ring of L;*
- *$\overline{\mu} = \mu_0$;*
- *$N_{L/F}(\mu) = -\theta$;*
- *$\alpha \cdot (\mu\pi^r) \in H^3(L, \mu_n^{\otimes 2})$ is unramified.*

Proof. Since ℓ is a prime and $[L_0 : \kappa] = \ell$, $L_0 = \kappa(\mu_0')$ for any $\mu_0' \in L_0 \backslash \kappa$. Let $g(X) = X^\ell + b_{\ell-1}X^{\ell-1} + \cdots + b_1 X + b_0 \in \kappa[X]$ be the minimal polynomial of μ_0' over κ. Let a_i be in the valuation ring of F mapping to b_i and $f(X) = X^\ell + a_{\ell-1}X^{\ell-1} + \cdots + a_1 X + a_0 \in F[X]$. Suppose $\mu_0 \notin \kappa$. Then we take $\mu_0' = \mu_0$. Since $N_{L_0/\kappa}(\mu_0) = -\overline{\theta}$, we have $b_0 = -(-1)^\ell \overline{\theta}$. Let $a_0 = -(-1)^\ell \theta$. Since $g(X)$ is irreducible in $\kappa[X]$, $f(X) \in F[X]$ is irreducible. Then $L = F[X]/(f)$. Let $\mu \in L$ be the class of X. Then the image of μ is μ_0 and $N_{L/F}(\mu) = -\theta$. Suppose $\mu_0 \in \kappa$. Then $-\overline{\theta} = N_{L_0/\kappa}(\mu_0) = \mu_0^\ell$. Since F is a complete discretely valued field and ℓ is coprime to $\mathrm{char}(\kappa)$, there exists $\mu \in F$ which is a unit in the valuation ring of F which maps to μ_0 and $\mu^\ell = -\theta$.

Since L/F, E/F and α' are unramified at the discrete valuation of F, we have $\partial_L(\alpha' \cdot (\mu\pi^r)) = r\overline{\alpha}' \otimes L_0$ and $\partial_L((E, \sigma, \pi) \cdot (\mu\pi^r)) = (E_0 \otimes L_0, \sigma_0 \otimes 1, (-1)^r \mu_0^{-1})$. Since $\alpha = \alpha' + (E, \sigma, \pi)$, we have

$$\begin{aligned} \partial_L(\alpha \cdot (\mu\pi^r)) &= \partial_L((\alpha' \otimes L) \cdot (\mu\pi^r)) + \partial_L((E, \sigma, \pi) \cdot (\mu\pi^r)) \\ &= r\overline{\alpha}' \otimes L_0 + (E_0 \otimes L_0, \sigma_0 \otimes 1, (-1)^r \mu_0^{-1}) \\ &= 0. \end{aligned}$$

□

Lemma 4.9. *Suppose that κ is a local field. Let ℓ be a prime not equal to $\mathrm{char}(k)$ and n a power of ℓ. Let $\alpha \in H^2(F, \mu_n)$ with $\alpha \neq 0$ and $\lambda \in F^*$. Suppose $\lambda \notin \pm F^{*\ell}$, $\alpha \neq 0$ and $\alpha \cdot (-\lambda) = 0$. Then $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\lambda})) < \mathrm{ind}(\alpha)$ and $\alpha \cdot (-\sqrt[\ell]{\lambda}) = 0 \in H^3(F(\sqrt[\ell]{\lambda}), \mu_n^{\otimes 2})$.*

Proof. Since $\lambda \notin F^{*\ell}$ and $N_{F(\sqrt[\ell]{\lambda})/F}(-\sqrt[\ell]{\lambda}) = -\lambda$, we have $\mathrm{cor}_{F(\sqrt[\ell]{\lambda})/F}(\alpha \cdot (-\sqrt[\ell]{\lambda})) = \alpha \cdot (-\lambda) = 0$. Hence, by Proposition 4.6, $\alpha \cdot (-\sqrt[\ell]{\lambda}) = 0 \in H^3(F(\sqrt[\ell]{\lambda}), \mu_n^{\otimes 2})$.

Suppose $r = \nu(\lambda)$ is coprime to ℓ. Then, by Lemma 4.7, we have $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\lambda})) < \mathrm{ind}(\alpha)$.

Suppose that $\nu(\lambda)$ is divisible by ℓ. Write $\lambda = \theta\pi^{\ell d}$, with $\theta \in F$ a unit in the valuation ring of F. Since $\lambda \notin \pm F^{*\ell}$, $\theta \notin \pm F^{*\ell}$.

Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1. Then $\mathrm{ind}(\alpha) = \mathrm{ind}(\alpha' \otimes E)[E : F]$ (cf. Lemma 4.2) and $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\theta})) \leqslant \mathrm{ind}(\alpha' \otimes E(\sqrt[\ell]{\theta}))[E(\sqrt[\ell]{\theta}) : F(\sqrt[\ell]{\theta})]$.

Suppose $\sqrt[\ell]{\theta} \in E$. Then $F(\sqrt[\ell]{\theta}) \subset E = E(\sqrt[\ell]{\theta})$. In particular, $[E(\sqrt[\ell]{\theta}) : F(\sqrt[\ell]{\theta})] = [E : F(\sqrt[\ell]{\theta})] < [E : F]$. Since θ is a unit in the valuation ring of F, $F(\sqrt[\ell]{\theta})/F$ is unramified and hence π is a parameter in $F(\sqrt[\ell]{\theta})$ and we have $\alpha \otimes F(\sqrt[\ell]{\theta}) = \alpha' \otimes F(\sqrt[\ell]{\theta}) + (E/F(\sqrt[\ell]{\theta}), \sigma^\ell, \pi)$. We have (cf. Lemma 4.2), $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\theta})) = \mathrm{ind}(\alpha' \otimes E)[E : F(\sqrt[\ell]{\theta})] = \mathrm{ind}(\alpha' \otimes E)[E : F]/\ell < \mathrm{ind}(\alpha)$.

Suppose that $\alpha' \otimes E = 0$. Then, by Lemma 4.4, $\alpha = (E, \sigma, u\pi)$ for some unit u in the valuation ring of F. Since $\alpha \cdot (-\lambda) = 0$, $(E, \sigma, u\pi) \cdot (-\lambda) = 0$. Since E/F is unramified with residue field E_0, u, θ are units in the valuation ring of F and π is a parameter, by taking the residue of $\alpha \cdot (-\lambda) = 0$, we see that $(E_0, \sigma_0, -(-1)^{\ell d}\overline{\theta}^{-1}\overline{u}^{\ell d}) = 0 \in H^2(\kappa, \mu_n)$ (cf. Lemma 4.7). In particular, $-(-1)^{\ell d}\overline{\theta}\overline{u}^{-\ell d}$ is a norm from E_0. Since $[E_0 : \kappa]$ is a power of ℓ and E_0/κ is cyclic, there exists a subextension L_0 of E_0 such that $[L_0 : \kappa] = \ell$. Then $-(-1)^{\ell d}\overline{\theta}\overline{u}^{-\ell d}$ is a norm from L_0 and hence $-\overline{\theta}$ is a norm from L_0. Since $\pm\overline{\theta}$ is not in $\kappa^{*\ell}$, by Lemma 2.5, $L_0 = \kappa(\sqrt[\ell]{\overline{\theta}})$. In particular, $\sqrt[\ell]{\overline{\theta}} \in E_0$ and hence $\sqrt[\ell]{\theta} \in E$. Also $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\theta})) < \mathrm{ind}(\alpha)$.

Suppose that $\sqrt[\ell]{\theta} \notin E$. Then, as above, $\alpha' \otimes E \neq 0$. Since E is an unramified extension of F and θ is a unit in the valuation ring of E, $E(\sqrt[\ell]{\theta})$ is an unramified extension of F with residue field $E_0(\sqrt[\ell]{\overline{\theta}})$, where E_0 is the residue field of E and $\overline{\theta}$ is the image of θ in the residue field. Since F is a complete discretely valued field and θ is not an ℓth power in E, $\overline{\theta}$ is not an ℓth power in E_0 and $[E_0(\sqrt[\ell]{\overline{\theta}}) : E_0] = \ell$. Since $\alpha' \otimes E \neq 0$, $\overline{\alpha}' \otimes E_0 \neq 0$. Since E_0 is a local field and $\mathrm{ind}(\overline{\alpha}')$ is a power of ℓ, $\mathrm{ind}(\overline{\alpha}' \otimes E_0(\sqrt[\ell]{\overline{\theta}})) < \mathrm{ind}(\overline{\alpha}' \otimes E_0)$ [CF67, p. 131]. Hence $\mathrm{ind}(\alpha' \otimes E(\sqrt[\ell]{\theta})) < \mathrm{ind}(\alpha' \otimes E)$ and $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\theta})) < \mathrm{ind}(\alpha)$ (cf. Lemma 4.2). □

Lemma 4.10. *Suppose κ is a local field. Let ℓ be a prime not equal to* $\mathrm{char}(\kappa)$ *and $n = \ell^d$. Let $\alpha \in H^2(F, \mu_n)$ and $\lambda \in F^*$. Suppose that κ contains a primitive ℓth root of unity. If $\alpha \neq 0$ and $\alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then there exist a cyclic field extension L/F of degree ℓ and $\mu \in L^*$ such that $N_{L/F}(\mu) = -\lambda$,* $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$ *and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$. Further, if $\nu(\lambda)$ is divisible by ℓ, then one can choose L/F unramified.*

Proof. Suppose $\lambda \notin \pm F^{*\ell}$. Let $L = F(\sqrt[\ell]{\lambda})$ and $\mu = -\sqrt[\ell]{\lambda}$. Then, by Lemma 4.9, $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$ and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$. Clearly $N_{L/F}(\mu) = -\lambda$, and if $\nu(\lambda)$ is a multiple of ℓ, then L/F is unramified.

Suppose $\lambda \in F^{*\ell}$ or $-\lambda \in F^{*\ell}$. Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1.

Suppose that $\alpha' \otimes E = 0$. Then, by Lemma 4.4, $\alpha = (E, \sigma, u\pi)$ for some $u \in F^*$ which is a unit in the valuation ring of F. Since $\alpha \neq 0$, $E \neq F$. Let L be the unique subfield of E with L/F of degree ℓ. Then $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$.

Suppose $-\lambda \in F^{*\ell}$. Then $-\lambda = \mu^\ell$ for some $\mu \in F^*$ and $N_{L/F}(\mu) = \mu^\ell = -\lambda$. Since $\mathrm{cor}_{L/F}(\alpha \cdot (\mu)) = \alpha \cdot (\mu^\ell) = \alpha \cdot (-\lambda) = 0$, by Proposition 4.6, we have $\alpha \cdot (\mu) = 0$ in $H^3(L, \mu_n^{\otimes 2})$.

Suppose $-\lambda \notin F^{*\ell}$. Then $\lambda \in F^{*\ell}$, $\ell = 2$ and $-1 \notin F^{*2}$. Write $\lambda = (\theta\pi^r)^2$ for some $\theta \in F^*$ with $\nu(\theta) = 0$. Since $\alpha \cdot (-\lambda) = 0$ and $\alpha = (E, \sigma, u\pi)$, by taking the residue of $\alpha \cdot (-\lambda)$, we see that $(E_0, \sigma_0) \cdot (-\overline{u}^{2r}\overline{\theta}^{-2}) = 0$. In particular, $-u^{2r}\theta^{-2}$ is a norm from E. Thus -1 is a norm from L. Let $v \in L$ such that $N_{L/F}(v) = -1$ and $\mu = v\theta\pi^r$. Then $N_{L/F}(\mu) = N_{L/F}(v)(\theta\pi^r)^2 = -\lambda$. Since $\mathrm{cor}(\alpha \cdot (\mu)) = \alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$ (cf. Proposition 4.6).

Suppose that $\alpha' \otimes E \neq 0$. Let E_0 be the residue field of E. Then E_0/κ is a cyclic field extension of κ of degree equal to the degree of E/F. Let $\overline{\alpha}'$ be the image of α' in $H^2(\kappa, \mu_n)$. Since $\lambda \in F^{*\ell}$ or $-\lambda \in F^{*\ell}$, $-\lambda = \epsilon\theta^\ell\pi^{r\ell}$ with $\epsilon = \pm 1$ and $\theta \in F^*$ a unit at ν. Since E is a complete discretely valued field, $\overline{\alpha}' \otimes E_0 \neq 0$. Since κ is a local field and contains a primitive ℓth root of unity, there exist a cyclic extension L_0/κ of degree ℓ and $\mu_0 \in L_0$ such that $N_{L_0/\kappa}(\mu_0) = \epsilon\overline{\theta}^\ell$ (cf. the proof of Lemma 2.8). Let L/F be the unramified extension of degree ℓ with residue field L_0. Since F is complete, $\epsilon\theta^\ell \in N_{L/F}(L^*)$. Let $\mu' \in L^*$ such that $N_{L/F}(\mu') = \epsilon\theta^\ell$ and $\mu = \mu'\pi^r$. Then $N_{L/F}(\mu) = -\lambda$. Suppose that $L_0 \not\subset E_0$. Since κ is a local field, $\mathrm{ind}(\overline{\alpha}' \otimes E_0 \otimes L_0) < \mathrm{ind}(\overline{\alpha}' \otimes E_0)$. Since E is a complete discretely valued field with residue field E_0, $\mathrm{ind}(\alpha \otimes E \otimes L) < \mathrm{ind}(\alpha \otimes E)$. Suppose that $L_0 \subset E_0$. Then $L \subset E$. Since L/F is unramified, $\partial(\alpha \otimes L) = \partial(\alpha) \otimes L_0$ (cf. [Col95, Proposition 3.3.1]) and hence the decomposition $\alpha \otimes L = \alpha' \otimes L + (E \otimes L, \sigma \otimes 1, \pi)$ is as in Lemma 4.1. Thus, by Lemma 4.2, $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$. Since $-\lambda = N_{L/F}(\mu)$, as above, we have $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$. □

Lemma 4.11. *Suppose that κ is a global field. Let ℓ be a prime not equal to* $\mathrm{char}(\kappa)$ *and $n = \ell^d$. Suppose that either n is odd or κ has no real places. Let $\alpha \in H^2(F, \mu_n)$ and $\lambda \in F^*$. If $\alpha \neq 0$ and $\alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then there exist a field extension L/F of degree ℓ and $\mu \in L^*$ such that $N_{L/F}(\mu) = -\lambda$,* $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$ *and $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$.*

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Proof. Suppose that $\nu(\lambda)$ is coprime to ℓ. Then, by Lemma 4.7, $L = F(\sqrt[\ell]{\lambda})$ and $\mu = -\sqrt[\ell]{\lambda}$ has the required properties.

Suppose that $\nu(\lambda)$ is divisible by ℓ. Let π be a parameter in F. Then $\lambda = \theta\pi^{r\ell}$ with $\nu(\theta) = 0$. Write $\alpha = \alpha' + (E, \sigma, \pi)$ as in Lemma 4.1. Let $\overline{\alpha}'$ be the image of α' in $H^2(\kappa, \mu_n)$ and θ_0 the image of θ in κ. Since $\alpha \cdot (-\lambda) = 0$, by Lemma 4.7, we have $r\ell\overline{\alpha}' = (E_0, \sigma_0, (-1)^{r\ell+1}\theta_0)$, where E_0 is the residue field of E and σ_0 induced by σ.

Suppose that $r\overline{\alpha}' \otimes E_0 \neq 0$. Then, by Proposition 3.4, there exist an extension L_0/κ of degree ℓ and $\mu_0 \in L_0$ such that $N_{L_0/\kappa}(\mu_0) = (-1)^{r\ell+1}\theta_0$, $\mathrm{ind}(\overline{\alpha}' \otimes E_0 \otimes L_0) < \mathrm{ind}(\overline{\alpha}' \otimes E_0)$ and $r\overline{\alpha}' \otimes L_0 = (E_0 \otimes L_0, \sigma_0, \mu_0)$.

Suppose that $r\overline{\alpha}' \otimes E_0 = 0$. Suppose that $E_0 \neq \kappa$. Let L_0 be the unique subfield of E_0 of degree ℓ over κ. Then, by Proposition 3.5, there exists $\mu_0 \in L_0$ such that $N_{L_0/\kappa}(\mu_0) = (-1)^{r\ell+1}\theta_0$ and $r\overline{\alpha}' \otimes L_0 = (E_0, \sigma_0, \mu_0)$. Suppose that $E_0 = \kappa$. Then, by Corollary 3.2, there exist a field extension L_0/κ of degree ℓ and $\mu_0 \in L_0$ such that $N_{L_0/k}(\mu_0) = (-1)^{r\ell+1}\theta_0$ and $\mathrm{ind}(\overline{\alpha}' \otimes L_0) < \mathrm{ind}(\overline{\alpha}')$. Let $\mu_1 = (-1)^r\mu_0$. Then $N_{L_0/\kappa}(\mu_1) = (-1)^{r\ell}N_{L_0/\kappa}(\mu_0) = (-1)^{r\ell}(-1)^{r\ell+1}\theta_0 = -\theta_0$. Since $(-1)^r\mu_1 = \mu_0$, we have $r\overline{\alpha}' \otimes L_0 = (E_0, \sigma_0, (-1)^r\mu_1)$.

Let L be the unramified extension of F of degree ℓ with residue field L_0. Then, as in the last paragraph of the proof of Lemma 4.10, $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$. By Lemma 4.8, there exists $\mu \in L$ with the required properties. $\square$

Theorem 4.12. *Let F be a complete discretely valued field with residue field κ. Suppose that κ is a local field or a global field. Suppose that either n is odd or κ has no real places. Let D be a central simple algebra over F of period n. Suppose that n is coprime to $\mathrm{char}(\kappa)$. Let $\alpha \in H^2(F, \mu_n)$ be the class of D and $\lambda \in F^*$. If $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, then λ is a reduced norm from D.*

Proof. Write $n = \ell_1^{d_1} \cdots \ell_r^{d_r}$, ℓ_i distinct primes, $d_i > 0$, $D = D_1 \otimes \cdots \otimes D_r$ with each D_i a central simple algebra over F of period power of ℓ_i [Alb61, ch. V, Theorem 18]. Let α_i be the corresponding cohomology class of D_i. Since the ℓ_i are distinct primes, $\alpha \cdot (\lambda) = 0$ if and only if $\alpha_i \cdot (\lambda) = 0$ and λ is a reduced norm from D if and only if λ is a reduced norm from each D_i. Thus without loss of generality we assume that $\mathrm{per}(D) = \ell^d$ for some prime ℓ.

We prove the theorem by induction on the index of D. Suppose that $\mathrm{ind}(D) = 1$. Then every element of F^* is a reduced norm from D. We assume that $\mathrm{ind}(D) = n = \ell^d \geqslant 2$.

Let $\lambda \in F^*$ with $\alpha \cdot (\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$. Let ρ be a primitive ℓth root of unity. Since $[F(\rho) : F]$ is coprime to n, λ is a reduced norm from F if and only if λ is a reduced norm from $D \otimes F(\rho)$. Thus, replacing F by $F(\rho)$, we assume that $\rho \in F$.

Since κ is either a local field or a global field, by Lemmas 4.10 and 4.11, there exist an extension L/F of degree ℓ and $\mu \in L^*$ such that $N_{L/F}(\mu) = \lambda$, $\alpha \cdot (\mu) = 0$ and $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$. Thus, by induction, μ is a reduced norm from $D \otimes L$. Since $N_{L/F}(\mu) = \lambda$, λ is a reduced norm from D. $\square$

The following technical lemma is used in § 6.

Lemma 4.13. *Let κ be a finite field and K a function field of a curve over κ. Let $u, v, w \in \kappa^*$ and $\lambda \in K^*$. Let ℓ be a prime not equal to $\mathrm{char}(\kappa)$ and $\theta = wu\lambda$. If κ contains a primitive ℓth root of unity and $w \notin \kappa^{*\ell}$, then for $r \geqslant 1$, the element $(v, \sqrt[\ell^r]{\theta})_\ell$ in $H^2(K(\sqrt[\ell^r]{\theta}), \mu_\ell)$ is trivial over $K(\sqrt[\ell^r]{\theta}, \sqrt[\ell]{v + u\lambda})$.*

Proof. Let $L = K(\sqrt[\ell^r]{\theta}, \sqrt[\ell]{v + u\lambda})$ and $\beta = (v, \sqrt[\ell^r]{\theta})_\ell$. Since L is a global field, to show that $\beta \otimes L$ is trivial, it is enough to show that $\beta \otimes L_\nu$ is trivial for every discrete valuation ν of L. Let ν be a

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discrete valuation of L. Since $v \in \kappa^*$, v is a unit at ν. If θ is a unit at ν, then $\beta \otimes L$ is unramified at ν and hence $\beta \otimes L_\nu$ is trivial. Suppose that θ is not a unit at ν. Since u and w are units at ν, λ is not a unit. Suppose that $\nu(\lambda) > 0$. Then $v \in L_\nu^{*\ell}$ and hence $\beta \otimes L_\nu$ is trivial. Suppose that $\nu(\lambda) < 0$. Then $\sqrt[\ell]{u\lambda} \in L_\nu$. Since $r \geqslant 1$, $\theta = uw\lambda$ and $\sqrt[\ell^r]{\theta} \in L_\nu$, we have $\sqrt[\ell]{\theta} = \sqrt[\ell]{wu\lambda} \in L_\nu$. Hence $\sqrt[\ell]{w} \in L_\nu$. Since $w \in \kappa^* \backslash \kappa^{*\ell}$, $v \in \kappa^*$ and κ is a finite field, $\sqrt[\ell]{v} \in \kappa(\sqrt[\ell]{w})$. Since $\kappa(\sqrt[\ell]{w}) \subset L_\nu$, $\beta \otimes L_\nu$ is trivial. □

We end this section with the following well-known fact.

Lemma 4.14. *Let L/F be a cyclic extension of degree n, τ a generator of* $\mathrm{Gal}(L/F)$ *and* $\theta \in F^*$. *If $\nu(\theta)$ is coprime to n and* $\mathrm{ind}(L/F, \tau, \theta) = [L : F]$, *then* $[L : F] = \mathrm{per}(\partial(L/F, \tau, \theta))$.

Proof. Let $\beta = (L/F, \tau, \theta)$ and $m = \mathrm{per}(\partial(\beta))$. Since $n = [L : F] = \mathrm{ind}(\beta)$, m divides n. Since $\nu(\theta)$ is coprime to n, $F(\sqrt[m]{\theta})/F$ is a totally ramified extension of degree m with residue field equal to the residue field κ of F. Since $\partial(\beta \otimes F(\sqrt[m]{\theta})) = m\partial(\beta)$, $\beta \otimes F(\sqrt[m]{\theta})$ is unramified. Since $F(\sqrt[n]{\theta})/F(\sqrt[m]{\theta})$ is totally ramified and $\beta \otimes F(\sqrt[n]{\theta})$ is trivial, $\beta \otimes F(\sqrt[m]{\theta})$ is trivial (cf. Lemma 4.3). Hence $n = m$. □

5. Brauer group: complete two-dimensional regular local rings

Let X be an integral regular scheme with function field F. For every point x of X, let $\mathscr{O}_{X,x}$ be the regular local ring at x and $\kappa(x)$ the residue field at x. Let $\hat{\mathscr{O}}_{X,x}$ be the completion of $\mathscr{O}_{X,x}$ at its maximal ideal m_x and F_x the field of fractions of $\hat{\mathscr{O}}_{X,x}$. Then every codimension one point x of X gives a discrete valuation ν_x on F. Let $n \geqslant 1$ be an integer which is a unit on X. For any $d \geqslant 1$, the residue homomorphism $H^d(F, \mu_n^{\otimes j}) \to H^{d-1}(\kappa(x), \mu_n^{\otimes(j-1)})$ at the discrete valuation ν_x is denoted by ∂_x. An element $\alpha \in H^d(F, \mu_n^{\otimes m})$ is said to be *ramified* at x if $\partial_x(\alpha) \neq 0$ and *unramified* at x if $\partial_x(\alpha) = 0$. If $X = \mathrm{Spec}(A)$ and x is a point of X given by (π), π a prime element, we also denote F_x by F_π and $\kappa(x)$ by $\kappa(\pi)$.

Throughout this section A denotes a complete regular local ring of dimension 2 with residue field κ and F its field of fractions. Let ℓ be a prime not equal to the characteristic of κ and $n = \ell^d$ for some $d \geqslant 1$. Let $m = (\pi, \delta)$ be the maximal ideal of A. For any prime $p \in A$, let F_p be the completion of the field of fractions of the completion of the local ring $A_{(p)}$ at p and $\kappa(p)$ the residue field at p.

Lemma 5.1. *Let E_π be an unramified Galois extension of F_π of degree coprime to* $\mathrm{char}(\kappa)$. *Then there exists a Galois extension E of F of degree $[E_\pi : F_\pi]$ which is unramified on A, except possibly at δ and* $\mathrm{Gal}(E/F) \simeq \mathrm{Gal}(E_\pi/F_\pi)$. *Further, if the residue field of E_π is unramified over $\kappa(\pi)$, then E/F can be chosen to be unramified on A.*

Proof. Since A is complete and $m = (\pi, \delta)$, $\kappa(\pi)$ is a complete discretely valued field with residue field κ and the image $\overline{\delta}$ of δ as a parameter. Let E_0 be the residue field of E_π. Then $E_0/\kappa(\pi)$ is a Galois extension with $\mathrm{Gal}(E_0/\kappa(\pi)) \simeq \mathrm{Gal}(E_\pi/F_\pi)$. Let L_0 be the maximal unramified extension of $\kappa(\pi)$ contained in E_0. Then L_0 is also a complete discretely valued field with $\overline{\delta}$ as a parameter and $L_0/\kappa(\pi)$ is Galois. Since E_0/L_0 is a totally ramified extension of degree coprime to $\mathrm{char}(\kappa)$, we have $E_0 = L_0(\sqrt[e]{v\overline{\delta}})$ for some $v \in L_0$ which is a unit at the discrete valuation of L_0 (cf. Lemma 2.4).

Since $E_0/\kappa(\pi)$ is a Galois extension, E_0/L_0 is a Galois extension. Let κ_0 be the residue field of E_0. Then the residue field of L_0 is also κ_0. Since κ_0 is a Galois extension of κ and A is complete, there exists a Galois extension L of F which is unramified on A with residue field κ_0. Let B be the integral closure of A in L. Then B is a regular local ring with residue field κ_0 (cf. [PS14, Lemma 3.1]). Let $u \in B$ be a lift of $\overline{v}$ in κ_0.

Let $E = L(\sqrt[e]{u\delta})$. Since L/F is unramified on A, E/F is unramified on A, except possibly at δ. In particular, E/F is unramified at π with residue field E_0. By construction, $[E : F] = [E_0 : \kappa(\pi)]$. Hence $E \otimes F_\pi \simeq E_\pi$.

Since L/F is a Galois extension which is unramified at π, we have $\mathrm{Gal}(L/F) \simeq \mathrm{Gal}(L_0/\kappa(\pi))$. Let $\tau \in \mathrm{Gal}(L/F)$ and $\overline{\tau} \in \mathrm{Gal}(L_0/\kappa(\pi))$ be the image of τ. Since $E_0/\kappa(\pi)$ is Galois and $E_0 = L_0(\sqrt[e]{v\overline{\delta}})$, by Lemma 2.3, E_0 contains a primitive eth root of unity ρ and $\overline{\tau}(v\overline{\delta}) \in E_0^e$. In particular, $\rho \in \kappa_0$. Since B is complete with residue field κ_0, $\rho \in B$ and hence $\rho \in L \subseteq E$. Since $\overline{\tau}(v\overline{\delta}) = \overline{\tau}(v)\overline{\delta}$ and $v\overline{\delta}$, $\overline{\tau}(v\overline{\delta}) \in E_0^e$, $\overline{\tau}(v)/v \in E_0^e$. Since $\overline{\tau}(v)$ and v are units at the discrete valuation of L_0 and E_0/L_0 is totally ramified, $\overline{\tau}(v)/v \in L_0^e$. Since B is complete and the image of $\tau(u)/u$ in L_0 is $\overline{\tau}(v)/v$, $\tau(u)/u \in L^e$. Since $E = L(\sqrt[e]{u\delta})$, $\tau(u\delta) \in E^e$. Thus, by Lemma 2.3, E/F is Galois. Since $E \otimes F_\pi \simeq E_\pi$, $\mathrm{Gal}(E/F) \simeq \mathrm{Gal}(E_\pi/F_\pi)$.

Further, if the residue field E_0 of E_π is unramified, then $E_0 = L_0$ and hence $E = L$ is unramified on A. □

Since A is complete and (π, δ) is the maximal ideal of A, $A/(\pi)$ is a complete discrete valuation ring with $\overline{\delta}$ as a parameter and $A/(\delta)$ is a complete discrete valuation ring with $\overline{\pi}$ as a parameter. The next lemma follows from [Kat86, Proposition 1.7].

Lemma 5.2 [Kat86, Proposition 1.7]. *Let $m \geqslant 1$ and $\alpha \in H^m(F, \mu_n^{\otimes(m-1)})$. Suppose that α is unramified on A, except possibly at π and δ. Then*

$$\partial_{\overline{\delta}}(\partial_\pi(\alpha)) = -\partial_{\overline{\pi}}(\partial_\delta(\alpha)).$$

Let $H^m_{nr}(F, \mu_n^{\otimes(m-1)})$ be the intersection of the kernels of the residue homomorphisms $\partial_\theta : H^m(F, \mu_n^{\otimes(m-1)}) \to H^{m-1}(\kappa(\theta), \mu_n^{\otimes(m-2)})$ for all primes $\theta \in A$. The next lemma follows from the purity theorem of Gabber.

Lemma 5.3. *For $m = 1, 2$, we have $H^m_{nr}(F, \mu_n^{\otimes(m-1)}) \simeq H^m(\kappa, \mu_n^{\otimes(m-1)})$. For $m \geqslant 3$, we have a surjection $H^m(\kappa, \mu_n^{\otimes(m-1)}) \to H^m_{nr}(F, \mu_n^{\otimes(m-1)})$. In particular, if κ is a finite field and $m \geqslant 2$, then $H^m_{nr}(F, \mu_n^{\otimes(m-1)}) = 0$.*

Proof. For $m \geqslant 1$, by the purity theorem of Gabber (cf. [Rio14, ch. XVI]), we have a surjection $H^m_{\text{ét}}(A, \mu_n^{\otimes(m-1)}) \to H^m_{nr}(F, \mu_n^{\otimes(m-1)})$. Since A is complete, we have $H^m_{\text{ét}}(A, \mu_n^{\otimes(m-1)}) \simeq H^m(\kappa, \mu_n^{\otimes(m-1)})$ (cf. [Mil80, Corollary 2.7, p. 224]). Thus we have a surjection $H^m(\kappa, \mu_n^{\otimes(m-1)}) \to H^m_{nr}(F, \mu_n^{\otimes(m-1)})$. For $m = 1$ and 2, since the map $H^m_{\text{ét}}(A, \mu_n^{\otimes(m-1)}) \to H^m_{nr}(F, \mu_n^{\otimes(m-1)})$ is injective (cf. [MO60, Theorem 7.2]), we have $H^m_{nr}(F, \mu_n^{\otimes(m-1)}) \simeq H^m(\kappa, \mu_n^{\otimes(m-1)})$.

Suppose κ is a finite field and $m \geqslant 2$. Since $H^m(\kappa, \mu_n^{\otimes(m-1)}) = 0$ (cf. [Ser79, § 3.3 p. 80]), we have $H^m_{nr}(F, \mu_n^{\otimes(m-1)}) = 0$. □

Lemma 5.4. *Let $1 \leqslant m \leqslant 3$ and $\alpha \in H^m(F, \mu_n^{\otimes(m-1)})$. Suppose that α is unramified, except possibly at π. Then there exist $\alpha_0 \in H^m(F, \mu_n^{\otimes(m-1)})$ and $\beta \in H^{m-1}(F, \mu_n^{\otimes(m-2)})$ which are unramified on A such that*

$$\alpha = \alpha_0 + \beta \cdot (\pi).$$

Proof. Let $\beta_0 = \partial_\pi(\alpha)$. By Lemma 5.2, $\beta_0 \in H^{m-1}(\kappa(\pi), \mu_n^{\otimes(m-2)})$ is unramified on $A/(\pi)$. Since $A/(\pi)$ is a complete discrete valuation ring with residue field κ, we have $H^{m-1}_{nr}(\kappa(\pi), \mu_n^{\otimes(m-2)}) \simeq H^{m-1}(\kappa, \mu_n^{\otimes(m-2)})$ (cf. Lemma 5.3). Since A is complete, we have $H^{m-1}_{nr}(F, \mu_n^{\otimes(m-1)}) \simeq H^{m-1}(\kappa, \mu_n^{\otimes(m-1)})$ (cf. Lemma 5.3). Thus, there exists $\beta \in H^{m-1}_{nr}(F, \mu^{\otimes(m-1)})$ which is the lift of β_0. Then $\alpha_0 = \alpha - \beta \cdot (\pi)$ is unramified on A. Hence $\alpha = \alpha_0 + \beta \cdot (\pi)$. □

COROLLARY 5.5. *Let $1 \leqslant m \leqslant 3$ and $\alpha \in H^m(F, \mu_n^{\otimes(m-1)})$ is unramified on A, except possibly at π and δ. If $\alpha \otimes F_\delta = 0$, then $\alpha = 0$. In particular, if $\alpha_1, \alpha_2 \in H^m(F, \mu_n^{\otimes(m-1)})$ unramified on A, except possibly at π and δ and $\alpha_1 \otimes F_\delta = \alpha_2 \otimes F_\delta$, then $\alpha_1 = \alpha_2$.*

Proof. Since $\alpha \otimes F_\delta = 0$, α is unramified at δ. Thus α is unramified on A, except possibly at π. By Lemma 5.4, we have $\alpha = \alpha_0 + \beta \cdot (\pi)$ for some $\alpha_0 \in H^m(F, \mu_n^{\otimes(m-1)})$ and $\beta \in H^{m-1}(F, \mu_n^{\otimes(m-2)})$ which are unramified on A. Since $\alpha \otimes F_\delta = 0$, we have $(\beta \cdot (\pi)) \otimes F_\delta = -\alpha_0 \otimes F_\delta$. Since $\beta \cdot (\pi)$ and α_0 are unramified at δ, we have $\overline{\beta} \cdot (\overline{\pi}) = -\overline{\alpha}_0$, where the bar denotes the image over $\kappa(\delta)$. Since $\kappa(\delta)$ is a complete discretely valued field with $\overline{\pi}$ as a parameter, by taking the residues, we see that the image of β is zero in $H^{m-1}(\kappa, \mu_n^{\otimes(m-2)})$. Since A is a complete regular local ring, $\beta = 0$ (cf. Lemma 5.3). Hence $\alpha = \alpha_0$ is unramified on A. Let $\alpha' \in H^m(\kappa, \mu_n^{\otimes m-1})$ which maps to α (cf. Lemma 5.3). Let $\hat{A}_{(\delta)}$ be the completion of the localization of A at (δ). Since $\hat{A}_{(\delta)}$ is a complete discrete valuation ring, the natural map $H^m_{\text{ét}}(\hat{A}_{(\delta)}, \mu_n^{\otimes(m-1)}) \to H^m(F_\delta, \mu_n^{\otimes m-1})$ is injective [Col95, § 3.6]. Thus, since $\alpha \otimes F_\delta = 0$, $\alpha' \otimes \hat{A}_{(\delta)} = 0 \in H^m_{\text{ét}}(\hat{A}_{(\delta)}, \mu_n^{\otimes(m-1)})$. In particular, $\alpha' \otimes A/(\delta) = 0 \in H^m_{\text{ét}}(A/(\delta), \mu_n^{\otimes(m-1)})$ and hence $\alpha' \otimes \kappa = 0 \in H^m(\kappa, \mu_n^{\otimes(m-1)})$. Since A is a complete regular local ring, $\alpha' = 0$ (cf. [Mil80, Corollary 2.7, p. 224]) and hence $\alpha = 0$. □

If $\text{char}(F) = \text{char}(\kappa)$, the above corollary follows from [Hu17, Lemma 2.2].

COROLLARY 5.6. *Let $1 \leqslant m \leqslant 3$ and $\alpha \in H^m(F, \mu_n^{m-1})$. If α is unramified on A, except possibly at π and δ, then $\text{per}(\alpha) = \text{per}(\alpha \otimes F_\pi) = \text{per}(\alpha \otimes F_\delta)$.*

Proof. Suppose $t = \text{per}(\alpha \otimes F_\delta)$. Then $t\alpha \otimes F_\delta = 0$ and hence, by Corollary 5.5, $t\alpha = 0$. Since $\text{per}(\alpha \otimes F_\delta) \leqslant \text{per}(\alpha)$, it follows that $\text{per}(\alpha) = \text{per}(\alpha \otimes F_\delta)$. Similarly, $\text{per}(\alpha) = \text{per}(\alpha \otimes F_\pi)$. □

COROLLARY 5.7. *Suppose that κ is a finite field. Let $\alpha \in H^2(F, \mu_n)$. If α is unramified, except at π and δ, then there exist a cyclic extension E/F and $\sigma \in \text{Gal}(E/F)$ a generator, $u \in A$ a unit, and $0 \leqslant i, j < n$ such that $\alpha = (E, \sigma, u\pi^i\delta^j)$ with E/F unramified on A, except at δ and $i = 1$, or E/F unramified on A, except at π and $j = 1$.*

Proof. Since n is a power of the prime ℓ and $n\alpha = 0$, $\text{per}(\partial_\pi(\alpha))$ and $\text{per}(\partial_\delta(\alpha))$ are powers of ℓ. Let d' be the maximum of $\text{per}(\partial_\pi(\alpha))$ and $\text{per}(\partial_\delta(\alpha))$. Then $\partial_\pi(d'\alpha) = d'\partial_\pi(\alpha) = 0$ and $\partial_\delta(d'\alpha) = d'\partial_\delta(\alpha) = 0$. In particular, $d'\alpha$ is unramified on A. Since κ is a finite field, $d'\alpha = 0$. Hence $\text{per}(\alpha)$ divides d' and $d' = \text{per}(\alpha)$. Thus $\text{per}(\alpha) = \text{per}(\partial_\pi(\alpha))$ or $\text{per}(\partial_\delta(\alpha))$.

Suppose that $\text{per}(\alpha) = \text{per}(\partial_\pi(\alpha))$. Since $\partial_\pi(\alpha \otimes F_\pi) = \partial_\pi(\alpha)$, we have $\text{per}(\partial_\pi(\alpha)) \leqslant \text{per}(\alpha \otimes F_\pi) \leqslant \text{per}(\alpha)$. Thus $\text{per}(\alpha \otimes F_\pi) = \text{per}(\partial_\pi(\alpha \otimes F_\pi))$. Let $(E_0, \sigma_0) = \partial_\pi(\alpha \otimes F_\pi)$ and $(E_\pi/F_\pi, \sigma)$ be the lift of (E_0, σ_0). Then $[E_\pi : F_\pi] = [E_0 : \kappa(\pi)] = \text{per}(\partial_\pi(\alpha \otimes F_\pi)) = \text{per}(\alpha \otimes F_\pi)$. Write $\alpha \otimes F_\pi = \alpha' + (E_\pi, \sigma, \pi)$ as in Lemma 4.1. Let $\overline{\alpha}'$ be the image of α' over $\kappa(\pi)$. Since $\kappa(\pi)$ is a local field and $\text{per}(\overline{\alpha}')$ divides $\text{per}(\alpha \otimes F_\pi) = [E_0 : \kappa(\pi)]$, we have $\overline{\alpha}' \otimes E_0 = 0$ and hence $\alpha' \otimes E_\pi = 0$. Since $\alpha \otimes E_\pi = \alpha' \otimes E_\pi = 0$, by Lemma 4.4, we have $\alpha \otimes F_\pi = (E_\pi/F_\pi, \sigma, \theta\pi)$ for some cyclic unramified extension E_π/F_π and $\theta \in F_\pi$ a unit in the valuation ring of F_π.

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By Lemma 5.1, there exists a Galois extension E/F which is unramified on A, except possibly at (δ), such that $E\otimes F_\pi \simeq E_\pi$. Since E_π/F_π is cyclic, E/F is cyclic. Since $\theta\in F_\pi$ is a unit in the valuation ring of F_π and the residue field of F_π is a complete discretely valued field with $\overline{\delta}$ as parameter, we can write $\theta = u\delta^j\theta_1^n$ for some unit $u\in A$, $\theta_1\in F_\pi$ and $0\leqslant j\leqslant n-1$. Then $\alpha\otimes F_\pi \simeq (E,\sigma,u\delta^j\pi)\otimes F_\pi$. Thus, by Corollary 5.5, we have $\alpha=(E,\sigma,u\delta^j\pi)$.

If $\mathrm{per}(\alpha)=\mathrm{per}(\partial_\delta(\alpha))$, then, as above, we get $\alpha=(E,\sigma,u\pi^i\delta)$ for some cyclic extension E/F which is unramified on A, except possibly at π. □

The following proposition is proved in [RS13, 2.4] under the assumption that F contains a primitive nth root of unity.

Proposition 5.8. *Suppose that κ is a finite field. Let $\alpha\in H^2(F,\mu_n)$. If α is unramified on A, except possibly at (π) and (δ), then* $\mathrm{ind}(\alpha)=\mathrm{ind}(\alpha\otimes F_\pi)=\mathrm{ind}(\alpha\otimes F_\delta)$.

Proof. Suppose that α is unramified on A, except possibly at (π) and (δ). Then, by Corollary 5.7, we assume without loss of generality that $\alpha=(E/F,\sigma,\pi\delta^j)$ with E/F unramified on A, except possibly at δ. Then $\mathrm{ind}(\alpha)\leqslant[E:F]$. Since E/F is unramified on A except possibly at δ, we have $[E:F]=[E_\pi:F_\pi]$ and $\mathrm{ind}(\alpha\otimes F_\pi)=[E_\pi:F_\pi]$. Thus $[E:F]=[E_\pi:F_\pi]=\mathrm{ind}(\alpha\otimes F_\pi)\leqslant \mathrm{ind}(\alpha)\leqslant[E:F]$ and hence $[E:F]=\mathrm{ind}(\alpha\otimes F_\pi)=\mathrm{ind}(\alpha)$. □

Corollary 5.9. *Suppose that κ is a finite field. Let $\alpha\in H^2(F,\mu_n)$. If α is unramified on A, except possibly at (π) and (δ), then* $\mathrm{ind}(\alpha)=\mathrm{per}(\alpha)$.

Proof. By Corollary 5.6, $\mathrm{per}(\alpha)=\mathrm{per}(\alpha\otimes F_\pi)$, and by Theorem 4.5, $\mathrm{ind}(\alpha\otimes F_\pi)=\mathrm{per}(\alpha\otimes F_\pi)$. Thus $\mathrm{per}(\alpha)=\mathrm{ind}(\alpha\otimes F_\pi)$. By Proposition 5.8), we have $\mathrm{ind}(\alpha)=\mathrm{per}(\alpha)$. □

Let $\mathscr{X}$ be an integral regular two-dimensional scheme with field of fractions F. For each $x\in\mathscr{X}$, let F_x denote the field of fractions of the completion of the local ring at x. The following proposition follows from [HHK15b].

Proposition 5.10. *Let $\alpha\in H^2(F,\mu_n)$. Let $\phi:\mathscr{X}\to\mathrm{Spec}(A)$ be a sequence of blow-ups and $V=\phi^{-1}(m)$. Then* $\mathrm{ind}(\alpha)=\mathrm{l.c.m.}\{\mathrm{ind}(\alpha\otimes F_x)\mid x\in V\}$.

Proof. Let η be the generic point of an irreducible component of an exceptional curves in $\mathscr{X}$. Then, arguing as in [HHK15a, Theorems 9.2 and 9.12], we get that $\mathrm{ind}(\alpha\otimes F_\eta)=\mathrm{ind}(\alpha\otimes F_U)$ for some nonempty open set U of the closure of η. Since A is a complete regular local ring of dimension 2, the proposition follows by [HHK15b, Lemma 4.6 and Example 4.16]. □

We end this section with the following well-known results.

Lemma 5.11. *Let E/F be a cyclic extension of degree ℓ^d for some $d\geqslant 1$. If E/F is unramified on A, except possibly at δ, then there exist a subextension E_{nr} of E/F and $w\in E_{nr}$ which is a unit in the integral closure of A in E_{nr} such that E_{nr}/F is unramified on A and $E=E_{nr}(\sqrt[\ell^e]{w\delta})$ for some $e\geqslant 0$. Further, if κ is a finite field containing a primitive ℓth root of unity and $0<e<d$, then $N_{E/F}(\sqrt[\ell^e]{w\delta})=w_1\delta^{\ell^{d-e}}$ with $w_1\in A$ a unit and not an ℓth power in A.*

Proof. Let $E(\pi)$ be the residue field of E at π. Since E/F is unramified at A, except possibly at δ, by Corollary 5.6 (with $m=1$), $[E(\pi):\kappa(\pi)]=[E:F]$. Since E/F is cyclic, $E(\pi)/\kappa(\pi)$ is cyclic. As in the proof of Lemma 5.1, there exist a cyclic extension E_0/F unramified on A and a unit w in the integral closure of A in E_0 such that the residue field of $E_0(\sqrt[\ell^e]{w\delta})$ at π is $E(\pi)$. By Corollary 5.5 (with $m = 1$), we have $E \simeq E_0(\sqrt[\ell^e]{w\delta})$. Let $E_{nr} = E_0$. Then E_{nr} has the required properties. Since $[E : F] = \ell^d$ and $[E : E_{nr}] = \ell^e$, we have $[E_{nr} : F] = \ell^f$, where $f = d - e$.

Suppose that κ is a finite field and contains a primitive ℓth root of unity. Let B be the integral closure of A in E_{nr}. Then B is a complete regular local ring with residue field κ' a finite extension of κ.

Let $w_0 = N_{E_{nr}/F}(w) \in A^*$ and $\overline{w}_0 \in \kappa^*$. Suppose that $w_0 \in A^{*\ell}$. Then $\overline{w}_0 \in \kappa^{*\ell}$. Since κ contains a primitive ℓth root of unity, we have $|\kappa'^*/\kappa'^{*\ell}| = |\kappa^*/\kappa^{*\ell}| = \ell$. Since it is surjective from κ' to κ, the norm map induces an isomorphism from $\kappa'^*/\kappa'^{*\ell}$ to $\kappa^*/\kappa^{*\ell}$. Thus the image of w in κ' is an ℓth power. Since B is a complete regular local ring, $w \in B^{*\ell}$. Suppose $0 < e < d$. Then $\sqrt[\ell]{\delta} \in E$. Since E_{nr}/F is a nontrivial unramified extension and $F(\sqrt[\ell]{\delta})/F$ is a nontrivial extension of F which is totally ramified at δ, we have two distinct subextensions of E/F of degree ℓ, in contradiction to the fact that E/F is cyclic. Hence $w_0 \notin A^{*\ell}$. Further, we have $N_{E/F}(\sqrt[\ell^e]{w\delta}) = N_{E_{nr}/F}((-1)^{\ell^e+1}w\delta) = (-1)^{(\ell^e+1)\ell^f} w_0\delta^{\ell^f}$. Since $f > 0$, $w_1 = (-1)^{(\ell^e+1)\ell^f} w_0$ is not an ℓth power in A. □

Lemma 5.12. *Suppose κ is a perfect field. Let L_π/F_π be an unramified field extension of degree N. Then there exists a field extension L/F of degree N such that $L \otimes F_\pi \simeq L_\pi$ and the integral closure of A in L is regular.*

Proof. Let $L(\pi)$ be the residue field of L_π. Suppose that $L(\pi)/\kappa(\pi)$ is unramified at the discrete valuation of $A/(\pi)$. Let κ' be the residue field of $L(\pi)$. Then κ'/κ is an extension of degree N. Write $\kappa' = \kappa[T]/(f(T))$ for some monic polynomial. Let $g(T) \in A[T]$ be a monic polynomial which is a lift of $f(T)$. Then clearly $L = F[T]/(g(T))$ has the required properties.

Suppose $L(\pi)/\kappa(\pi)$ is ramified. Let $L(\pi)_{nr}$ be the maximal unramified extension of $\kappa(\pi)$ contained in $L(\pi)$. Let $\tilde{L}_\pi$ be the subextension of L_π with residue field $L(\pi)_{nr}$. Then, as above, there exists a field extension $\tilde{L}/F$ such that $\tilde{L} \otimes F_\pi \simeq \tilde{L}_\pi$. Let $\tilde{A}$ be the integral closure of A in $\tilde{L}$. Then $\tilde{A}$ is a regular local ring with (π, δ) as the maximal ideal. Thus, replacing F by $\tilde{L}_\pi$, we assume that $L(\pi)/\kappa(\pi)$ is totally ramified. Hence $L(\pi) = \kappa(\pi)[T]/(f(T))$ with $f(T) = T^N + \overline{a}_{N-1}\overline{\delta}T^{N-1} + \cdots + \overline{a}_1\overline{\delta}T + \overline{v}\overline{\delta}$ for some $a_i \in A$ and a unit $v \in A$, where the bar denotes the image in $A/(\pi)$. Let $g(T) = T^N + a_{N-1}\delta T^{N-1} + \cdots + a_1\delta T + v\delta \in A[T]$. Let $L = F[T]/(g(T))$ and $B = A[T]/(g(T))$. Let $\tilde{m}$ be a maximal ideal of B. Let t be the image of T in B. We have $t(t^{N-1} + a_{N-1}\delta t^{N-2} + \cdots + a_1\delta) = -v\delta$. Since $\delta \in m \subset \tilde{m}$, it follows that $t \in \tilde{m}$. Since $B/(\pi, t) \simeq \kappa$, $\tilde{m} = (\pi, t)$ is the unique maximal ideal of B and hence B is a regular local ring. In particular, B is integrally closed and hence B is the integral closure of A in L. □

Remark 5.13. Let L_π/F_π be an unramified extension of degree N and L/F be the extension of degree N as in the proof of Lemma 5.12. Let B be the integral closure of A in L. Then, by the construction of L, (π, δ') is the maximal ideal of B for some $\delta' \in B$ such that δ' is the only prime in B lying over δ and $N_{L/F}(\delta') = v\delta^f$ for some unit $v \in A$ and $f \geqslant 1$.

6. Reduced norms: complete two-dimensional regular local rings

Throughout this section we fix the following notation:

- A a complete two-dimensional regular local ring;
- F the field of fractions of A;
- $m = (\pi, \delta)$ the maximal ideal of A;

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- $\kappa = A/m$ a finite field;
- ℓ a prime not equal to char(κ);
- $n = \ell^d$;
- $\alpha \in H^2(F, \mu_n)$ is unramified on A, except possibly at (π) and (δ);
- $\lambda = w\pi^s\delta^t$, $w \in A$ a unit and $s, t \in \mathbb{Z}$ with $1 \leqslant s, t < n$.

The aim of this section is to prove that if $\alpha \neq 0$ and $\alpha \cdot (\lambda) = 0$, then there exist an extension L/F of degree ℓ and $\mu \in L$ such that $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$ and $N_{L/F}(\mu) = \lambda$. We assume that:

- F contains a primitive ℓth root of unity.

We begin with the following lemma.

Lemma 6.1. *If $\alpha \cdot (-\lambda) = 0$, then $s\alpha = (E, \sigma, (-1)^{s+1}\lambda)$ for some cyclic extension E of F which is unramified on A, except possibly at δ. In particular, if s is coprime to ℓ, then $\alpha = (E', \sigma', (-1)^{s+1}\lambda)$ for some cyclic extension E' of F which is unramified on A, except possibly at δ.*

Proof. By Lemma 4.7, there exists an unramified cyclic extension E_π of F_π such that $s\alpha \otimes F_\pi = (E_\pi, \sigma, (-1)^{s+1}\lambda)$. By Lemma 5.1, there exists a cyclic extension E of F which is unramified on A, except possibly at δ with $E \otimes F_\pi \simeq E_\pi$. Since E/F is unramified on A, except possibly at δ and $\lambda = w\pi^s\delta^t$ with w a unit in A, $(E, \sigma, (-1)^{s+1}\lambda)$ is unramified on A, except possibly at (π) and (δ). Since α is unramified on A, except possibly at (π) and (δ), $s\alpha - (E, \sigma, (-1)^{s+1}\lambda)$ is unramified on A, except possibly at (π) and (δ). Since $s\alpha \otimes F_\pi = (E_\pi, \sigma, (-1)^{s+1}\lambda) = (E, \sigma, (-1)^{s+1}\lambda) \otimes F_\pi$, by Corollary 5.5, $s\alpha = (E, \sigma, (-1)^{s+1}\lambda)$. □

Lemma 6.2. *Suppose that $\alpha \cdot (-\lambda) = 0$ and $\lambda \notin \pm F^{*\ell}$. If $\alpha \neq 0$, then $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\lambda})) < \mathrm{ind}(\alpha)$ and $\alpha \cdot (-\sqrt[\ell]{\lambda}) = 0 \in H^3(F(\sqrt[\ell]{\lambda}), \mu_n^{\otimes 2})$.*

Proof. Suppose that s is coprime to ℓ. Then, by Lemma 6.1, $\alpha = (E', \sigma', (-1)^{s+1}\lambda)$ for some cyclic extension E' of F which is unramified on A, except possibly at δ. Since $\nu_\pi(\lambda) = s$ is coprime to ℓ and E'/F is unramified at π, it follows that $\mathrm{ind}(\alpha) = [E' : F]$. In particular, $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{(-1)^{s+1}\lambda})) \leqslant [E' : F]/\ell < \mathrm{ind}(\alpha)$. Since s is coprime to ℓ, we have $(-1)^s = -(\epsilon)^\ell$ for some $\epsilon = \pm 1$ and hence $F(\sqrt[\ell]{(-1)^{s+1}\lambda}) = F(\sqrt[\ell]{\lambda})$. Similarly, if t is coprime to ℓ, then $\mathrm{ind}(\alpha \otimes F(\sqrt[\ell]{\lambda})) < \mathrm{ind}(\alpha)$. Further, $\alpha \cdot (-\sqrt[\ell]{\lambda}) = (E', \sigma', \lambda) \cdot (-\sqrt[\ell]{\lambda}) = 0$.

Suppose that s and t are divisible by ℓ. Since $\lambda = w\pi^s\delta^t$, we have $F(\sqrt[\ell]{\lambda}) = F(\sqrt[\ell]{w})$. Let $L = F(\sqrt[\ell]{w})$ and B be the integral closure of A in L. Since w is a unit in A, by [PS14, Lemma 3.1], B is a complete regular local ring with maximal ideal generated by π and δ. Since $\lambda \notin \pm F^{*\ell}$ and A is a complete regular local ring, the images of $\pm w$ in A/m are not ℓth powers. Since $A/(\pi)$ is also a complete regular local ring with residue field A/m, the images of $\pm w$ in $A/(\pi)$ are not ℓth powers. Since F_π is a complete discretely valued field with residue field the field of fractions of $A/(\pi)$, $\pm w$ are not ℓth powers in F_π. Since $\alpha \cdot (-\lambda) = 0$ and the residue field of F_π is a local field, by Lemma 4.9, $\mathrm{ind}(\alpha \otimes L_\pi) < \mathrm{ind}(\alpha)$. Hence, by Proposition 5.8, $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$.

Since $L_\pi = L \otimes F_\pi$ and $L_\delta = L \otimes F_\delta$ are field extensions of degree ℓ over F_π and F_δ respectively, and $\mathrm{cores}(\alpha \cdot (-\sqrt[\ell]{\lambda})) = \alpha \cdot (-\lambda) = 0$, by Proposition 4.6, $(\alpha \cdot (-\sqrt[\ell]{\lambda})) \otimes L_\pi = 0$ and $(\alpha \cdot (-\sqrt[\ell]{\lambda})) \otimes L_\delta = 0$. Hence, by Corollary 5.5, $\alpha \cdot (-\sqrt[\ell]{\lambda}) = 0$. □

Lemma 6.3. *Suppose $\alpha = (E/F, \sigma, u\pi\delta^{\ell m})$ for some $m \geqslant 0$, u a unit in A, E/F a cyclic extension of degree ℓ^d which is unramified on A, except possibly at δ, and σ a generator of $\mathrm{Gal}(E/F)$. Let ℓ^e be the ramification index of E/F at δ and $f = d - e$. Let $i \geqslant 1$ be such that $\ell^f + \ell^{di} > \ell m$. Let $v \in A$ be a unit which is not in $F^{*\ell}$ and $L = F(\sqrt[\ell]{v\delta^{\ell^f + \ell^{di} - \ell m} + u\pi})$. If $f > 0$, then $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$.*

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Proof. Let B be the integral closure of A in L and $r = \ell^f + \ell^{di} - \ell m$. Since $\ell^f + \ell^{di} > \ell m$, $L = F(\sqrt[\ell]{v\delta^r + u\pi})$ and $v\delta^r + u\pi$ is a regular prime in A. Thus B is a complete regular local ring (cf. [PS14, Lemma 3.2]) and π, δ remain primes in B. Note that π and δ may not generate the maximal ideal of B. Let L_π and L_δ be the completions of L at the discrete valuations given by π and δ, respectively. Since $v \notin F^{*\ell}$, $F(\sqrt[\ell]{v})$ is the unique extension of F of degree ℓ, which is unramified on A. Since $f > 0$, there is a subextension of E of degree ℓ over F which is unramified on A and hence $F(\sqrt[\ell]{v}) \subset E$.

Since E/F is unramified on A, except possibly at δ, $[E : F] = [E_\pi : F_\pi]$ and hence $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha) = [E : F]$ (Proposition 5.8).

Since r is divisible by ℓ, $L_\pi \simeq F_\pi(\sqrt[\ell]{v})$ and hence $L_\pi \subset E_\pi$. Thus $\mathrm{ind}(\alpha \otimes L_\pi) < \mathrm{ind}(\alpha)$. Since $r > 0$, $L_\delta \simeq F_\delta(\sqrt[\ell]{u\pi})$. Since $\alpha = (E/F, \sigma, u\pi\delta^{\ell m})$, $\mathrm{ind}(\alpha \otimes L_\delta) < [E \otimes L_\delta : L_\delta] \leqslant [E : F]$. In particular, $\mathrm{per}(\alpha \otimes L_\pi) < \mathrm{ind}(\alpha)$ and $\mathrm{per}(\alpha \otimes L_\delta) < \mathrm{ind}(\alpha)$. Since $\alpha \otimes L$ is unramified on B, except possibly at π and δ, and $H^2(B, \mu_\ell) = 0$, $\mathrm{per}(\alpha \otimes L) < \mathrm{ind}(\alpha)$. If $d = 1$, then $\mathrm{per}(\alpha \otimes L) < \mathrm{ind}(\alpha) = \ell$ and hence $\mathrm{per}(\alpha \otimes L) = \mathrm{ind}(\alpha \otimes L) = 1 < \mathrm{ind}(\alpha)$. Suppose that $d \geqslant 2$.

Let $\phi : \mathscr{X} \to \mathrm{Spec}(B)$ be a sequence of blow-ups such that the ramification locus of $\alpha \otimes L$ is a union of regular curves with normal crossings. Let $V = \phi^{-1}(P)$. To show that $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$, by Proposition 5.10, it is enough to show that for every point x of V, $\mathrm{ind}(\alpha \otimes L_x) < \mathrm{ind}(\alpha)$.

Let $x \in V$ be a closed point. Then, by Corollary 5.9, $\mathrm{ind}(\alpha \otimes L_x) = \mathrm{per}(\alpha \otimes L_x)$. Since $\mathrm{per}(\alpha \otimes L_x) < \mathrm{ind}(\alpha)$, $\mathrm{ind}(\alpha \otimes L_x) < \mathrm{ind}(\alpha)$.

Let $x \in V$ be a codimension zero point. Then $\phi(x)$ is the closed point of $\mathrm{Spec}(B)$. Let $\tilde{\nu}$ be the discrete valuation of L given by x. Then $\kappa(\tilde{\nu}) \simeq \kappa'(t)$ for some finite extension κ' over κ and a variable t over κ. Let ν be the restriction of $\tilde{\nu}$ to F.

Suppose that $\nu(\delta^r) < \nu(\pi)$. Then $L \otimes F_\nu = F_\nu(\sqrt[\ell]{v\delta^r})$. Since ℓ divides r, $L \otimes F_\nu = F_\nu(\sqrt[\ell]{v})$. Since $F(\sqrt[\ell]{v}) \subset E$, $\mathrm{ind}(\alpha \otimes L \otimes F_\nu) < \mathrm{ind}(\alpha)$. Suppose that $\nu(\delta^r) > \nu(\pi)$. Then $L \otimes F_\nu = F_\nu(\sqrt[\ell]{u\pi})$ and, as above, $\mathrm{ind}(\alpha \otimes L \otimes F_\nu) < \mathrm{ind}(\alpha)$. Suppose that $\nu(\delta^r) = \nu(\pi)$. Let $g = \pi/\delta^r$. Then g is a unit at ν and $L_{\tilde{\nu}} = F_\nu(\sqrt[\ell]{v + ug})$. We have $u\pi\delta^{\ell m} = ug\delta^{r+\ell m} = ug\delta^{\ell^f + \ell^{di}}$ and

$$\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, u\pi\delta^{\ell m}) = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, ug\delta^{\ell^f + \ell^{di}}).$$

Since $[E : F] = \ell^d$, $\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, ug\delta^{\ell^f})$. Suppose that $f = d$. Then E/F is unramified and hence every element of A^* is a norm from E. Thus $\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, w_0 ug)$ for any $w_0 \in A^*$. Suppose that $f < d$. Then $e = d - f > 0$ and hence, by Lemma 5.11, we have $E = E_{nr}(\sqrt[\ell^e]{w\delta})$, for some unit w in the integral closure of A in E_{nr}, with $N_{E/F}(\sqrt[\ell^e]{w\delta}) = w_1\delta^{\ell^f}$ with $w_1 \in A^* \backslash A^{*\ell}$. Thus

$$\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, ug\delta^{\ell^f}) = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, w_0 ug),$$

with $w_0 = w_1^{-1}$. Hence, in either case, we have $\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, w_0 ug)$ with $w_0 \notin A^{*\ell}$.

If $E \otimes F_\nu$ is not a field, then $\mathrm{ind}(\alpha \otimes F_\nu) < [E : F]$. Suppose $E \otimes F_\nu$ is a field. Let $\theta = w_0 ug$. Since $\alpha \otimes F_\nu = (E \otimes F_\nu/F_\nu, \sigma \otimes 1, \theta)$, $\mathrm{ind}(\alpha \otimes L \otimes F_\nu) \leqslant \mathrm{ind}(\alpha \otimes L \otimes F_\nu(\sqrt[\ell^{d-1}]{\theta})) \cdot [L \otimes F_\nu(\sqrt[\ell^{d-1}]{\theta}) : L \otimes F_\nu]$. Since $[L \otimes F_\nu(\sqrt[\ell^{d-1}]{\theta}) : L \otimes F_\nu] \leqslant \ell^{d-1} < [E : F]$, it is enough to show that $\alpha \otimes L \otimes F_\nu(\sqrt[\ell^{d-1}]{\theta})$ is trivial.

Since $F(\sqrt[\ell]{v})/F$ is the unique subextension of E/F of degree ℓ and $[E : F] = \ell^d$, we have $\alpha \otimes F_\nu(\sqrt[\ell^{d-1}]{\theta}) = (F_\nu(\sqrt[\ell^{d-1}]{\theta}, \sqrt[\ell]{v})/F_\nu(\sqrt[\ell^{d-1}]{\theta}), \sigma, \sqrt[\ell^{d-1}]{\theta})$ (cf. Lemma 2.1). Let $M = F_\nu(\sqrt[\ell^{d-1}]{\theta})$. Since κ contains a primitive ℓth root of unity, we have $\alpha \otimes M = (v, \sqrt[\ell^{d-1}]{\theta})_\ell$. Then M is a complete discretely valued field. Since g is a unit at ν, θ is a unit at ν. Hence the residue field of M

is $\kappa(\nu)(\sqrt[\ell^{d-1}]{\overline{\theta}})$. Since θ and v are units at ν, $\alpha \otimes M = (v, \sqrt[\ell^{d-1}]{\theta})$ is unramified at the discrete valuation of M. Hence it is enough to show that the specialization β of $\alpha \otimes M$ is trivial over $\kappa(\nu)(\sqrt[\ell^{d-1}]{\overline{\theta}}) \otimes L_0$, where L_0 is the residue field of $L \otimes F_\nu$ at ν.

Suppose that $L_{\tilde{\nu}}/F_\nu$ is ramified. Since $L_{\tilde{\nu}} = F_\nu(\sqrt[\ell]{v + ug})$, $v + ug$ is not a unit at ν. Thus $v = -ug$ modulo $F_\nu^{*\ell^d}$ and $\theta = w_0 ug = -w_0 v$ modulo $F_\nu^{*\ell^d}$. In particular, $\sqrt[\ell^{d-1}]{\theta} = \sqrt[\ell^{d-1}]{-w_0 v}$ modulo $M^{*\ell}$. Since $\overline{v}, \overline{w_0} \in \kappa$ and κ a finite field, $\beta = (\overline{v}, \sqrt[\ell^{d-1}]{\overline{\theta}}) = (\overline{v}, \sqrt[\ell^{d-1}]{-\overline{w_0}\overline{v}})$ is trivial.

Suppose that $L_{\tilde{\nu}}/F_\nu$ is unramified. Then $L_0 = \kappa(\nu)(\sqrt[\ell]{\overline{v} + \overline{ug}})$. Since $\kappa(\nu)$ is a global field of positive characteristic and $d - 1 \geqslant 1$, by Lemma 4.13, $\beta \otimes L_0(\sqrt[\ell^{d-1}]{\overline{\theta}}) = 0$. □

Lemma 6.4. *Suppose L_π/F_π and L_δ/F_δ are unramified cyclic field extensions of degree ℓ and $\mu_\pi \in L_\pi$, $\mu_\delta \in L_\delta$ such that:*
- $-\lambda = N_{L_\pi/F_\pi}(\mu_\pi)$ *and* $-\lambda = N_{L_\delta/F_\delta}(\mu_\delta)$;
- $\alpha \cdot (\mu_\pi) = 0 \in H^3(L_\pi, \mu_n^{\otimes 2})$, $\alpha \cdot (\mu_\delta) = 0 \in H^3(L_\delta, \mu_n^{\otimes 2})$;
- $\alpha = 0$ *or* $\alpha \neq 0$, $\mathrm{ind}(\alpha \otimes L_\pi) < \mathrm{ind}(\alpha)$ *and* $\mathrm{ind}(\alpha \otimes L_\delta) < \mathrm{ind}(\alpha)$.

Then there exist a cyclic extension L/F of degree ℓ and $\mu \in L$ such that:
- $-\lambda = N_{L/F}(\mu)$;
- $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$;
- $L \otimes F_\pi \simeq L_\pi$ *and* $L \otimes F_\delta \simeq L_\delta$;
- *if* $\alpha \neq 0$, *then* $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$.

Proof. Since $\alpha \cdot (\mu_\pi) = 0 \in H^3(L_\pi, \mu_n^{\otimes 2})$ and $-\lambda = N_{L_\pi/F_\pi}(\mu_\pi)$, by taking the corestriction, we see that $\alpha \cdot (-\lambda) = 0 \in H^3(F_\pi, \mu_n^{\otimes 2})$. Since $\alpha \cdot (-\lambda)$ is unramified on A, except possibly at π and δ, by Corollary 5.5, $\alpha \cdot (-\lambda) = 0$.

Suppose that $\lambda \notin \pm F^{*\ell}$. Then, by Lemmas 2.6 and 6.2, $L = F(\sqrt[\ell]{\lambda})$ and $\mu = -\sqrt[\ell]{\lambda}$ have the required properties.

Suppose that $\lambda \in F^{*\ell}$ or $-\lambda \in F^{*\ell}$. Let $L(\pi)$ and $L(\delta)$ be the residue fields of L_π and L_δ, respectively. Since L_π/F_π and L_δ/F_δ are unramified cyclic extensions of degree ℓ, $L(\pi)/\kappa(\pi)$ and $L(\delta)/\kappa(\delta)$ are cyclic extensions of degree ℓ. Since F contains a primitive ℓth root of unity, we have $L(\pi) = \kappa(\pi)[X]/(X^\ell - a)$ and $L(\delta) = \kappa(\delta)[X]/(X^\ell - b)$ for some $a \in \kappa(\pi)$ and $b \in \kappa(\delta)$. Since $\kappa(\pi)$ is a complete discretely valued field with $\overline{\delta}$ a parameter, without loss of generality we assume that $a = \overline{u_1}\overline{\delta}^\epsilon$ for some unit $u_1 \in A$ and $\epsilon = 0$ or 1. Similarly, we have $b = \overline{u_2 \pi}^{\epsilon'}$ for some unit $u_2 \in A$ and $\epsilon' = 0$ or 1.

Suppose $\alpha = 0$. If $-\lambda \in F^{*\ell}$, then $L = F(\sqrt[\ell]{u_1\delta^{\epsilon+\ell} + u_2\pi^{\epsilon'+\ell}})$ and $\mu = \sqrt[\ell]{-\lambda} \in F \subset L$ have the required properties. Suppose $-\lambda \notin F^{*\ell}$. Then $\lambda \in F^{*\ell}$ and hence $\ell = 2$ and $-1 \notin F^{*2}$. In particular, $-1 \notin \kappa(\pi)^{*2}$ and $-1 \notin \kappa(\delta)^{*2}$. Since $-\lambda$ is a norm from L_π and L_δ, -1 is a norm from L_π and L_δ. Thus -1 is a norm from the extensions $L(\pi)/\kappa(\pi)$ and $L(\delta)/\kappa(\delta)$. Hence $L(\pi)/\kappa(\pi)$ and $L(\delta)/\kappa(\delta)$ are unramified and hence $\epsilon = \epsilon' = 0$. Let L be the degree two extension of F which is unramified on A. Then -1 is a norm from L. Hence there exists $\mu \in L$ such that $N_{L/F}(\mu) = -\lambda$ and L, μ have the required properties.

Suppose that $\alpha \neq 0$. Then $\mathrm{ind}(\alpha \otimes L_\pi) < \mathrm{ind}(\alpha)$ and $\mathrm{ind}(\alpha \otimes L_\delta) < \mathrm{ind}(\alpha)$.

By Corollary 5.7, we assume that $\alpha = (E/F, \sigma, u\pi\delta^j)$ for some cyclic extension E/F which is unramified on A, except possibly at δ, u a unit in A and $j \geqslant 0$. Then $\mathrm{ind}(\alpha) = [E : F]$. Let E_0 be the residue field of E at π. Then $[E : F] = [E_0 : \kappa(\pi)]$. Since $\partial_\pi(\alpha) = (E_0/\kappa(\pi), \overline{\sigma})$, $\mathrm{per}(\partial_\pi(\alpha)) = [E : F] = \mathrm{ind}(\alpha)$. Since L_π/F_π is an unramified extension of degree ℓ, π is a parameter in L_π and hence $\mathrm{ind}(\alpha \otimes L_\pi) = [EL_\pi : L_\pi]$. Since $\mathrm{ind}(\alpha \otimes L_\pi) < \mathrm{ind}(\alpha) = [E_\pi : F_\pi]$, $[EL_\pi : L_\pi] < [E_\pi : F_\pi]$ and hence $L_\pi \subseteq E_\pi$. Thus the residue field $L(\pi)$ of L_π is the unique subextension of $E_0/\kappa(\pi)$ of degree ℓ.

Suppose that $\epsilon = \epsilon' = 0$. Since L_π and L_δ are fields, u_1 and u_2 are not ℓth powers. Let L/F be the unique cyclic field extension of degree ℓ which is unramified on A. Then $L \otimes F_\pi \simeq L_\pi$ and $L \otimes F_\delta \simeq L_\delta$. Let B be the integral closure of A in L. Then B is a regular local ring with maximal ideal (π, δ) and hence, by Proposition 5.8, $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$.

Suppose $\epsilon = 1$. Then $L_\pi = F_\pi(\sqrt[\ell]{u_1\delta})$ and $L(\pi) = \kappa(\pi)(\sqrt[\ell]{\overline{u_1}\overline{\delta}})$. Since $E_0/\kappa(\pi)$ is a cyclic extension containing a totally ramified extension, $E_0/\kappa(\pi)$ is a totally ramified cyclic extension. Thus $\kappa(\pi)$ contains a primitive ℓ^dth root of unity and $E_0 = \kappa(\pi)(\sqrt[\ell^d]{\overline{u_1}\overline{\delta}})$ (cf. Lemmas 2.3 and 2.4). In particular, F contains a primitive ℓ^dth root of unity and $\alpha = (u_1\delta, u\pi\delta^j) = (u_1\delta, u'\pi)$. Then $\partial_\delta(\alpha) = \kappa(\delta)(\sqrt[\ell^d]{(\overline{u'\pi})})$. Since L_δ/F_δ is an unramified extension of degree ℓ with $\mathrm{ind}(\alpha \otimes L_\delta) < \mathrm{ind}(\alpha)$, the residue field $L(\delta)$ of L_δ is the unique subfield of $\kappa(\delta)(\sqrt[\ell^d]{\overline{u'\pi}})$ of degree ℓ over $\kappa(\delta)$. Hence $L(\delta) = \kappa(\delta)(\sqrt[\ell]{\overline{u'\pi}})$. Since $L(\delta) = \kappa(\delta)(\sqrt[\ell]{\overline{u_2\pi^{\epsilon'}}})$, we have $\epsilon' = 1$ and $u' = u_2$ modulo $F^{*\ell}$. Hence $\alpha = (u_1\delta, u_2\pi)$. Let $L = F(\sqrt[\ell]{u_1\delta + u_2\pi})$. Then $L \otimes F_\pi \simeq L_\pi$ and $L \otimes F_\delta \simeq L_\delta$. Since for any $a, b \in F^*$, $(a, b) = (a+b, -a^{-1}b)$, we have $\alpha = (u_1\delta + u_2\pi, -u_1^{-1}\delta^{-1}u_2\pi)$. In particular, $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$.

Suppose that $\epsilon = 0$ and $\epsilon' = 1$. Suppose j is coprime to ℓ. Then, by Lemma 4.14, $\mathrm{ind}(\alpha) = \mathrm{per}(\partial_\delta(\alpha))$, and, as in the proof of Corollary 5.7, we have $\alpha = (E'/F, \sigma', v\delta\pi^{j'})$ for some cyclic extension E'/F which is unramified on A, except possibly at π. Thus, we have the required extension as in the case $\epsilon = 1$.

Suppose j is divisible by ℓ. Since $\epsilon = 0$, $L_\pi = F_\pi(\sqrt[\ell]{u_1})$. Since the residue field $L(\pi)$ of L_π is contained in the residue field E_0 of E at π, $F(\sqrt[\ell]{u_1}) \subset E$ and hence E/F is not totally ramified at δ. Since E/F is unramified on A, except possibly at δ, by Lemma 5.11, $E = E_{nr}(\sqrt[\ell^e]{w\delta})$ for some unit w in the integral closure of A in E_{nr}. Suppose $e = 0$. Then $E = E_{nr}/F$ is unramified on A. Since κ is a finite field and A is complete, every unit in A is a norm from E/F. Thus, multiplying $u\pi\delta^j$ by a norm from E/F, we assume that $\alpha = (E/F, \sigma, u_2\pi\delta^j)$. Suppose that $e > 0$. Then, by Lemma 5.11, $N_{E/F}(\sqrt[\ell^e]{w\delta}) = w_1\delta^{\ell f}$ with $w_1 \in A^* \backslash A^{*\ell}$. Since $A^*/A^{*\ell^d}$ is a cyclic group of order dividing ℓ^d, we have $u^{-1}u_2 = w_1^{j'}$ modulo $A^{*\ell^d}$. In particular, $N_{E/F}((\sqrt[\ell^e]{w\delta})^{j'}) = w_1^{j'}\delta^{\ell f j'} = u^{-1}u_2\delta^{\ell f j'}$ modulo $A^{*\ell^d}$. Hence, we have $\alpha = (E/F, \sigma, u_2\pi\delta^{j + j'\ell^f})$ for some j'. Since j is divisible by ℓ and $f \geqslant 1$, $j + j'\ell^f$ is divisible by ℓ. Hence, we assume that $\alpha = (E/F, \sigma, u_2\pi\delta^{\ell m})$ for some m. Thus, by Lemma 6.3, there exists $i \geqslant 0$ such that $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$ for $L = F(\sqrt[\ell]{u_1\delta^{\ell f + \ell^{d}i} + u_2\pi\delta^{\ell m}})$.

By choice, we have that L/F is the unique unramified extension or $L = F(\sqrt[\ell]{u_1\delta + u_2\pi})$ or $L = F(\sqrt[\ell]{u_1\delta^{\ell f + \ell^{d}i} + u_2\pi\delta^{\ell m}})$ with $\ell f + \ell^{d}i > \ell m$. Let B be the integral closure of A in L. Then B is a complete regular local ring with π and δ remain prime in B.

Suppose $-\lambda \in F^{*\ell}$. Since $-\lambda = -w\pi^s\delta^t$, we have $-\lambda = w_0^\ell\pi^{\ell s_1}\delta^{\ell t_1}$ for some unit $w_0 \in A$. Let $\mu = w_0\pi^{s_1}\delta^{t_1} \in F$. Then $N_{L/F}(\mu) = \mu^\ell = -\lambda$. Since $\alpha \cdot (-\lambda) = 0$, by Proposition 4.6, $\alpha \cdot (\mu) = 0$ in $H^3(L_\pi, \mu_n^{\otimes 2})$ and $H^3(L_\delta, \mu_n^{\otimes 2})$. Hence $\alpha \cdot (\mu)$ is unramified at all height one prime ideals of B. Since B is a complete regular local ring with residue field finite, $\alpha \cdot (\mu) = 0$ (Lemma 5.3).

Suppose that $-\lambda \notin F^{*\ell}$. Then $\lambda \in F^{*\ell}$, $\ell = 2$ and $-1 \notin F^{*\ell}$. Hence $-1 \notin F_\pi^{*2}$ and $-1 \notin F_\delta^{*2}$. In particular, $-1 \notin \kappa(\pi)^{*2}$, $-1 \notin \kappa(\delta)^{*2}$. Since $\lambda \in F^{*2}$ and $-\lambda$ is a norm from L_π and L_δ, -1 is a norm from L_π and L_δ. Hence -1 is a norm from $L(\pi)$ and $L(\delta)$. Since $\kappa(\pi)$ and $\kappa(\delta)$ are local fields with residue fields of characteristic not equal to 2, we have $L(\pi) \simeq \kappa(\pi)(\sqrt{-1})$ and $L(\delta) \simeq \kappa(\delta)(\sqrt{-1})$. Let $L = F(\sqrt{-1})$. Since κ is a finite field of characteristic not equal to 2, -1 is a norm from L. Since $\lambda \in F^{*2}$, there exists $\mu \in L$ such that $N_{L/F}(\mu) = -\lambda$. Further, L and μ have the required properties. $\square$

Lemma 6.5. *Suppose that $\nu_\pi(\lambda)$ is divisible by ℓ, α is unramified on A, except possibly at π and δ, and $\alpha \cdot (-\lambda) = 0$. Let L_π be a finite product of unramified finite field extensions of F_π with $\dim_{F_\pi}(L_\pi) = \ell$, $\mu_\pi \in L_\pi$ and $d_0 \geqslant 2$ such that:*

- $N_{L_\pi/F_\pi}(\mu_\pi) = -\lambda$;
- $\mathrm{ind}(\alpha \otimes L_\pi) < d_0$;
- $\alpha \cdot (\mu_\pi) = 0$ *in* $H^3(L_\pi, \mu_n^{\otimes 2})$.

Then there exist an étale algebra L over F of degree ℓ and $\mu \in L$ such that:

- $N_{L/F}(\mu) = -\lambda$;
- $\mathrm{ind}(\alpha \otimes L) < d_0$;
- $\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$; *and*
- *there is an isomorphism $\phi : L_\pi \to L \otimes F_\pi$ with*

$$\phi(\mu_\pi)(\mu \otimes 1)^{-1} \in (L \otimes F_\pi)^{\ell^m},$$

for all $m \geqslant 1$.

Further, if L_π/F_π is a field extension with the residue field of L_π unramified over $\kappa(\pi)$, then L can be chosen to be a field extension with L/F unramified on A.

Proof. Since $\nu_\pi(\lambda)$ is divisible by ℓ, $\lambda = w\pi^{s_1\ell}\delta^t$ for some $w \in A$ a unit. Write $L_\pi = \prod_1^q L_{\pi,i}$ with $L_{\pi,i}/F_\pi$ a finite unramified extension and $\mu_\pi = (\mu_1, \ldots, \mu_q)$ with $\mu_i \in L_{\pi,i}$. Since $L_{\pi,i}/F_\pi$ is unramified, π is a parameter in $L_{\pi,i}$ for all i. Write $\mu_i = \theta_i\pi^{r_i}$ for some $\theta_i \in L_{\pi,i}$ a unit at π. Let $\theta = (\theta_1, \ldots, \theta_q) \in L_\pi$. Since $N_{L_\pi/F_\pi}(\mu_\pi) = \lambda = w\pi^{s_1\ell}\delta^t$, we have $N_{L_\pi/F_\pi}(\theta) = w\delta^t$.

For each i, let L_i/F be a field extension with $L_i \otimes F_\pi \simeq L_{\pi,i}$ as in Lemma 5.12. Let B_i be the integral closure of A in L_i. Then each B_i is regular local ring with maximal ideal (π, δ_i) for some prime δ_i with $N_{L_i/F}(\delta_i) = v_i\delta^{f_i}$ for some unit $v_i \in A$ and $f_i \geqslant 1$ (Remark 5.13). Then the residue field $L_i(\pi)$ of L_i at the discrete valuation given by π is the field of fractions of $B_i/(\pi)$. In particular, $L_i(\pi)$ is a complete discrete valued field with $\overline{\delta}_i \in B_i/(\pi)$ as a parameter. We identify $L_{\pi,i}$ with $L_i \otimes F_\pi$ and assume that $\mu_i \in L_i \otimes F_\pi$.

For $1 \leqslant i \leqslant q$, let $\bar{\theta}_i$ be the image of θ_i in $L_i(\pi)$. Then $\bar{\theta}_i = \overline{w_i}\bar{\delta}_i^{t_i}$ for some unit $w_i \in B_i$ and $t_i \in \mathbb{Z}$. Since $N_{L_\pi/F_\pi}(\theta) = w\delta^t$ and $N_{L_{\pi,i}/F_\pi}(\delta_i) = v_i\delta^{f_i}$, we have $\prod_1^q N_{L_i(\pi)/\kappa(\pi)}(\overline{\theta}_i) = \prod_1^q N_{L_i(\pi)/\kappa(\pi)}(\overline{w}_i)\prod_1^q(\overline{v_i}^{t_i}\overline{\delta}^{f_it_i}) = \overline{w}\delta^t$. Hence

$$\sum f_it_i = t \quad \text{and} \quad N_{L_1(\pi)/\kappa(\pi)}(w_1) = \overline{w}\prod_2^q N_{L_i(\pi)/\kappa(\pi)}(\overline{w}_i)^{-1}\prod_1^q \overline{v_i}^{-t_i}.$$

Since A is complete, there exists $w_1' \in B_1$ such that $\overline{w'}_1 = \overline{w}_1 \in B_1/(\pi)$ and $N_{L_1/F}(w_1') = w\prod_2^q N_{L_i/F_\pi}(w_i)^{-1}\prod_1^q v_i^{-t_i}$. Let $L = \prod_1^q L_i$ and $\mu = (w_1'\delta_1^{t_1}\pi^{s_1}, w_2\delta_2^{t_2}\pi^{s_1}, \ldots, w_q\delta_q^{t_q}\pi^{s_1}) \in L$. Then we claim that L and μ have the required properties.

By the choice of w_1', we have $N_{L/F}(\mu) = \lambda$. Since $L_i \otimes F_\pi \simeq L_{\pi,i}$, we have $L \otimes F_\pi \simeq L_\pi$. Since $\overline{w'}_1 = \overline{w}_1 \in B_1/(\pi)$, we have $\overline{\mu^{-1}\mu_\pi} = 1 \in B/(\pi)$. Since B is complete, we have $\mu^{-1}\mu_\pi \in (L \otimes F_\pi)^{\ell^m}$ for all $m \geqslant 1$.

Since α is unramified on A, except possibly at π and δ, $\alpha \otimes L_i$ is unramified on B_i, except possibly at π and δ_i for each i. Since $\mathrm{ind}(\alpha \otimes L_{\pi,i}) < d_0$, by Proposition 5.8, $\mathrm{ind}(\alpha \otimes L_i) = \mathrm{ind}(\alpha \otimes L_{\pi,i}) < d_0$. Hence $\mathrm{ind}(\alpha \otimes L) < d_0$.

Since $\mu^{-1}\mu_\pi \in (L \otimes F_\pi)^{\ell^m}$ for all $m \geqslant 1$, $\alpha \cdot (\mu) = \alpha \cdot (\mu_\pi) = 0 \in H^3(L_\pi, \mu_n^{\otimes 2})$. Since α is unramified on A, except possibly at π and δ, and $\mu = (w_1'\delta_1^{t_1}\pi^{s_1}, w_2\delta_2^{t_2}\pi^{s_1}, \ldots, w_q\delta_q^{t_q}\pi^{s_1})$ with w_1' and w_i units in B, by Corollary 5.5, we have $\alpha \cdot (\mu) = 0$ in $H^3(L, \mu_n^{\otimes 2})$. Thus L and μ have the required properties.

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Further, if L_π/F_π is a field extension such that the residue field $L_\pi(\pi)$ of L_π is an unramified extension of $\kappa(\pi)$, then by the choice of L, L/F is a field extension with L/F unramified on A (see the proof of Lemma 5.12). □

LEMMA 6.6. *Suppose that $\alpha = (E/F, \sigma, u\pi\delta^m)$ for some cyclic extension E/F which is unramified on A, except possibly at δ. Let E_δ be the lift of the residue of α at δ. If $t_1\alpha \otimes E_\delta = 0$ for some t_1, then there exists an integer $r_1 \geqslant 0$ such that $w_1\delta^{mr_1 - t_1}$ is a norm from the extension E/F for some unit $w_1 \in A$.*

Proof. Write $\alpha \otimes F_\delta = \alpha' + (E_\delta/F_\delta, \sigma_\delta, \delta)$ as in Lemma 4.1. Suppose that $t_1\alpha \otimes E_\delta = 0$. Since $\alpha \otimes E_\delta = \alpha' \otimes E_\delta$, $t_1\alpha' \otimes E_\delta = 0$. Hence $t_1\alpha' = (E_\delta, \sigma_\delta, \theta)$ for some $\theta \in F_\delta$. Since α' and E_δ/F_δ are unramified at δ, we assume that $\theta \in F_\delta$ is a unit at δ. Since the residue field $\kappa(\delta)$ of F_δ is a complete discretely valued field with the image of π as a parameter, without loss of generality we assume that $\theta = w_0\pi^{r_1}$ for unit $w_0 \in A$ and $r_1 \geqslant 0$. Let $\lambda_1 = w_0\pi^{r_1}\delta^{t_1}$. Since $t_1\alpha' = (E_\delta, \sigma_\delta, \theta)$, by Lemma 4.7, $\partial_\delta(\alpha \cdot (\lambda_1)) = 0$. Since $\kappa(\delta)$ is a local field, $\alpha \cdot (\lambda_1) = 0 \in H^3(F_\delta, \mu_n^{\otimes 2})$ (cf. the proof of Proposition 4.6). Since α is unramified on A , except possibly at π, δ and $\lambda_1 = w_0\pi^{r_1}\delta^{t_1}$ with $w_0 \in A$ a unit, $\alpha \cdot (\lambda_1)$ is unramified in A, except possibly at π and δ. Hence, by Corollary 5.5, $\alpha \cdot (\lambda_1) = 0 \in H^3(F, \mu_n^{\otimes 2})$. We have

$$0 = \partial_\pi(\alpha \cdot (\lambda_1)) = \partial_\pi((E/F, \sigma, u\pi\delta^m) \cdot (w_0\pi^{r_1}\delta^{t_1})) = (E(\pi)/\kappa(\pi), \overline{\sigma}, (-1)^{r_1}\overline{u}^{r_1}\overline{w}_0^{-1}\overline{\delta}^{mr_1 - t_1}).$$

Since $(E/F, \sigma, (-1)^{r_1}u^{r_1}w_0^{-1}\delta^{mr_1 - t_1})$ is unramified on A, except possibly at π and δ, by Corollary 5.5, $(E/F, \sigma, (-1)^{r_1}u^{r_1}w_0^{-1}\delta^{mr_1 - t_1}) = 0$. In particular, $(-1)^{r_1}u^{r_1}w_0^{-1}\delta^{mr_1 - t_1}$ is a norm from the extension E/F. □

LEMMA 6.7. *Suppose that $\alpha \cdot (-\lambda) = 0$ and $\lambda = w\pi^s\delta^{t_1\ell}$ for some unit $w \in A$ and s coprime to ℓ. Let E_δ be the lift of the residue of α at δ. If $t_1\alpha \otimes E_\delta = 0$, then there exists $\theta \in A$ such that:*
- *$\alpha \cdot (\theta) = 0$;*
- *$\nu_\pi(\theta) = 0$;*
- *$\nu_\delta(\theta) = t_1$.*

Proof. Since s is coprime to ℓ, by Lemma 6.1, $\alpha = (E/F, \sigma, (-1)^{s+1}\lambda)$ for some cyclic extension E/F which is unramified on A, except possibly at δ. Let $r = [E : F]$. Since r is a power of ℓ and s is coprime to ℓ, there exists an integer $s' \geqslant 1$ such that $ss' \equiv 1$ modulo r. We have

$$\begin{aligned}\alpha = \alpha^{ss'} &= (E/F, \sigma, (-1)^{s+1}w\pi^s\delta^{t_1\ell})^{ss'} \\ &= (E/F, \sigma)^s \cdot ((-1)^{s+1}w\pi^s\delta^{t_1\ell})^{s'} \\ &= (E/F, \sigma)^s \cdot ((-1)^{s'}w^{s'}\pi\delta^{s't_1\ell}).\end{aligned}$$

Since s is coprime to ℓ, we also have $(E/F, \sigma)^s = (E/F, \sigma^{s'})$ (cf. § 2) and hence $\alpha = (E/F, \sigma^{s'}, ((-1)^{s'}w^{s'}\pi\delta^{s't_1\ell}))$. Thus, by Lemma 6.6, there exist a unit $w_1 \in A$ and $r_1 \geqslant 0$ such that $w_1\delta^{s't_1\ell r_1 - t_1}$ is a norm from E/F. Since $s'\ell r_1 - 1$ is coprime to ℓ, $s'\ell r_1 - 1$ is coprime to r and hence there exists an integer $r_2 \geqslant 0$ such that $(s'\ell r_1 - 1)r_2 \equiv 1$ modulo r. In particular, $w_1^{r_2}\delta^{t_1} \equiv (w_1\delta^{s't_1\ell r_1 - t_1})^{r_2}$ modulo F^{*r} and hence $w_1^{r_2}\delta^{t_1}$ is a norm from E/F. Thus $\theta = w_1^{r_2}\delta^{t_1}$ has the required properties. □

LEMMA 6.8. *Let E_π and E_δ be the lift of the residues of α at π and δ, respectively. Suppose that $\lambda = w\pi^{s_1\ell}\delta^{t_1\ell}$ for some unit $w \in A$. If $\alpha \cdot (-\lambda) = 0$, $s_1\alpha \otimes E_\pi = 0$ and $t_1\alpha \otimes E_\delta = 0$, then there exists $\theta \in A$ such that:*

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- $\alpha \cdot (\theta) = 0$;
- $\nu_\pi(\theta) = s_1$;
- $\nu_\delta(\theta) = t_1$.

Proof. By Corollary 5.7, we assume that $\alpha = (E/F, \sigma, u\pi\delta^m)$ for some extension E/F which is unramified on A, except possibly at δ and $m \geqslant 0$. Without loss of generality, we assume that $0 \leqslant m < [E : F]$. By Lemma 6.6, there exist an integer $r_1 \geqslant 0$ and a unit $w_1 \in A$ such that $w_1\delta^{mr_1 - t_1}$ is a norm from E/F. Let $r = [E : F]$ and $\theta = (-u\pi + \delta^{r-m})^{r_1 - s_1} w_1^{-1}(-u)^{s_1}\pi^{s_1}\delta^{t_1}$. Since $r - m > 0$, we have $\nu_\pi(\theta) = s_1$ and $\nu_\delta(\theta) = t_1$.

Now we show that $\alpha \cdot (\theta) = 0$. Let γ be a prime in A with $(\gamma) \neq (\pi)$ and $(\gamma) \neq (\delta)$. Since α is unramified on A, except possibly at π and δ, if γ does not divide θ, then $\alpha \cdot (\theta)$ is unramified at γ. Suppose γ divides θ. Then $\gamma = -u\pi + \delta^{r-m}$. Thus $u\pi\delta^m \equiv \delta^r$ modulo γ. Since $\partial_\gamma(\alpha \cdot (\theta)) = (E(\gamma), \overline{\sigma}, \overline{u\pi}\overline{\delta}^m)^{r_1 - s_1}$, where $E(\gamma)$ is the residue field of E at γ and bar denotes the image modulo γ, we have $\partial_\gamma(\alpha \cdot (\theta)) = (E(\gamma), \overline{\sigma}, \overline{u\pi}\overline{\delta}^m)^{r_1 - s_1} = (E(\gamma), \overline{\sigma}, \overline{\delta}^r)^{r_1 - s_1} = 0$. Hence $\alpha \cdot (\theta)$ is unramified on A, except possibly at π and δ.

We have $(-u\pi + \delta^{r-m})^{r_1 - s_1} \equiv \delta^{r(r_1 - s_1) + m(s_1 - r_1)}$ modulo π and hence

$$\theta \equiv \delta^{r(r_1 - s_1) + m(s_1 - r_1)} w_1^{-1}(-u)^{s_1}\pi^{s_1}\delta^{t_1} \equiv (-u\pi\delta^m)^{s_1}(w_1\delta^{mr_1 - t_1})^{-1} \quad \text{modulo } F_\pi^{*r}.$$

Since $w_1\delta^{mr_1 - t_1}$ is a norm from E/F and $r = [E : F]$, we have

$$\begin{aligned}(\alpha \cdot (\theta)) \otimes F_\pi &= (E/F, \sigma, u\pi\delta^m) \cdot ((-u\pi\delta^m)^{s_1}(w_1\delta^{mr_1 - t_1})^{-1}) \otimes F_\pi \\ &= (E/F, \sigma, u\pi\delta^m) \cdot ((-u\pi\delta^m)^{s_1}) \otimes F_\pi = 0.\end{aligned}$$

Thus, by Corollary 5.5, we have $\alpha \cdot (\theta) = 0$. □

7. Patching

We fix the following data:

- R a complete discrete valuation ring;
- K the field of fractions of R;
- κ the residue field of R;
- ℓ a prime not equal to char(κ) and $n = \ell^d$ for some $d \geqslant 1$;
- X a smooth projective geometrically integral curve over K;
- F the function field of X;
- $\alpha \in H^2(F, \mu_n)$, $\alpha \neq 0$;
- $\lambda \in F^*$ with $\alpha \cdot (-\lambda) = 0$;
- $\mathscr{X}$ a normal proper model of X over R and X_0 the reduced special fiber of $\mathscr{X}$;
- $\mathscr{P}_0$ the finite set of closed points of X_0 consisting of all the points of intersection of irreducible components of X_0.

We recall the following notation from [HH10, § 6] and [HHK09, § 3.3]. For $x \in \mathscr{X}$, let $\hat{A}_x$ be the completion of the local ring A_x at x on $\mathscr{X}$, F_x the field of fractions of $\hat{A}_x$ and $\kappa(x)$ the residue field at x. Let η be a codimension zero point of X_0 and $U \subset \eta$ be a nonempty open subset. Let A_U be the ring of all those functions in F which are regular at every closed point of U. Let t be a parameter in R. Then $t \in A_U$. Let $\hat{A}_U$ be the (t)-adic completion of A_U and F_U be the field of fractions of $\hat{A}_U$. Then $F \subseteq F_U \subseteq F_\eta$.

Let $\eta \in X_0$ be a codimension zero point and $P \in X_0$ be a closed point such that P is in the closure of η. By an abuse of notation, we denote the closure of η by η and say that P is a point of η. A *branch* is a height one prime ideal $\wp$ of $\hat{A}_P$ containing t. Let $\wp$ be a branch. Let $\hat{A}_\wp$ be

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the completion of the localization of $\hat{A}_P$ at $\wp$ and $F_\wp$ the field of fractions of $\hat{A}_\wp$. The contraction $\wp \cap A_P$ of $\wp$ to A_P is a height one prime ideal and hence a branch $\wp$ uniquely determines an irreducible component η of X_0 containing P.

Suppose further that $\mathscr{X}$ is a regular proper model of X over R and X_0 is a union of regular curves with normal crossings. Then A_x, $\hat{A}_x$ are regular local rings. Every branch $\wp$ is uniquely determined by a pair (P, η) where η is a codimension zero point of X_0 and $P \in \eta$ is a closed point. In this case, $F_\wp$ is the completion of F_P at the discrete valuation of F_P given by η. We also denote $F_\wp$ by $F_{P,\eta}$. Note that the residue field $\kappa(\eta)_P$ of $\hat{A}_\wp$ is the completion of the residue field $\kappa(\eta)$ at the discrete valuation given by P.

We begin with the following result, which follows from [HHK15a, Theorem 9.11] (cf. the proof of [PS15, Theorem 2.4]).

PROPOSITION 7.1. *For each irreducible component X_η of X_0, let U_η be a nonempty proper open subset of X_η and $\mathscr{P} = X_0 \backslash \cup_\eta U_\eta$, where η runs over the codimension zero points of X_0. Suppose that $\mathscr{P}_0 \subseteq \mathscr{P}$. Let L be a finite extension of F. Suppose that there exists $N \geqslant 1$ such that for each codimension zero point η of X_0, $\operatorname{ind}(\alpha \otimes L \otimes F_{U_\eta}) \leqslant N$, and for every closed point $P \in \mathscr{P}$, $\operatorname{ind}(\alpha \otimes L \otimes F_P) \leqslant N$. Then $\operatorname{ind}(\alpha \otimes L) \leqslant N$.*

Proof. Let $\mathscr{Y}$ be the integral closure of $\mathscr{X}$ in L and $\phi : \mathscr{Y} \to \mathscr{X}$ be the induced map. Let $\mathscr{P}'$ be a finite set of closed points of $\mathscr{Y}$ containing the inverse image of $\mathscr{P}$ under ϕ. Let U be an irreducible component of $Y_0 \backslash \mathscr{P}'$. Then $\phi(U) \subset U_\eta$ for some U_η and there is a homomorphism of algebras from $L \otimes F_{U_\eta}$ to L_U. (Note that $L \otimes F_{U_\eta}$ may be a product of fields.) Since $\operatorname{ind}(\alpha \otimes L \otimes F_{U_\eta}) \leqslant N$, we have $\operatorname{ind}(\alpha \otimes L_U) \leqslant N$. Let $Q \in \mathscr{P}'$. Suppose $\phi(Q) = P \in \mathscr{P}$. Then there is a homomorphism of algebras from $L \otimes F_P$ to L_Q. (Once again note that $L \otimes F_P$ may be a product of fields.) Since $\operatorname{ind}(\alpha \otimes L \otimes F_P) \leqslant N$, $\operatorname{ind}(\alpha \otimes L_Q) \leqslant N$. Suppose that $\phi(Q) \in U_\eta$ for some U_η. Then there is a homomorphism of algebras from $L \otimes F_{U_\eta}$ to L_Q. Thus $\operatorname{ind}(\alpha \otimes L_Q) \leqslant N$. Therefore, by [HHK15a, Theorem 9.11], $\operatorname{ind}(\alpha \otimes L) \leqslant N$. □

LEMMA 7.2. *Let η be a codimension zero point of X_0. Suppose there exist a field extension or split extension L_η / F_η of degree ℓ and $\mu_\eta \in L_\eta$ such that:*

(1) $N_{L_\eta / F_\eta}(\mu_\eta) = -\lambda$;
(2) $\operatorname{ind}(\alpha \otimes L_\eta) < \operatorname{ind}(\alpha)$;
(3) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$.

Then there exist a nonempty open subset U_η of η, a split or field extension L_{U_η} / F_{U_η} of degree ℓ and $\mu_{U_\eta} \in L_{U_\eta}$ such that:

(1) $N_{L_{U_\eta} / F_{U_\eta}}(\mu_{U_\eta}) = -\lambda$;
(2) $\operatorname{ind}(\alpha \otimes L_{U_\eta}) < \operatorname{ind}(\alpha)$;
(3) $\alpha \cdot (\mu_{U_\eta}) = 0 \in H^3(L_{U_\eta}, \mu_n^{\otimes 2})$;
(4) *there is an isomorphism $\phi_{U_\eta} : L_{U_\eta} \otimes F_\eta \to L_\eta$ with $\phi_{U_\eta}(\mu_{U_\eta} \otimes 1)\mu_\eta^{-1} \in L_\eta^{*\ell^m}$ for all $m \geqslant 1$.*

Further, if L_η / F_η is cyclic, then L_{U_η} / F_{U_η} is cyclic.

Proof. Suppose $L_\eta = \prod F_\eta$ is the split extension of degree ℓ. Write $\mu_\eta = (\mu_1, \ldots, \mu_\ell)$ with $\mu_i \in F_\eta$. Then $-\lambda = N_{L_\eta / F_\eta}(\mu_\eta) = \mu_1 \cdots \mu_\ell$. Since $\operatorname{ind}(\alpha \otimes L_\eta) = \operatorname{ind}(\alpha \otimes F_\eta) < \operatorname{ind}(\alpha)$, by [HHK15a, Proposition 5.8], [KMRT98, Proposition 1.17], there exists a nonempty open subset U_η of η such that $\operatorname{ind}(\alpha \otimes F_{U_\eta}) < \operatorname{ind}(\alpha)$. Since F_η is the completion of F at the discrete valuation given by η, there exist $\theta_i \in F^*$, $1 \leqslant i \leqslant \ell$, such that $\theta_i \mu_i^{-1} \equiv 1$ modulo the maximal ideal of $\hat{R}_\eta$. Let $L_{U_\eta} = \prod F_{U_\eta}$ and $\mu_{U_\eta} = (-\lambda(\theta_2 \cdots \theta_\ell)^{-1}, \theta_2, \ldots, \theta_\ell) \in L_{U_\eta}$. Then $N_{L_{U_\eta/F_{U_\eta}}}(\mu_{U_\eta}) = -\lambda$. Since $\alpha \cdot (\theta_i) \in H^3(F_{U_\eta}, \mu_n^{\otimes 2})$ and $\alpha \cdot (\theta_i) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$, by [HHK14, Proposition 3.2.2], there exists a nonempty open subset $V_\eta \subseteq U_\eta$ such that $\alpha \cdot (\theta_i) = 0 \in H^3(F_{V_\eta}, \mu_n^{\otimes 2})$. By replacing U_η by V_η, we have the required L_{U_η} and $\mu_{U_\eta} \in L_{U_\eta}$.

Suppose that L_η/F_η is a field extension of degree ℓ. Let F_η^h be the henselization of F at the discrete valuation η. Then there exists a field extension L_η^h/F_η^h of degree ℓ with an isomorphism $\phi_\eta^h : L_\eta^h \otimes_{F_\eta^h} F_\eta \to L_\eta$. We identify L_η^h with a subfield of L_η through ϕ_η^h. Further, if L_η/F_η is a cyclic extension, then L_η^h/F_η^h is also a cyclic extension. Let $\tilde{\pi}_\eta \in L_\eta^h$ be a parameter. Then $\tilde{\pi}_\eta$ is also a parameter in L_η. Write $\mu_\eta = u_\eta \tilde{\pi}_\eta^r$ for some $u_\eta \in L_\eta$ a unit at η. Since $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$, we have $-\lambda = N_{L_\eta/F_\eta}(u_\eta) N_{L_\eta/F_\eta}(\tilde{\pi}_\eta)$. Since $u_\eta \in L_\eta$ is a unit at η, $N_{L_\eta/F_\eta}(u_\eta) \in F_\eta$ is a unit at η. By [Art69, Theorem 1.10], there exists $u_\eta^h \in L_\eta^h$ such that $N_{L_\eta^h/F_\eta^h}(u_\eta^h) = N_{L_\eta/F_\eta}(u_\eta)$ and $u_\eta^h \equiv u_\eta$ modulo the maximal ideal of the valuation ring of L_η^h. Let $\mu_\eta^h = u_\eta^h \tilde{\pi}_\eta^r \in L_\eta^h$. Then $\alpha \cdot (\mu_\eta^h) = \alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$ and hence $\alpha \cdot (\mu_\eta^h) = 0 \in H^3(L_\eta^h, \mu_n^{\otimes 2})$ (cf. the proof of [HHK14, Proposition 3.2.2]). Since F_η^h is the filtered direct limit of the fields F_V, where V ranges over the nonempty open subset of η [HHK14, Lemma 3.2.1], there exist a nonempty open subset U_η of η, a field extension L_{U_η}/F_{U_η} of degree ℓ and $\mu_{U_\eta} \in L_{U_\eta}$ such that $N_{L_{U_\eta}/F_{U_\eta}}(\mu_{U_\eta}) = -\lambda$ and there is an isomorphism $\phi_{U_\eta}^h : L_{U_\eta} \otimes F_\eta^h \simeq L_\eta^h$ with $\phi_{U_\eta}^h(\mu_{U_\eta}) = \mu_\eta^h$. Since $u_\eta^h \equiv u_\eta$ modulo the maximal ideal of the valuation ring of L_η, $\mu_\eta = u_\eta \tilde{\pi}_\eta^r$ and $\mu_\eta^h = u_\eta^h \tilde{\pi}_\eta^t$, it follows that $\phi_{U_\eta}(\mu_{U_\eta} \otimes 1)\mu_\eta^{-1} \in L_\eta^{*\ell^m}$ for all $m \geqslant 1$. By shrinking U_η, we assume that $\alpha \cdot (\mu_{U_\eta}) = 0 \in H^3(L_{U_\eta}, \mu_n^{\otimes 2})$ [HHK14, Proposition 3.2.2]. Further, if L_η/F_η is cyclic, by shrinking U_η, we can assume that L_{U_η}/F_{U_η} is cyclic. □

For the rest of this section we assume that for each point x of X_0, there exist an étale algebra L_x/F_x of degree ℓ and $\mu_x \in L_x$ such that:

(1) $N_{L_x/F_x}(\mu_x) = -\lambda$;
(2) $\alpha \cdot (\mu_x) = 0 \in H^3(L_x, \mu_n^{\otimes 2})$;
(3) $\mathrm{ind}(\alpha \otimes L_x) < \mathrm{ind}(\alpha)$;
(4) for any branch (P, η) there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ such that $\phi_{P,\eta}(\mu_\eta)\mu_P^{-1} \in (L_P \otimes F_{P,\eta})^{*\ell^m}$ for all $m \geqslant 1$;
(5) if $x = \eta$ is a codimension zero point of X_0, then L_η/F_η is either a field or the split extension.

Lemma 7.3. *There exist:*

- *a field extension L/F of degree ℓ;*
- *a nonempty open proper subset U_η of η for every codimension zero point η of X_0 and $\mu'_{U_\eta} \in L \otimes F_{U_\eta}$;*
- *for every $P \in \mathscr{P} = X_0 \setminus \cup U_\eta$, $\mu'_P \in L \otimes F_P$,*

such that:

(1) *$\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$;*
(2) *$N_{L \otimes F_{U_\eta}/F_{U_\eta}}(\mu'_{U_\eta}) = -\lambda$ and $\alpha \cdot (\mu'_{U_\eta}) = 0 \in H^3(L \otimes F_{U_\eta}, \mu_n^{\otimes 2})$ for all codimension zero points η of X_0;*
(3) *$N_{L \otimes F_P/F_P}(\mu'_P) = -\lambda$ and $\alpha \cdot (\mu'_P) = 0 \in H^3(L \otimes F_P, \mu_n^{\otimes 2})$ for all $P \in \mathscr{P}$;*
(4) *for any branch (P, η), $\mu'_{U_\eta} \mu_P'^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^m}$ for all $m \geqslant 1$.*

Further, if for each $x \in X_0$, L_x/F_x is cyclic or split, then L/F is cyclic.

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Proof. Let η be a codimension zero point of X_0. By assumption, there exist a field or split extension L_η/F_η and $\mu_\eta \in L_\eta$ such that $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$, $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$ and $\mathrm{ind}(\alpha \otimes L_\eta) < \mathrm{ind}(\alpha)$. By Lemma 7.2, there exist a nonempty open set U_η of η, a field or split extension L_{U_η}/F_{U_η} of degree ℓ and $\mu_{U_\eta} \in L_{U_\eta}$ such that $N_{L_{U_\eta}/F_{U_\eta}}(\mu_{U_\eta}) = -\lambda$, $\alpha \cdot (\mu_{U_\eta}) = 0 \in H^3(L_{U_\eta}, \mu_n^{\otimes 2})$, $\mathrm{ind}(\alpha \otimes L_{U_\eta}) < \mathrm{ind}(\alpha)$, $\phi_\eta : L_{U_\eta} \otimes F_\eta \to L_\eta$ an isomorphism $\phi_{U_\eta}(\mu_{U_\eta} \otimes 1)\mu_\eta^{-1} \in L_\eta^{\ell m}$ for all $m \geqslant 1$. By shrinking U_η, if necessary, we assume that $\mathscr{P}_0 \cap U_\eta = \emptyset$.

Let $\mathscr{P} = X_0 \backslash \cup_\eta U_\eta$ and $P \in \mathscr{P}$. Then, by assumption, we have an étale algebra L_P/F_P of degree ℓ and for every branch (P, η) there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$. Thus $\phi_{P,U_\eta} = \phi_{P,\eta}(\phi_\eta \otimes 1) : L_{U_\eta} \otimes F_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ is an isomorphism. Thus, by [HH10, Theorem 7.1], there exists an extension L/F of degree ℓ with isomorphisms $\phi_{U_\eta} : L \otimes F_{U_\eta} \to L_{U_\eta}$ for all codimension zero points η of X_0 and $\phi_P : L \otimes F_P \to L_P$ for all $P \in \mathscr{P}$ with the following commutative diagram:

$$\begin{array}{ccc} L \otimes F_{U_\eta} \otimes F_{P,\eta} & \xrightarrow{\phi_{U_\eta} \otimes 1} & L_{U_\eta} \otimes F_\eta \otimes F_{P,\eta} \\ \downarrow & & \downarrow \phi_{P,U_\eta} \\ L \otimes F_P \otimes F_{P,\eta} & \xrightarrow{\phi_P \otimes 1} & L_P \otimes F_{P,\eta} \end{array}$$

where the vertical arrow on the left is the natural map. Further, if each L_x/F_x is cyclic or split for all $x \in X_0$, then L/F is cyclic [HH10, Theorem 7.1].

Since $\mathrm{ind}(\alpha \otimes L \otimes F_{U_\eta}) < \mathrm{ind}(\alpha)$ for all codimension zero points of X_0 and $\mathrm{ind}(\alpha \otimes L \otimes F_P) < \mathrm{ind}(\alpha)$ for all $P \in \mathscr{P}$, by Proposition 7.1, $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$. In particular, L is a field.

For every codimension zero point η of X_0, let $\mu'_{U_\eta} = (\phi_{U_\eta})^{-1}(\mu_{U_\eta}) \in L \otimes F_{U_\eta}$, and for every $P \in \mathscr{P}$, let $\mu'_P = (\phi_P)^{-1}(\mu_P) \in L \otimes F_P$. Since ϕ_{U_η} and ϕ_P are isomorphisms, we have the required properties. □

PROPOSITION 7.4. *Suppose that for every branch $\wp = (P, \eta)$, there exists $t_\wp \geqslant 0$ such that $F_{P,\eta}$ has no primitive $\ell^{t_\wp}$th root of unity. Let L/F be a cyclic field extension of degree ℓ. Suppose that:*

- *for every codimension zero point η of X_0, there exist a nonempty open proper subset U_η of η and $\mu'_{U_\eta} \in L \otimes F_{U_\eta}$;*
- *for every $P \in \mathscr{P} = X_0 \backslash \cup U_\eta$, $\mu'_P \in L \otimes F_P$,*

such that:

(1) *$N_{L\otimes F_{U_\eta}/F_{U_\eta}}(\mu'_{U_\eta}) = -\lambda$ and $\alpha \cdot (\mu'_{U_\eta}) = 0 \in H^3(L \otimes F_{U_\eta}, \mu_n^{\otimes 2})$ for all codimension zero points η of X_0;*
(2) *$N_{L\otimes F_P/F_P}(\mu'_P) = -\lambda$ and $\alpha \cdot (\mu'_P) = 0 \in H^3(L \otimes F_P, \mu_n^{\otimes 2})$ for all $P \in \mathscr{P}$;*
(3) *for any branch (P, η), $\mu'_{U_\eta}\mu'^{-1}_P \in (L_P \otimes F_{P,\eta})^{\ell m}$ for all $m \geqslant 1$.*

Then there exists $\mu \in L^*$ such that:

- *$N_{L/F}(\mu) = -\lambda$; and*
- *$\alpha \cdot (\mu) = 0 \in H^3(L, \mu_n^{\otimes 2})$.*

Proof. Let σ be a generator of $\mathrm{Gal}(L/F)$. Let $\wp = (P, \eta)$ be a branch. Since $N_{L\otimes F_{P,\eta}/F_{P,\eta}}(\mu'_{U_\eta}) = N_{L\otimes F_{P,\eta}/F_{P,\eta}}(\mu'_P)$, by Lemma 2.7, there exists $\theta_{P,\eta} \in L \otimes F_{P,\eta}$ such that $\mu'_{U_\eta}\mu'^{-1}_P = \theta_{P,\eta}^{-\ell^d}\sigma(\theta_{P,\eta}^{\ell^d})$. Applying [HHK09, Theorem 3.6] for the rational group $R_{L/F}\mathbf{G}_m$, there exist $\theta_{U_\eta} \in L \otimes F_{U_\eta}$ and $\theta_P \in L \otimes F_P$ for every codimension zero point η of X_0 and $P \in \mathscr{P}$ such that for every branch (P, η), $\theta_{P,\eta} = \theta_{U_\eta}\theta_P$.

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Let $\mu''_{U_\eta} = \mu'_{U_\eta}\theta_{U_\eta}^{\ell^d}\sigma(\theta_{U_\eta}^{-\ell^d}) \in L \otimes F_{U_\eta}$ and $\mu''_P = \mu'_P\theta_P^{-\ell^d}\sigma(\theta_P^{\ell^d}) \in L \otimes F_P$. If (P,η) is a branch, then we have

$$\begin{aligned}\mu''_{U_\eta} &= \mu'_{U_\eta}\theta_{U_\eta}^{\ell^d}\sigma(\theta_{U_\eta}^{-\ell^d})\\ &= \mu'_P\theta_{P,\eta}^{-\ell^d}\sigma(\theta_{P,\eta}^{\ell^d})\theta_{U_\eta}^{\ell^d}\sigma(\theta_{U_\eta}^{-\ell^d})\\ &= \mu'_P\theta_P^{-\ell^d}\sigma(\theta_P^{\ell^d})\\ &= \mu''_P \in L \otimes F_{P,\eta}.\end{aligned}$$

Hence, by [HH10, Proposition 6.3], there exists $\mu \in L$ such that $\mu = \mu''_{U_\eta}$ and $\mu = \mu''_P$ for every codimension zero point η of X_0 and $P \in \mathscr{P}$. Clearly, $N_{L/F}(\mu) = -\lambda$ over F. Let $P \in \mathscr{P}$. Since $\alpha\cdot(\mu'_P) = 0$ and $\alpha\cdot(\theta_P^{\ell^d}) = 0$, $\alpha\cdot(\mu) = 0 \in H^3(L\otimes F_P, \mu_n^{\otimes 2})$. Similarly, $\alpha\cdot(\mu) = 0 \in H^3(L\otimes F_{U_\eta}, \mu_n^{\otimes 2})$ for every codimension zero point η of X_0. Let $\mathscr{Y}$ be the normal closure of $\mathscr{X}$ in L and Y_0 the reduced special fiber of $\mathscr{Y}$. Let η' be a codimension zero point of Y_0. Then the image η of η' in X is a codimension zero point. Since $F_\eta \subset L_{\eta'}$, we have a map $L \otimes F_{U_\eta} \to L_{\eta'}$ and hence $\alpha\cdot(\mu) = 0 \in H^3(L_{\eta'}, \mu_n^{\otimes 2})$. Let Q be a closed point of Y_0 and P its image in X_0. Suppose $P \in U_\eta$ for some η. Since $F_{U_\eta} \subset F_P \subset L_Q$, it follows that $\alpha\cdot(\mu) = 0 \in H^3(L_Q, \mu_n^{\otimes 2})$. Suppose $P \in \mathscr{P}$. Since $F_P \subset L_Q$, we have $\alpha\cdot(\mu) = 0 \in H^3(L_Q, \mu_n^{\otimes 2})$. Hence, by [HHK14, Theorem 3.2.3], $\alpha\cdot(\mu) = 0$ in $H^3(L, \mu_n^{\otimes 2})$. □

Proposition 7.5. *Suppose that for every branch $\wp = (P,\eta)$, there exists $t_\wp \geqslant 0$ such that $F_{P,\eta}$ has no primitive $\ell^{t_\wp}$th root of unity. Let L/F be an extension of degree ℓ as in Lemma 7.3. Then there exist a field extension N/F of degree coprime to ℓ and $\mu \in (L\otimes N)^*$ such that:*
- *$N_{L\otimes N/N}(\mu) = -\lambda$; and*
- *$\alpha\cdot(\mu) = 0 \in H^3(L\otimes N, \mu_n^{\otimes 2})$.*

Proof. Let L/F, U_η, $\mathscr{P}$, μ'_{U_η} and μ'_P be as in Lemma 7.3. Since L/F is an extension of degree ℓ, there exists a field extension N/F of degree coprime to ℓ such that $L\otimes N$ is a cyclic extension field extension N of degree ℓ.

Let $\mathscr{Y}$ be the integral closure of $\mathscr{X}$ in N and Y_0 the reduced special fiber of $\mathscr{Y}$. Let $\phi : Y_0 \to X_0$ be the induced morphism.

Let $\eta' \in Y_0$ be a codimension zero point. Then $\eta = \phi(\eta') \in X_0$ is a codimension zero point. Let $U_{\eta'} = \phi^{-1}(U_\eta) \cap \overline{\eta'} \in Y_0$. Then $U_{\eta'}$ is a proper open subset of $\overline{\eta'}$ and we have an inclusion $F_{U_\eta} \subset N_{U_{\eta'}}$. Let $\mu'_{U_{\eta'}} \in (L\otimes_F N)\otimes_N N_{U_{\eta'}}$ be the image of μ'_{U_η} under the natural map $L\otimes_F F_{U_\eta} \to L\otimes_F N_{U_{\eta'}} \simeq (L\otimes_F N)\otimes_N N_{U_{\eta'}}$. Then we have $N_{(L\otimes_F N)\otimes_N N_{U_{\eta'}}/N_{U_{\eta'}}}(\mu'_{U_{\eta'}}) = -\lambda$ and $\alpha\cdot(\mu'_{U_{\eta'}}) = 0 \in H^3((L\otimes_F N)\otimes_N N_{U_{\eta'}}, \mu_n^{\otimes 2})$.

Let $\mathscr{P}' = Y_0 \setminus \cup_{\eta'} U_{\eta'}$. Let $Q \in \mathscr{P}'$ and $P = \phi(Q) \in X_0$. Then $P \in \mathscr{P}$ and $F_P \subset N_Q$. Let $\mu'_Q \in (L\otimes_F N)\otimes_N N_Q$ be the image of μ'_P under the natural map $L\otimes_F F_P \to L\otimes_F N_Q \simeq (L\otimes_F N)\otimes_N N_Q$. Then we have $N_{(L\otimes_F N)\otimes_N N_Q/N_Q}(\mu'_Q) = -\lambda$ and $\alpha\cdot(\mu'_Q) = 0 \in H^3((L\otimes_F N)\otimes_N N_Q, \mu_n^{\otimes 2})$.

Let $\wp' = (Q,\eta')$ be a branch in Y_0 and $P = \phi(Q)$, $\eta = \phi(\eta')$. Then (P,η) is a branch in X_0. Since $\mu'_{U_\eta}\mu_P'^{-1} \in (L_P\otimes F_{P,\eta})^{\ell^m}$ for all $m \geqslant 1$, it follows that $\mu'_{U_{\eta'}}\mu_Q'^{-1} \in ((L\otimes_F N)\otimes_N N_{Q,\eta'})^{\ell^m}$ for all $m \geqslant 1$. Since there exists $t_\wp \geqslant 0$, such that $F_{P,\eta}$ has no primitive $\ell^{t_\wp}$th root of unity and $N_{Q,\eta'}/F_{P,\eta}$ is a finite extension, there exists $t_{\wp'} \geqslant 0$ such that $N_{Q,\eta'}$ contains no primitive $\ell^{t_{\wp'}}$th root of unity.

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Since $L \otimes_F N$ is a cyclic extension of degree ℓ, by Proposition 7.4, there exist $\mu' \in L \otimes_F N$ such that $N_{L\otimes_F N/N}(\mu') = -\lambda$ and $\alpha \cdot (\mu') = 0 \in H^3(L \otimes_F N, \mu_n^{\otimes 2})$. □

8. Types of points, special points and type 2 connections

Let F, $\alpha \in H^2(F, \mu_n)$, $\lambda \in F^*$ with $\alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, $\mathscr{X}$ and X_0 be as in § 7. Further, assume that:

- κ is a finite field;
- $\mathscr{X}$ is regular such that $\mathrm{ram}_{\mathscr{X}}(\alpha) \cup \mathrm{supp}_{\mathscr{X}}(\lambda) \cup X_0$ is a union of regular curves with normal crossings;
- the intersection of any two distinct irreducible curves in X_0 is at most one closed point.

We fix the following notation:

- $\mathscr{P}$ is the set of points of intersection of distinct irreducible curves in X_0;
- $\mathscr{O}_{\mathscr{X},\mathscr{P}}$ is the semi-local ring at the points of $\mathscr{P}$ on $\mathscr{X}$;
- if a codimension zero point η of X_0 contains a closed point $P \in \mathscr{P}$, then $\pi_\eta \in \mathscr{O}_{\mathscr{X},\mathscr{P}}$ is a prime defining η on $\mathscr{O}_{\mathscr{X},\mathscr{P}}$.

Let η be a codimension zero point of X_0. For the rest of this paper, let (E_η, σ_η) denote the lift of the residue of α at η. Since $\alpha \in H^2(F, \mu_n)$ with n a power of ℓ, $[E_\eta : F_\eta]$ is a power of ℓ. If α is unramified at η, then $E_\eta = F_\eta$ and let $M_\eta = F_\eta$. If α is ramified at η, then $E_\eta \neq F_\eta$ and there is a unique subextension of E_η of degree ℓ and we denote it by M_η.

Remark 8.1. Let η be a codimension zero point of X_0. Suppose α is ramified at η. Since $\mathrm{ind}(\alpha \otimes F_\eta) = \mathrm{ind}(\alpha \otimes E_\eta)[E_\eta : F_\eta]$ (cf. Lemma 4.2) and $M_\eta \subset E_\eta$, it follows that $\mathrm{ind}(\alpha \otimes M_\eta) < \mathrm{ind}(\alpha)$.

We divide the codimension zero points η of X_0 as follows.
Type 1: $\nu_\eta(\lambda)$ is coprime to ℓ and $\mathrm{ind}(\alpha \otimes F_\eta) = \mathrm{ind}(\alpha)$.
Type 2: $\nu_\eta(\lambda)$ is coprime to ℓ and $\mathrm{ind}(\alpha \otimes F_\eta) < \mathrm{ind}(\alpha)$.
Type 3: $\nu_\eta(\lambda) = r\ell$, $r\alpha \otimes E_\eta \neq 0$ and $\mathrm{ind}(\alpha \otimes F_\eta) = \mathrm{ind}(\alpha)$.
Type 4: $\nu_\eta(\lambda) = r\ell$, $r\alpha \otimes E_\eta \neq 0$ and $\mathrm{ind}(\alpha \otimes F_\eta) < \mathrm{ind}(\alpha)$.
Type 5: $\nu_\eta(\lambda) = r\ell$, $r\alpha \otimes E_\eta = 0$ and $\mathrm{ind}(\alpha \otimes F_\eta) = \mathrm{ind}(\alpha)$.
Type 6: $\nu_\eta(\lambda) = r\ell$, $r\alpha \otimes E_\eta = 0$ and $\mathrm{ind}(\alpha \otimes F_\eta) < \mathrm{ind}(\alpha)$.

Let P be a closed point of $\mathscr{X}$. Suppose P is the point of intersection of two distinct codimension zero points η_1 and η_2 of X_0. We say that the point P is a:

(1) *special point of type I* if η_1 is of type 1 and η_2 is of type 2;
(2) *special point of type II* if η_1 is of type 1 and η_2 is of type 4;
(3) *special point of type III* if η_1 is of type 3 or 5 and η_2 is of type 4;
(4) *special point of type IV* if η_1 is of type 1, 3 or 5 and η_2 is of type 5 with $M_{\eta_2} \otimes F_{P,\eta_2}$ not a field.

Lemma 8.2. *Suppose that η is a codimension zero point of X_0 and P a point of η. Suppose that α is ramified at η. Let (E_η, σ_η) be the lift of residue of α at η. If $E_\eta \otimes F_{P,\eta}$ is not a field, then $\mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$.*

Proof. Suppose that $E_\eta \otimes F_{P,\eta}$ is not a field. Since E_η/F_η is a cyclic extension, $E_\eta \otimes F_{P,\eta} \simeq \prod E_{\eta,P}$ with $[E_{\eta,P} : F_{P,\eta}] < [E_\eta : F_\eta]$. We have $(E_\eta, \sigma_\eta, \pi_\eta) \otimes F_{P,\eta} = (E_{\eta,P}, \sigma_\eta, \pi_\eta)$ (cf. § 2).

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Write $\alpha \otimes F_\eta = \alpha_1 + (E_\eta, \sigma_\eta, \pi_\eta)$ as in Lemma 4.1. Then $\alpha \otimes F_{P,\eta} = \alpha_1 \otimes F_{P,\eta} + (E_{\eta,P}, \sigma_\eta, \pi_\eta)$. By Lemma 4.2, we have $\mathrm{ind}(\alpha \otimes F_\eta) = \mathrm{ind}(\alpha_1 \otimes E_\eta)[E_\eta : F_\eta]$. We have

$$\begin{aligned}\mathrm{ind}(\alpha \otimes F_{P,\eta}) &\leqslant \mathrm{ind}(\alpha_1 \otimes E_{\eta,P})[E_{\eta,P} : F_{P,\eta}]\\ &\leqslant \mathrm{ind}(\alpha_1 \otimes E_\eta)[E_{\eta,P} : F_{P,\eta}]\\ &< \mathrm{ind}(\alpha_1 \otimes E_\eta)[E_\eta : F_\eta]\\ &= \mathrm{ind}(\alpha \otimes F_\eta).\end{aligned}$$

Thus, by Proposition 5.8, $\mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$. □

Lemma 8.3. *Let $\eta \in X_0$ be a point of codimension zero and P a closed point on η. Let $\mathscr{X}_P \to \mathscr{X}$ be the blow-up at P and γ the exceptional curve in $\mathscr{X}_P$. If $E_\eta \otimes F_{P,\eta}$ is not a field or η is of type 2, 4 or 6, then γ is of type 2, 4 or 6.*

Proof. If $E_\eta \otimes F_{P,\eta}$ is not a field, then by Lemma 8.2, $\mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$. If η is of type 2, 4 or 6, then $\mathrm{ind}(\alpha \otimes F_\eta) < \mathrm{ind}(\alpha)$ and hence, by Proposition 5.8, $\mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$. Since $F_P \subset F_\gamma$, we have $\mathrm{ind}(\alpha \otimes F_\gamma) \leqslant \mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$. Hence γ is of type 2, 4 or 6. □

Lemma 8.4. *Let η_1 and η_2 be two distinct codimension zero points of X_0 intersecting at a closed point P. Suppose that η_1 is of type 1 or 2 and η_2 is of type 2. Then there exists a sequence of blow-ups $\psi : \mathscr{X}' \to \mathscr{X}$ such that if $\tilde{\eta}_i$ are the strict transforms of η_i, then:*

(1) *$\psi : \mathscr{X}' \backslash \psi^{-1}(P) \to \mathscr{X} \backslash \{P\}$ is an isomorphism;*
(2) *$\psi^{-1}(P)$ is the union of irreducible regular curves $\gamma_1, \ldots, \gamma_m$;*
(3) *$\tilde{\eta}_1 \cap \gamma_1 = \{P_0\}$, $\gamma_i \cap \gamma_{i+1} = \{P_i\}$, $\gamma_m \cap \tilde{\eta}_2 = \{P_m\}$, $\tilde{\eta}_1 \cap \gamma_i = \emptyset$ for all $i > 1$, $\tilde{\eta}_2 \cap \gamma_i = \emptyset$ for all $i < m$, $\tilde{\eta}_1 \cap \tilde{\eta}_2 = \emptyset$, $\gamma_i \cap \gamma_j = \emptyset$ for all $i < j \neq i+1$;*
(4) *γ_1 and γ_m are of type 6 and γ_i, $1 < i < m$, are of type 2, 4 or 6;*
(5) *$\psi^{-1}(P)$ has no special points.*

Proof. Let $\mathscr{X}_P \to \mathscr{X}$ be the blow-up of $\mathscr{X}$ at P and γ the exceptional curve in $\mathscr{X}_P$. Let $\tilde{\eta}_i$ be the strict transform of η_i. Then $\tilde{\eta}_1$ intersects γ only at one point P_0 and $\tilde{\eta}_2$ intersects γ at only one point P_1. Since η_2 is of type 2, by Lemma 8.3, γ is of type 2, 4 or 6 and hence P_1 is not a special point.

Let $s_1 = \nu_{\eta_1}(\lambda)$, $s_2 = \nu_{\eta_2}(\lambda)$. Then $\nu_\gamma(\lambda) = s_1 + s_2$. Suppose $s_1 + s_2 = \ell^{d+1} r_0$ for some integer r_0, where $\ell^d = \mathrm{ind}(\alpha)$. Since $\ell^d \alpha = 0$, $\ell^d r_0 \alpha = 0$. Thus, γ is of type 6. Hence P_0 is not a special point and $\mathscr{X}_P$ has all the required properties.

Suppose $s_1 + s_2 = \ell^t r_0$ with $t \leqslant d$ and r_0 coprime to ℓ. Then blow up the points P_0 and P_1 and let γ_1 and γ_2 be the exceptional curves in this blow-up. Then we have $\nu_{\gamma_1}(\lambda) = 2s_1 + s_2$ and $\nu_{\gamma_2}(\lambda) = s_1 + 2s_2$. If $2s_1 + s_2$ is not of the form $\ell^{d+1} r_1$ for some $r_1 \geqslant 1$, then blow up the point of intersection of the strict transforms of η_1 and γ_1. If $s_1 + 2s_2$ is not of the form $\ell^{d+1} r_2$ for some $r_2 \geqslant 1$, then blow up the point of intersection of the strict transforms of η_2 and γ_2. Since s_1 and s_2 are coprime to ℓ, there exist i and j such that $is_1 + s_2 = \ell^{d+1} r$ and $s_1 + js_2 = \ell^{d+1} r'$ for some $r, r' \geqslant 1$. Thus, we get the required finite sequence of blow-ups. □

Proposition 8.5. *There exists a regular proper model of F with no special points.*

Proof. Let $P \in \mathscr{P}$. Then there exist two codimension zero points η_1 and η_2 of X_0 intersecting at P.

Suppose that P is a special point of type I. Let $\psi : \mathscr{X}' \to \mathscr{X}$ be a sequence of blow-ups as in Lemma 8.4. Then there are no special points in $\psi^{-1}(P)$. Since there are only finitely many

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special points in $\mathscr{X}$, replacing $\mathscr{X}$ by a finite sequence of blow-ups at all special points of type I, we assume that $\mathscr{X}$ has no special points of type I.

Suppose P is a special point of type II. Without loss of generality we assume that η_1 is of type 1 and η_2 is of type 4. Let $\mathscr{X}_P \to \mathscr{X}$ be the blow-up of $\mathscr{X}$ at P and γ the exceptional curve in $\mathscr{X}_P$. Since η_2 is of type 4, by Lemma 8.3, γ is of type 2, 4 or 6. Since η_1 is of type 1 and η_2 is of type 4, $\nu_{\eta_1}(\lambda)$ is coprime to ℓ and $\nu_{\eta_2}(\lambda)$ is divisible by ℓ. Since $\nu_\gamma(\lambda) = \nu_{\eta_1}(\lambda) + \nu_{\eta_2}(\lambda)$, $\nu_\gamma(\lambda)$ is coprime to ℓ and hence γ is of type 2. Let $\tilde{\eta}_i$ be the strict transform of η_i in $\mathscr{X}_P$. Then $\tilde{\eta}_i$ and γ intersect at only one point Q_i. Since γ is of type 2, Q_1 is a special point of type I and Q_2 is not a special point. Thus, as above, by replacing $\mathscr{X}$ by a sequence of blow-ups of $\mathscr{X}$, we assume that $\mathscr{X}$ has no special points of type I or II.

Suppose P is a special point of type III. Without loss of generality assume that η_1 is of type 3 or 5 and η_2 of type 4. Let $\mathscr{X}_P \to \mathscr{X}$ be the blow-up of $\mathscr{X}$ at P, γ, $\tilde{\eta}_i$, and Q_i be as above. Since η_2 is of type 4, by Lemma 8.3, γ is of type 2, 4 or 6. Since $\nu_{\eta_1}(\lambda)$ and $\nu_{\eta_2}(\lambda)$ are divisible by ℓ, $\nu_\gamma(\lambda) = \nu_{\eta_1}(\lambda) + \nu_{\eta_2}(\lambda)$ is divisible by ℓ. Thus γ is of type 4 or 6. Hence Q_2 is not a special point. By Corollary 5.7, $\alpha \otimes F_P = (E_P, \sigma, u\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2})$ for some cyclic extension E_P/F_P, $u \in \hat{A}_P$ a unit, and at least one of the d_i is coprime to ℓ (in fact equal to 1). In particular, $\alpha \otimes F_P$ is split by the extension $F_P(\sqrt[m]{u\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2}})$, where m is the degree of E_P/F_P which is a power of ℓ. Suppose $d_1 + d_2$ is coprime to ℓ. Since $\nu_\gamma(\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2}) = d_1 + d_2$, $F_P(\sqrt[m]{u\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2}})$ is totally ramified at γ. Thus, by Lemma 4.3, γ is of type 6. Hence Q_1 is not a special point. Suppose that $d_1 + d_2$ is divisible by ℓ. Let π_γ be a prime defining γ at Q_1. Then we have $u\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2} = w_1\pi_{\eta_1}^{d_1}\pi_\gamma^{d_1+d_2}$ for some unit w_1 at Q_1. Since one of d_i is coprime to ℓ and $d_1 + d_2$ is divisible by ℓ, the d_i are not divisible by ℓ. In particular, $2d_1 + d_2$ is coprime to ℓ. Let $\mathscr{X}_{Q_1}$ be the blow-up of $\mathscr{X}_P$ at Q_1 and γ' be the generic point of the exceptional curve in $\mathscr{X}_{Q_1}$. Then $\nu_{\gamma'}(u\pi_{\eta_1}^{d_1}\pi_{\eta_2}^{d_2}) = \nu_{\gamma'}(w_1\pi_{\eta_1}^{d_1}\pi_\gamma^{d_1+d_2}) = 2d_1 + d_2$. Since $2d_1 + d_2$ is coprime to ℓ, once again by Lemma 4.3, γ' is of type 6. In particular, no point on the exceptional curve in $\mathscr{X}_{Q_1}$ is a special point. Thus, replacing $\mathscr{X}$ by a sequence of blow-ups, we assume that $\mathscr{X}$ has no special points of type I, II or III.

Suppose P is a special point of type IV. Without loss of generality assume that η_1 is of type 1, 3 or 5 and η_2 is of type 5, with $M_{\eta_2} \otimes F_{P,\eta_2}$ not a field. Let $\mathscr{X}_P \to \mathscr{X}$ be the blow-up of $\mathscr{X}$ at P and γ, $\tilde{\eta}_i$, Q_i be as above. Since $M_{\eta_2} \otimes F_{P,\eta_2}$ is not a field, by Lemma 8.3, γ is of type 2, 4 or 6. If γ is of type 6, then Q_1 and Q_2 are not special points. Suppose γ is of type 2 or 4. Then Q_1 and Q_2 are special points of type I, II or III. Thus, as above, by replacing $\mathscr{X}$ by a sequence of blow-ups of $\mathscr{X}$, we assume that $\mathscr{X}$ has no special points. □

Let η and η' be two codimension zero points of X_0 (need not be distinct). A *type 2 connection* from η to η' is a sequence of distinct codimension zero points $\eta_1, \ldots, \eta_n$ of X_0 of type 2 such that η intersects η_1, η' intersects η_n, η_i intersects η_{i+1} for all $1 \leqslant i \leqslant n-1$, η does not intersect η_i for $i > 1$, η' does not intersect η_i for $i < n$, η_i does not intersect η_j for $i < j \neq i+1$ and if $\eta = \eta'$, then $n \geqslant 2$.

We note that if η is a codimension zero point of X_0 of type 2 and η' is any other codimension zero point of X_0 intersecting η at a closed point, then there is a type 2 connection from η to η'. This can be seen by taking $n = 1$ and $\eta_1 = \eta$.

PROPOSITION 8.6. *There exists a regular proper model $\mathscr{X}$ of F such that:*

1. *$\mathscr{X}$ has no special points;*
2. *if η_1 and η_2 are two (not necessarily distinct) codimension zero points of X_0 with η_1 of type 3 or 5 and η_2 of type 3, 4 or 5, then there is no type 2 connection between η_1 and η_2.*

Proof. Let $\mathscr{X}$ be a regular proper model with no special points (Proposition 8.5). Let $m(\mathscr{X})$ be the number of type 2 connections between a point of type 3 or 5 and a point of type 3, 4 or 5. We prove the proposition by induction on $m(\mathscr{X})$. Suppose $m(\mathscr{X}) \geqslant 1$. We show that there is a sequence of blow-ups $\mathscr{X}'$ of $\mathscr{X}$ with no special points and $m(\mathscr{X}') < m(\mathscr{X})$.

Let η be a codimension zero point of X_0 of type 3 or 5 and η' a codimension zero point of X_0 of types 3, 4 or 5. Suppose there is a type 2 connection from η to η'. Then there exist distinct codimension zero points $\eta_1, \dots, \eta_n$ of X_0 of type 2 with η intersecting η_1, η' intersecting η_n and η_i intersecting η_{i+1} for $i = 1, \dots, n-1$.

Suppose $n = 1$. Let Q be the point of the intersection of η and η_1. Let $\mathscr{X}_Q \to \mathscr{X}$ be the blow-up of $\mathscr{X}$ at Q and γ the exceptional curve in $\mathscr{X}_Q$. Since η_1 is of type 2, by Lemma 8.3, γ is of type 2, 4 or 6. Since η is of type 3 or 5 and η_1 is of type 2, ℓ divides $\nu_\eta(\lambda)$ and ℓ does not divide $\nu_{\eta_1}(\lambda)$. Since $\nu_\gamma(\lambda) = \nu_\eta(\lambda) + \nu_{\eta_1}(\lambda)$, $\nu_\gamma(\lambda)$ is not divisible by ℓ and hence γ is of type 2. Let $\tilde{\eta}$ and $\tilde{\eta}_1$ be the strict transform of η and η_1 in $\mathscr{X}_Q$. Since γ is a point of type 2, the points of intersection of $\tilde{\eta}$ and $\tilde{\eta}_1$ with γ are not special points. Hence $\mathscr{X}_Q$ has no special points. Replacing $\mathscr{X}$ by $\mathscr{X}_Q$, we assume that $n \geqslant 2$ and $\mathscr{X}$ has no special points.

Let P be the point of intersection of η_1 and η_2. Let $\mathscr{X}'$ be as in Lemma 8.4. Then $\mathscr{X}'$ has no special points and all the exceptional curves in $\mathscr{X}'$ are of type 2, 4 or 6 and the exceptional curves which intersect the strict transforms of η_1 and η_2 are of type 6. In particular, the number of type 2 connections between the strict transforms of η and η' is one less than the number of type 2 connections between η and η'. Since all the exceptional curves in $\mathscr{X}'$ are of type 2, 4 or 6, $m(\mathscr{X}') = m(\mathscr{X}) - 1$. Thus, by induction, we have a regular proper model with the required properties. □

Lemma 8.7. *Let $\mathscr{X}$ be as in Proposition 8.6 and X_0 the special fiber of $\mathscr{X}$. Let η be a codimension zero point of X_0 of type 2 and η' a codimension zero point of X_0 of type 3 or 5. Suppose there is a type 2 connection from η to η'. If there is a type 2 connection from η to a type 3 or 5 point η'', then $\eta' = \eta''$. Further, if $\eta_1, \dots, \eta_n$ are codimension zero points of X_0 of type 2 giving a type 2 connection from η to η' and $\gamma_1, \dots, \gamma_m$ codimension zero points of X_0 of type 2 giving another type 2 connection from η to η', then $n = m$ and $\eta_i = \gamma_i$ for all i.*

Proof. Suppose η'' is a codimension zero point of X_0 of type 3 or 5 with type 2 connection to η. Since η is of type 2, there is a type 2 connection from η' to η''. Since no two points of type 3 or 5 have a type 2 connection (cf. Proposition 8.6), $\eta' = \eta''$. Suppose $\gamma_1, \dots, \gamma_m$ is of type 2 connection from η to η'. If $m \neq n$ or $\eta_i \neq \gamma_i$ for some i, then we will have a type 2 connection from η' to η' and hence a contradiction to the choice of $\mathscr{X}$ (cf. Proposition 8.6). Thus $n = m$ and $\eta_i = \gamma_i$ for all i. □

Let η be a codimension zero point of X_0 of type 2 and η' be a codimension zero point of X_0 of type 3 or 5. Suppose there is a type 2 connection $\eta_1, \dots, \eta_n$ from η to η'. Then, by Lemma 8.7, η' and η_n are uniquely defined by η. We call this point of intersection of η_n with η' *the point of type 2 intersection of η and η'*. Once again note that such a closed point is uniquely defined by η.

9. Choice of L_P and μ_P at closed points

Let F, $\alpha \in H^2(F, \mu_n)$, $\lambda \in F^*$ with $\alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, $\mathscr{X}$ and X_0 be as in (§§ 7 and 8). Throughout this section we assume that $\mathscr{X}$ has no special points and if η_1 and η_2 are two (not necessarily distinct) codimension zero points of X_0 with η_1 is of type 3 or 5 and η_2 is of type 3, 4 or 5, then there is no type 2 connection between η_1 and η_2. Further, assume that F contains a primitive ℓth root of unity.

Let η be a codimension zero point of X_0 of type 5. Then we call η of *type 5a* if α is unramified at η and of *type 5b* if α is ramified at η. Suppose η is of type 5b. Then α is ramified and hence M_η is the unique subextension of E_η of degree ℓ, where (E_η, σ_η) is the lift of the residue of α.

Lemma 9.1. *Let η be a codimension zero point of X_0 of type 5b. Then $\mathrm{ind}(\alpha \otimes M_\eta) < \mathrm{ind}(\alpha)$ and there exists $\mu_\eta \in M_\eta$ such that $N_{M_\eta/F_\eta}(\mu_\eta) = -\lambda$ and $\alpha \cdot (\mu_\eta) = 0 \in H^3(M_\eta, \mu_n^{\otimes 2})$.*

Proof. Since η is of type 5b, α is ramified at η, $\nu_\eta(\lambda) = r\ell$, $r\alpha \otimes E_\eta = 0$ and $E_\eta \neq F_\eta$. Thus, as in the proof of Lemma 4.11, there exists $\mu_\eta \in M_\eta$ such that $N_{M_\eta/F_\eta}(\mu_\eta) = -\lambda$ and $\alpha \cdot (\mu_\eta) = 0$. □

Lemma 9.2. *Let $P \in \mathscr{P}$, and η_1 and η_2 be codimension zero points of X_0 containing P. Suppose that η_1 and η_2 are of type 5. Then there exist a cyclic field extension L_P/F_P of degree ℓ and $\mu_P \in L_P$ such that:*

1. *$N_{L_P/F_P}(\mu_P) = -\lambda$;*
2. *$\mathrm{ind}(\alpha \otimes L_P) < \mathrm{ind}(\alpha)$;*
3. *$\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$;*
4. *if η_i is of type 5a, then $L_P \otimes F_{P,\eta_i}/F_{P,\eta_i}$ is an unramified field extension;*
5. *if η_i is of type 5b, then $L_P \otimes F_{P,\eta_i} \simeq M_{\eta_i} \otimes F_{P,\eta_i}$.*

Proof. Since $\mathscr{X}$ has no special points, P is not a special point of type IV. Since η_1 and η_2 are of type 5 intersecting at P, $M_{\eta_1} \otimes F_{P,\eta_1}$ and $M_{\eta_2} \otimes F_{P,\eta_2}$ are fields. Suppose η_i is of type 5a. If $\alpha \otimes F_{P,\eta_i} = 0$, then let $L_{P,\eta_i}/F_{P,\eta_i}$ be any cyclic unramified field extension with $-\lambda$ a norm and $\mu_{\eta_i} \in L_{P,\eta_i}$ with $N_{L_{P,\eta_i}/F_{P,\eta_i}}(\mu_{\eta_i}) = -\lambda$. If $\alpha \otimes F_{P,\eta_i} \neq 0$, then let $L_{P,\eta_i}/F_{P,\eta_i}$ be a cyclic unramified field extension of degree ℓ and μ_{η_i} be as in Lemma 4.10. Suppose η_i is of type 5b. Let $L_{P,\eta_i} = M_{\eta_i} \otimes F_{P,\eta_i}$ and $\mu_{\eta_i} \in M_{\eta_i}$ be as in Lemma 9.1. Then, by choice $L_{P,\eta_i}/F_{P,\eta_i}$ are unramified field extensions. By applying Lemma 6.4 to L_{P,η_i} and μ_{η_i}, there exist a cyclic field extension L_P/F_P and $\mu_P \in L_P$ with the required properties. □

Lemma 9.3. *Let η be a codimension zero point of X_0 with $\nu_\eta(\lambda)$ a multiple of ℓ and P a closed point on η. Then there exists a cyclic unramified field extension $L_{P,\eta}/F_{P,\eta}$ of degree ℓ and $\mu_{P,\eta} \in L_{P,\eta}$ such that $N_{L_{P,\eta}/F_{P,\eta}}(\mu_{P,\eta}) = -\lambda$ and $\alpha \cdot (\mu_{P,\eta}) = 0$. Further, if η is of type 3 or 4, then $\mathrm{ind}(\alpha \otimes E_\eta \otimes L_{P,\eta}) < \mathrm{ind}(\alpha \otimes E_\eta)$.*

Proof. Since $\nu_\eta(\lambda)$ is divisible by ℓ, write $\lambda = \theta\pi_\eta^{r\ell}$ for some $\theta \in F_\eta$ a unit at η and integer r. Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)$ as in Lemma 4.1. Let $\overline{\alpha}'$ be the image of α' in $H^2(\kappa(\eta), \mu_n)$ and θ_0 be the image of θ in $\kappa(\eta)$. Since $\kappa(\eta)_P$ is a local field containing a primitive ℓth root of unity, there exists a cyclic field extension $L(\eta)_P/\kappa(\eta)_P$ of degree ℓ such that $-\theta_0$ is a norm from $L(\eta)_P$ (cf. the proof of Lemma 2.8). Let $L_{P,\eta}/F_{P,\eta}$ be the unramified extension of degree ℓ with residue field $L(\eta)_P$. Since $-\overline{\theta}$ is a norm from $L(\eta)_P$, $-\theta$ is a norm from $L_{P,\eta}$ and hence $-\lambda = -\theta\pi_\eta^{r\ell}$ is a norm from $L_{P,\eta}$. Since $N_{L_{P,\eta}/F_{P,\eta}}(\mu_{P,\eta}) = -\lambda$, $L_{P,\eta}/F_{P,\eta}$ is a field extension and $\alpha \cdot (-\lambda) = 0$, by Proposition 4.6, we have $\alpha \cdot (\mu_{P,\eta}) = 0$.

Suppose η is of type 3 or 4. Then $r\alpha' \otimes E_\eta = r\alpha \otimes E_\eta \neq 0$ and hence $r\overline{\alpha}' \otimes E(\eta) \neq 0$. Thus, by Lemma 3.3, $\mathrm{ind}(\overline{\alpha}' \otimes E(\eta) \otimes L(\eta)_P) < \mathrm{ind}(\overline{\alpha}' \otimes E(\eta))$. Suppose $\alpha \otimes E_\eta \otimes F_{P,\eta} \neq 0$. Since $\alpha \otimes E_\eta = \alpha' \otimes E_\eta$, $\alpha' \otimes E_\eta \neq 0$ and hence $\overline{\alpha}' \otimes E(\eta) \neq 0$. Thus, by the choice of $L(\eta)_P$, $\mathrm{ind}(\overline{\alpha}' \otimes E(\eta) \otimes L(\eta)_P) < \mathrm{ind}(\overline{\alpha}' \otimes E(\eta))$. In particular, $\mathrm{ind}(\alpha \otimes E_\eta \otimes L_{P,\eta}) = \mathrm{ind}(\alpha' \otimes E_\eta \otimes L_{P,\eta}) = \mathrm{ind}(\overline{\alpha}' \otimes E(\eta) \otimes L(\eta)_P) < \mathrm{ind}(\overline{\alpha}' \otimes E(\eta)) = \mathrm{ind}(\alpha' \otimes E_\eta) = \mathrm{ind}(\alpha \otimes E_\eta)$. □

Lemma 9.4. *Let $P \in \mathscr{P}$, and η_1 and η_2 be codimension zero points of X_0 containing P. Suppose that η_1 is of type 2 and η_2 is of type 5 or 6. Then there exist $\mu_i \in F_P$, $1 \leqslant i \leqslant \ell$, such that:*

(1) $\mu_1 \cdots \mu_\ell = -\lambda$;
(2) $\nu_{\eta_1}(\mu_1) = \nu_{\eta_1}(\lambda)$, $\nu_{\eta_1}(\mu_i) = 0$ *for* $i \geqslant 2$;
(3) $\nu_{\eta_2}(\mu_i) = \nu_{\eta_2}(\lambda)/\ell$ *for all* $i \geqslant 1$;
(4) $\alpha \cdot (\mu_i) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$.

Proof. Since η_1 is of type 2 and η_2 is of type 5 or 6, we have $\lambda = w\pi_{\eta_1}^{r_1}\pi_{\eta_2}^{r_2\ell}$ with r_1 coprime to ℓ and $r_2\alpha \otimes E_{\eta_2} = 0$. Hence, by Lemma 6.7, there exists $\theta \in F_P$ such that $\alpha \cdot (\theta) = 0$, $\nu_{\eta_1}(\theta) = 0$ and $\nu_{\eta_2}(\theta) = r_2$. For $i \geqslant 2$, let $\mu_i = \theta$ and $\mu_1 = -\lambda\theta^{1-\ell}$. Then the μ_i have the required properties. □

Lemma 9.5. *Let $P \in \mathscr{P}$, and η_1 and η_2 be codimension zero points of X_0 containing P. Suppose that η_1 and η_2 are of type 5 or 6. Then there exist $\mu_i \in F_P$, $1 \leqslant i \leqslant \ell$, such that:*

(1) $\mu_1 \cdots \mu_\ell = -\lambda$;
(2) $\nu_{\eta_j}(\mu_i) = \nu_{\eta_j}(\lambda)/\ell$ *for all* $i \geqslant 0$ *and* $j = 1, 2$;
(3) $\alpha \cdot (\mu_i) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$.

Proof. Since η_1 and η_2 are of type 5 or 6, by Lemma 6.8, there exists $\theta \in F_P$ such that $\alpha \cdot (\theta) = 0$ and $\nu_{\eta_i}(\theta) = \nu_{\eta_i}(\lambda)/\ell$ for $i = 1, 2$. For $i \geqslant 2$, let $\mu_i = \theta \in F_P$ and $\mu_1 = -\lambda\theta^{1-\ell} \in F_P$. Then the μ_i have the required properties. □

Lemma 9.6. *Let $P \in \mathscr{P}$, η_1 be a codimension zero point of X_0 of type 3 and η_2 a codimension zero point of X_0 of type 5. Suppose η_1 and η_2 intersect at P. Then there exist a cyclic field extension L_P/F_P of degree ℓ and $\mu_P \in L_P$ such that:*

(1) $N_{L_P/F_P}(\mu_P) = -\lambda$;
(2) $\operatorname{ind}(\alpha \otimes L_P) < \operatorname{ind}(\alpha)$;
(3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$;
(4) $L_P \otimes F_{P,\eta_i}/F_{P,\eta_i}$ *is an unramified field extension for* $i = 1, 2$;
(5) *if* $\lambda \in F_P^{*\ell}$ *or* $-\lambda \in F_P^{*\ell}$, *then* $\operatorname{ind}(\alpha \otimes (E_{\eta_1} \otimes F_{P,\eta_1}) \otimes (L_P \otimes F_{P,\eta_1})) < \operatorname{ind}(\alpha \otimes E_{\eta_1})$;
(6) *if* η_2 *is of type 5b, then* $L_P \otimes F_{P,\eta_2} \simeq M_{\eta_2} \otimes F_{P,\eta_2}$.

Proof. Suppose $\lambda \notin \pm F_P^{*\ell}$. Let $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$. Then $N_{L_P/F_P}(\mu_P) = -\lambda$ and, by Lemma 6.2, (2) and (3) are satisfied. Since η_i is of type 3 or 5, $\nu_{\eta_i}(\lambda)$ is divisible by ℓ and hence (4) is satisfied. Since $\lambda \notin F_P^{*\ell}$, case (5) does not arise. Suppose that η_2 is of type 5b. Since $\mathscr{X}$ has no special points, $M_{\eta_2} \otimes F_{P,\eta_2}$ is a field. Since $-\lambda$ is a norm from M_{η_2} (Lemma 9.1), by Lemma 2.6, we have $L_P \otimes F_{P,\eta_2} \simeq M_{\eta_2} \otimes F_{P,\eta_2}$.

Suppose that $\lambda \in F_P^{*\ell}$ or $-\lambda \in F_P^{*\ell}$. Let L_{P,η_1} and $\mu_{P,\eta_1} \in L_{P,\eta_1}$ be as in Lemma 9.3. Write $\alpha \otimes F_{\eta_1} = \alpha_1 + (E_{\eta_1}, \sigma_1, \pi_{\eta_1})$ as in Lemma 4.1. Then, by Lemma 4.2, we have $\operatorname{ind}(\alpha \otimes F_{\eta_1}) = \operatorname{ind}(\alpha \otimes E_{\eta_1})[E_{\eta_1} : F_{\eta_1}]$. Since η_1 is of type 3, by the choice of L_{P,η_1} (cf. Lemma 9.3), $\operatorname{ind}(\alpha \otimes E_{\eta_1} \otimes L_{P,\eta_1}) < \operatorname{ind}(\alpha \otimes E_{\eta_1})$. We have $\operatorname{ind}(\alpha \otimes L_{P,\eta_1}) \leqslant \operatorname{ind}(\alpha \otimes E_{\eta_1} \otimes L_{P,\eta_1})[E_{\eta_1} \otimes L_{P,\eta_1} : L_{P,\eta_1}] < \operatorname{ind}(\alpha \otimes E_{\eta_1})[E_{\eta_1} : F_{\eta_1}] = \operatorname{ind}(\alpha)$.

Suppose that η_2 is of type 5a. Let L_{P,η_2} and $\mu_{P,\eta_2} \in L_{P,\eta_2}$ be as in Lemma 9.3. Since η_2 is of type 5a, α is unramified at η_2. Since $L_{P,\eta_2}/F_{P,\eta_2}$ is an unramified field extension, $\operatorname{ind}(\alpha \otimes L_{P,\eta_2}) < \operatorname{ind}(\alpha)$.

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Suppose η_2 is of type 5b. Since $\mathscr{X}$ has no special points, $M_{\eta_2} \otimes F_{P,\eta_2}$ is a field. Let $L_{P,\eta_2} = M_{\eta_2} \otimes F_{P,\eta_2}$. Then, by Lemma 9.1, there exists $\mu_{P,\eta_2} \in L_{P,\eta_2}$ such that $N_{L_{P,\eta_2}/F_{P,\eta_2}}(\mu_{P,\eta_2}) = -\lambda$, $\operatorname{ind}(\alpha \otimes L_{P,\eta_2}) < \operatorname{ind}(\alpha)$ and $\alpha \cdot (\mu_{P,\eta_2}) = 0$.

Then, by Lemma 6.4, there exist L_P and μ_P with the required properties. □

LEMMA 9.7. *Let $P \in \mathscr{P}$, and η_1 and η_2 be codimension zero points of X_0 of type 3, 4 or 6. Suppose η_1 and η_2 intersect at P. Then there exist a cyclic field extension L_P/F_P of degree ℓ and $\mu_P \in L_P$ such that:*

(1) $N_{L_P/F_P}(\mu_P) = -\lambda$;
(2) $\operatorname{ind}(\alpha \otimes L_P) < \operatorname{ind}(\alpha)$;
(3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$;
(4) $L_P \otimes F_{P,\eta_i}/F_{P,\eta_i}$ *is an unramified field extension;*
(5) *if η_i is of type 3, $\lambda \in F_P^{*\ell}$ or $-\lambda \in F_P^{*\ell}$, then* $\operatorname{ind}(\alpha \otimes (E_{\eta_i} \otimes F_{P,\eta_i}) \otimes (L_P \otimes F_{P,\eta_i})) < \operatorname{ind}(\alpha \otimes E_{\eta_i})$.

Proof. Suppose $\lambda \notin \pm F_P^{*\ell}$. Then, as in the proof of Lemma 9.6, $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$ have the required properties.

Suppose that $\lambda \in F_P^{*\ell}$ or $-\lambda \in F_P^{*\ell}$. For $i = 1, 2$, let L_{P,η_i} and $\mu_{P,\eta_i} \in L_{P,\eta_i}$ be as in Lemma 9.3. If η_i is of type 3, then as in the proof of Lemma 9.6, $\operatorname{ind}(\alpha \otimes L_{P,\eta_i}) < \operatorname{ind}(\alpha)$. Suppose η_i is of type 4 or 6. Then $\operatorname{ind}(\alpha \otimes F_{\eta_i}) < \operatorname{ind}(\alpha)$ and hence $\operatorname{ind}(\alpha \otimes L_{P,\eta_i}) < \operatorname{ind}(\alpha)$.

Then, by Lemma 6.4, there exist L_P and μ_P with the required properties. □

PROPOSITION 9.8. *Let $P \in \mathscr{P}$. Then there exist a cyclic field extension or split extension L_P/F_P of degree ℓ and $\mu_P \in L_P$ such that:*

(1) $N_{L_P/F_P}(\mu_P) = -\lambda$;
(2) $\operatorname{ind}(\alpha \otimes L_P) < \operatorname{ind}(\alpha)$;
(3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$.

Further, suppose η is a codimension zero point of X_0 containing P.

(4) *If η is of type 1, then $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$.*

(5) *Suppose η is of type 2 with a type 2 connection to a type 5 point η'. Let Q be the type 2 intersection point of η and η'. If $M_{\eta'} \otimes F_{Q,\eta'}$ is not a field, then $L_P = \prod F_P$ and $\mu_P = (\theta_1, \ldots, \theta_\ell)$ with $\theta_i \in F_P$, $\nu_\eta(\theta_1) = \nu_\eta(\lambda)$ and $\nu_\eta(\theta_i) = 0$ for $i \geqslant 2$.*

(6) *Suppose η is of type 2 with a type 2 connection to a type 5 point η'. Let Q be the type 2 intersection point of η and η'. If $M_{\eta'} \otimes F_{Q,\eta'}$ is a field, then $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$.*

(7) *Suppose η is of type 2 and there is no type 2 connection from η to any type 5 point. Then $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$.*

(8) *If η is of type 3, then $L_P \otimes F_{P,\eta}/F_{P,\eta}$ is an unramified field extension. Further, if $\lambda \in F_P^{*\ell}$ or $-\lambda \in F_P^{*\ell}$, then $\operatorname{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_P \otimes F_{P,\eta})) < \operatorname{ind}(\alpha \otimes E_\eta)$.*

(9) *If η is of type 4, then $L_P \otimes F_{P,\eta}/F_{P,\eta}$ is an unramified field extension.*

(10) *If η is of type 5a, then $L_P \otimes F_{P,\eta}/F_{P,\eta}$ is an unramified field extension.*

(11) *If η is of type 5b, then $L_P \otimes F_{P,\eta} \simeq M_\eta \otimes F_{P,\eta}$, and if $L_P = \prod F_P$, then $\mu_P = (\theta_1, \ldots, \theta_\ell)$ with $\nu_\eta(\theta_i) = \nu_\eta(\lambda)/\ell$.*

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(12) *If η is of type 6, then either $L_P \otimes F_{P,\eta}/F_{P,\eta}$ is an unramified field extension or $L_P = \prod F_P$, with $\mu_P = (\theta_1, \ldots, \theta_\ell)$ and $\nu_\eta(\theta_i) = \nu_\eta(\lambda)/\ell$.*

Proof. Let η_1 and η_2 be two codimension zero points of X_0 intersecting at P. By the choice of $\mathscr{X}$, X_0 is a union of regular curves with normal crossings and hence there are no other codimension zero points of X_0 passing through P.

Case I. Suppose that either η_1 or η_2, say η_1, is of type 1. Then $\nu_{\eta_1}(\lambda)$ is coprime to ℓ and hence $\lambda \notin \pm F_P^{*\ell}$. Let $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$. Then, by Lemma 6.2, L_P and μ_P satisfy (1), (2) and (3). By choice (4) is satisfied. Since $\mathscr{X}$ has no special points, η_2 is not of type 2 or 4. Thus (5), (6), (7) and (9) do not arise. Suppose η_2 is of type 3, 5 or 6. Then $\nu_{\eta_2}(\lambda)$ is divisible by ℓ and hence $L_P \otimes F_{P,\eta_2}/F_{P,\eta_2}$ is an unramified field extension. Thus (8), (10) and (12) are satisfied. Suppose η_2 is of type 5b. Since $\mathscr{X}$ has no special points and η_1 is of type 1, $M_{\eta_2} \otimes F_{P,\eta_2}$ is a field. Since $-\lambda$ is a norm from the extension M_{η_2}/F_{η_2} (Lemma 9.1) and $\lambda \notin \pm F_{P,\eta_2}^{*\ell}$ (Corollary 5.6), by (Lemma 2.6), $M_{\eta_2} \otimes F_{P,\eta_2} \simeq F_{P,\eta_2}(\sqrt[\ell]{\lambda})$ and hence (11) is satisfied.

Case II. Suppose neither η_1 nor η_2 is of type 1. Suppose either η_1 or η_2 is of type 2, say η_1 is of type 2. Then $\nu_{\eta_1}(\lambda)$ is coprime to ℓ and hence $\lambda \notin \pm F_P^{*\ell}$.

Suppose that η_1 has type 2 connection to a codimension zero point η' of X_0 of type 5. Let Q be the closed point on η' which is the type 2 intersection point of η_1 and η'. By the choice of $\mathscr{X}$ (cf. Proposition 8.6), η_2 is of type 2, 5 or 6. Note that if η_2 is also of type 2, then Q is also the point of type 2 intersection of η_2 and η'. Thus if both η_1 and η_2 are of type 2, η' and Q do not depend on whether we start with η_1 or η_2.

Suppose that $M_{\eta'} \otimes F_{Q,\eta'}$ is not a field. Let $L_P = \prod F_P$. Suppose η_2 is of type 2. Then let $\mu_P = (\lambda, 1, \ldots, 1) \in L_P = \prod F_P$. Suppose η_2 is of type 5. Then by the assumption on $\mathscr{X}$, $\eta_2 = \eta'$, $Q = P$. Thus $M_{\eta_2} \otimes F_{P,\eta_2} = M_{\eta'} \otimes F_{Q,\eta'}$ is not a field and hence η_2 is of type 5b. Let $\mu_i \in F_P$ be as in Lemma 9.4, and $\mu_P = (\mu_1, \ldots, \mu_\ell)$. Suppose η_2 is of type 6. Let $\mu_i \in F_P$ be as in Lemma 9.4, and $\mu_P = (\mu_1, \ldots, \mu_\ell) \in L_P$. Then L_P and μ_P satisfy (1) and (3). Since η_1 is of type 2, $\mathrm{ind}(\alpha \otimes F_{\eta_1}) < \mathrm{ind}(\alpha)$ and hence, by Proposition 5.8, $\mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$ and (2) is satisfied. Since neither η_1 nor η_2 is of type 1, case (4) does not arise. By choice L_P satisfies (5). Since there is only one type 5 point with a type 2 connection to η_1 or η_2, case (6) does not arise. Clearly case (7) does not arise. Since η_2 is not of type 3, 4 or 5a, cases (8), (9) and (10) do not arise. By the choice of L_P and μ_P, (11) and (12) are satisfied.

Suppose $M_{\eta'} \otimes F_{Q,\eta'}$ is a field. Let $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$. Since $\lambda \notin F_P^{*\ell}$, by Lemma 6.2, L_P and μ_P satisfy (1), (2) and (3). As above, cases (4), (5), (7), (8) and (9) do not arise. By choice (6) is satisfied. Suppose η_2 is of type 5. Then $\eta_2 = \eta'$, $Q = P$ and $\nu_{\eta_2}(\lambda)$ is divisible by ℓ and hence (10) is satisfied. Suppose η_2 is of type 5b. Since $M_{\eta_2} \otimes F_{P,\eta_2}$ is a field, as in case I, $M_{\eta_2} \otimes F_{P,\eta_2} \simeq L_P \otimes F_{P,\eta_2}$ and hence (11) is satisfied. If η_2 is of type 6, then $\nu_{\eta_2}(\lambda)$ is divisible by ℓ and $L_P \otimes F_{P,\eta_2}/F_{P,\eta_2}$ is an unramified field extension and hence (12) is satisfied.

Suppose that η_1 has no type 2 connection to a point of type 5. In particular, η_2 is not of type 5. Then, let $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$. Then, by Lemma 6.2, L_P and μ_P satisfy (1), (2) and (3). Since neither η_1 nor η_2 is of type 1, case (4) does not arise. Since neither η_1 nor η_2 has type 2 connection to a point of type 5, (5) and (6) do not arise. By the choice of L_P and μ_P, (7) is satisfied. If η_2 is of type 3, 4 or 6, then $\nu_{\eta_2}(\lambda)$ is divisible by ℓ and (8), (9) and (12) are satisfied. Since neither η_1 nor η_2 is of type 5, (10) and (11) do not arise.

Case III. Suppose neither of η_i is of type 1 or 2. Suppose that one of the η_i, say η_1, is of type 3. Since $\mathscr{X}$ has no special points, η_2 is not of type 4 and hence η_2 is of type 3, 5 or 6. If η_2 is of type 5, let L_P and μ_P be as in Lemma 9.6. If η_2 is of type 3 or 6, let L_P and μ_P be as

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in Lemma 9.7. Then, (1), (2), (3), (8), (9), (10), (11) and (12) are satisfied and the other cases do not arise.

Case IV. Suppose neither of η_i is of type 1, 2 or 3. Suppose that one of the η_i, say η_1, is of type 4. Since $\mathscr{X}$ has no special points, η_2 is not of type 5. Hence η_2 is of type 4 or 6. Let L_P and μ_P be as in Lemma 9.7. Then L_P and μ_P have the required properties.

Case V. Suppose neither of η_i is of type 1, 2, 3 or 4. Suppose that one of the η_i is of type 5, say η_1 is of type 5. Then η_2 is of type 5 or 6. Suppose that η_2 is of type 5. Since $\mathscr{X}$ has no special points, $M_{\eta_i} \otimes F_{P,\eta_i}$ are fields for $i = 1, 2$. Let L_P and μ_P be as in Lemma 9.2. Then L_P and μ_P have the required properties.

Suppose that η_2 is of type 6. Suppose that η_1 is of type 5a. Let L_{P,η_i} and μ_{P,η_i} be as in Lemma 4.10. Since $\nu_i(\lambda)$ is divisible by ℓ, by the construction of L_{P,η_i}, $L_{P,\eta_i}/F_{P,\eta_i}$ are unramified. Let $L_P, \mu_P \in L_P$ be as in Lemma 6.4. Then L_P, μ_P have the required properties. Suppose that η_1 is of type 5b. Suppose $M_{\eta_1} \otimes F_{P,\eta_1}$ is a field with the residue field $M(\eta_1)_P$ of $M_{\eta_1} \otimes F_{P,\eta_1}$ unramified over $\kappa(\eta_1)_P$. Let $L_{P,\eta_1} = M_{\eta_1} \otimes F_{P,\eta_1}$ and $\mu_{\eta_1} \in M_{\eta_1}$ with $N_{M_{\eta_1}/F_{\eta_1}}(\mu_{\eta_1}) = -\lambda$ (cf. Lemma 9.1). Let L_P and μ_P be as in Lemma 6.5 with $L_P \otimes F_{P,\eta_1} \simeq L_{P,\eta_1}$. Then L_P is a field with L_P/F_P unramified on A_P (cf. Lemma 6.5) and hence L_P and μ_P have the required properties. Suppose that $M_{\eta_1} \otimes F_{P,\eta_1}$ is a field extension and the residue field $M(\eta_1)_P$ of $M_{\eta_1} \otimes F_{P,\eta_1}$ is ramified over $\kappa(\eta_1)_P$. Then $M_{\eta_1} \otimes F_{P,\eta_1} = F_{P,\eta_1}(\sqrt[\ell]{v_P \pi_{\eta_2}})$ for some unit v_P at P (cf. proof of Lemma 6.4). Since $\lambda = w_P \pi_{\eta_1}^{r_1 \ell} \pi_{\eta_2}^{r_2 \ell}$ for some unit w_P at P and $-\lambda$ is a norm from $M_{\eta_1} \otimes F_{P,\eta_1}$, it follows that the image $-\overline{w}_P$ of w_P in $\kappa(\eta_1)_P$ is a norm from $M(\eta_1)_P$. Since w_P is a unit and $M(\eta_1)_P/\kappa(\eta_1)_P$ is a ramified extension, it follows that $-w_P \in F_{P,\eta_1}^{\ell}$ and hence $-w_P \in F_P^{*\ell}$. Let $L_P = F_P(\sqrt[\ell]{v_P \pi_{\eta_2} + \pi_{\eta_1}})$ and $\mu_P = \sqrt[\ell]{-\lambda} \in F_P$. Then $N_{L_P/F_P}(\mu_P) = -\lambda$. Since η_2 is of type 6, $\mathrm{ind}(\alpha \otimes F_{\eta_2}) < \mathrm{ind}(\alpha)$ and hence, by Proposition 5.8, $\mathrm{ind}(\alpha \otimes F_P) < \mathrm{ind}(\alpha)$. In particular, $\mathrm{ind}(\alpha \otimes L_P) < \mathrm{ind}(\alpha)$. Let B_P be the integral closure of the local ring A_P at P in L_P. Since the maximal ideal m_P at P is equal to $(\pi_{\eta_1}, \pi_{\eta_2})$, $v_P \pi_{\eta_2} + \pi_{\eta_1}$ is a regular prime and hence B_P is a regular local ring. Since $\mathrm{cor}_{L_P \otimes F_{P,\eta_i}/F_{P,\eta_i}}(\alpha \cdot (\mu_P)) = \alpha \cdot (-\lambda) = 0$ and $L_{P,\eta_i}/F_{P,\eta_i}$ is a field extension, by Proposition 4.6, $\alpha \cdot (\mu_P) = 0$ in $H^3(L_P \otimes F_{P,\eta_i}, \mu_n^{\otimes 2})$ for $i = 1, 2$. In particular, $\alpha \cdot (\mu_P)$ is unramified on B_P and hence $\alpha \cdot (\mu_P) = 0$ (cf. Lemma 5.3). Thus L_P and μ_P satisfy the required properties.

Suppose that $M_{\eta_1} \otimes F_{P,\eta_1}$ is not a field. Let $L_P = \prod F_P$ and $\mu_i \in F_P$ be as in Lemma 9.5, and $\mu_P = (\mu_1, \dots, \mu_\ell) \in L_P$. Then L_P and μ_P have the required properties.

Case VI. Suppose neither of η_i is of type 1, 2, 3, 4 or 5. Then, η_1 and η_2 are of type 6. Let L_P and μ_P be as in Lemma 9.7. Then L_P and μ_P have the required properties. □

10. Choice of L_η and μ_η at codimension zero points

Let F, $n = \ell^d$, $\alpha \in H^2(F, \mu_n)$, $\lambda \in F^*$ with $\alpha \neq 0$, $\alpha \cdot (-\lambda) = 0 \in H^3(F, \mu_n^{\otimes 2})$, $\mathscr{X}$, X_0 and $\mathscr{P}$ be as in §§ 7– 9). Assume that $\mathscr{X}$ has no special points and that there is no type 2 connection between a codimension zero point of X_0 of type 3 or 5 and a codimension zero point of X_0 of type 3, 4 or 5.

For a codimension zero point η of X_0, let $\mathscr{P}_\eta = \eta \cap \mathscr{P}$.

Proposition 10.1. *Let η be a codimension zero point of X_0 of type 1. For each $P \in \mathscr{P}_\eta$, let (L_P, μ_P) be chosen as in Proposition 9.8, and $L_\eta = F_\eta(\sqrt[\ell]{\lambda})$ and $\mu_\eta = -\sqrt[\ell]{\lambda} \in L_\eta$. Then:*

(1) $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$;
(2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$;

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(3) $\mathrm{ind}(\alpha \otimes L_\eta) < \mathrm{ind}(\alpha)$;

(4) *for* $P \in \mathscr{P}_\eta$, *there is an isomorphism* $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ *and*

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} = 1.$$

Proof. By choice, we have $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$. Since η is of type 1, $\nu_\eta(\lambda)$ is coprime to ℓ and hence by Lemma 4.7, L_η and μ_η satisfy (2) and (3). Let $P \in \mathscr{P}_\eta$. Since η is of type 1, by the choice of L_P and μ_P (cf. Proposition 9.8(4)), we have $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$. Hence L_η and μ_η satisfy (4). □

Lemma 10.2. *Let η be a codimension zero point of X_0. For each $P \in \mathscr{P}_\eta$, let $\theta_P \in F_P$ with $\alpha \cdot (\theta_P) = 0 \in H^3(F_{P,\eta}, \mu_n^{\otimes 2})$. Suppose $\nu_\eta(\theta_P) = 0$ for all $P \in \mathscr{P}_\eta$. Then there exists $\theta_\eta \in F_\eta$ such that:*

(1) $\alpha \cdot (\theta_\eta) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$;

(2) *for* $P \in \mathscr{P}_\eta$, $\theta_P^{-1}\theta_\eta \in F_{P,\eta}^{\ell^m}$ *for all* $m \geqslant 1$.

Proof. Let $\pi_\eta \in F_\eta$ be a parameter. Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)$ as in Lemma 4.1. Let $E(\eta)$ be the residue field of E_η. Since $\alpha \cdot (\theta_P) = 0 \in H^3(F_{P,\eta}, \mu_n^{\otimes 2})$ and $\nu_\eta(\theta_P) = 0$, by Lemma 4.7, we have $(E(\eta) \otimes \kappa(\eta)_P, \sigma_0, \overline{\theta}_P) = 0 \in H^2(\kappa(\eta)_P, \mu_n)$, where $\overline{\theta}_P$ is the image of $\theta_P \in \kappa(\eta)_P$. Hence $\overline{\theta}_P$ is a norm from $E(\eta) \otimes \kappa(\eta)_P$ for all $P \in \mathscr{P}_\eta$. For $P \in \mathscr{P}_\eta$, let $\tilde{\theta}_P \in E(\eta) \otimes \kappa(\eta)_P$ with $N_{E(\eta)\otimes\kappa(\eta)_P/\kappa(\eta)_P}(\tilde{\theta}_P) = \overline{\theta}_P$. By weak approximation, there exists $\tilde{\theta} \in E(\eta) \otimes \kappa(\eta)$ which is sufficiently close to $\tilde{\theta}_P$ for all $P \in \mathscr{P}_\eta$. Let $\theta_0 = N_{E(\eta)/\kappa(\eta)}(\tilde{\theta}) \in \kappa(\eta)$. Then θ_0 is sufficiently close to $\overline{\theta}_P$ for all $P \in \mathscr{P}_\eta$. In particular, $\theta_0^{-1}\overline{\theta}_P \in \kappa(\eta)_P^{\ell^m}$ for all $m \geqslant 1$. Let $\theta_\eta \in F_\eta$ have image θ_0 in $\kappa(\eta)$. Then $(E_\eta, \sigma_\eta, \theta_\eta) = 0$ and hence, by Lemma 4.7, $\alpha \cdot (\theta_\eta) = 0$. Since $\theta_0^{-1}\overline{\theta}_P \in \kappa(\eta)_P^{\ell^m}$ for all $m \geqslant 1$ and $F_{P,\eta}$ is a complete discretely valued field with residue field $\kappa(\eta)_P$, it follows that $\theta_\eta^{-1}\theta_P \in F_{P,\eta}^{\ell^m}$ for all $m \geqslant 1$. □

Proposition 10.3. *Let η be a codimension zero point of X_0 of type 2. Suppose there is a type 2 connection between η and a codimension zero point η' of X_0 of type 5. Let Q be the point of type 2 intersection of η and η'. Suppose that $M_{\eta'} \otimes F_{Q,\eta'}$ is not a field. For each $P \in \mathscr{P}_\eta$, let $\mu_P = (\theta_1^P, \ldots, \theta_\ell^P) \in L_P = \prod F_P$ be as in Proposition 9.8(5). Let $L_\eta = \prod F_\eta$. Then there exists $\mu_\eta = (\theta_1^\eta, \ldots, \theta_\ell^\eta) \in L_\eta$ such that:*

(1) $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$;

(2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$;

(3) $\mathrm{ind}(\alpha \otimes L_\eta) < \mathrm{ind}(\alpha)$;

(4) $\mu_P^{-1}\mu_\eta \in (L_\eta \otimes F_{P,\eta})^{\ell^m}$ *for all* $P \in \mathscr{P}_\eta$ *and* $m \geqslant 1$.

Proof. Let $i \geqslant 2$. By choice (cf. Proposition 9.8(5)), we have $\nu_\eta(\theta_i^P) = 0$ and $\alpha \cdot (\theta_i^P) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$ for all $P \in \mathscr{P}_\eta$. By Lemma 10.2, there exists $\theta_i^\eta \in F_\eta$ such that $\alpha \cdot (\theta_i^\eta) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$ and $(\theta_i^P)^{-1}\theta_i^\eta \in F_{P,\eta}^{\ell^m}$ for all $P \in \mathscr{P}_\eta$ and $m \geqslant 1$. Let $\theta_1^\eta = -\lambda(\theta_2^\eta \cdots \theta_\ell^\eta)^{-1}$. Then $\theta_1^\eta \cdots \theta_\ell^\eta = -\lambda$ and $(\theta_1^P)^{-1}\theta_1^\eta \in F_{P,\eta}^{\ell^m}$ for all $m \geqslant 1$. Since $\alpha\cdot(-\lambda) = 0$ and $\alpha\cdot(\theta_i^\eta) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$ for $i \geqslant 2$, we have $\alpha \cdot (\theta_1^\eta) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$. Let $L_\eta = \prod F_\eta$ and $\mu_\eta = (\theta_1^\eta, \ldots, \theta_\ell^\eta) \in L_\eta$. Since η is of type 2, $\mathrm{ind}(\alpha \otimes F_\eta) < \mathrm{ind}(\alpha)$ and hence L_η, μ_η have the required properties. □

Proposition 10.4. *Let η be a codimension zero point of X_0 of type 2. For each $P \in \mathscr{P}_\eta$, let (L_P, μ_P) be chosen as in Proposition 9.8. Suppose one of the following holds:*

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- *there is a type 2 connection between η and codimension zero point η' of X_0 of type 5 with Q the point of type 2 intersection of η and η', and $M_{\eta'} \otimes F_{Q,\eta'}$ is a field;*
- *there is no type 2 connection between η and any codimension zero point of X_0 of type 5.*

Let $L_\eta = F_\eta(\sqrt[\ell]{\lambda})$ and $\mu_\eta = -\sqrt[\ell]{\lambda}$. Then:

(1) $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$;
(2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$;
(3) $\mathrm{ind}(\alpha \otimes L_\eta) < \mathrm{ind}(\alpha)$;
(4) *for $P \in \mathscr{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and*

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} = 1.$$

Proof. Since $\nu_\eta(\lambda)$ is coprime to ℓ, by Lemma 4.7, $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$ and $\mathrm{ind}(\alpha \otimes L_\eta) < \mathrm{ind}(\alpha)$. Clearly, $N_{L_\eta/F_\eta}(\mu_\eta) = \lambda$. By the choice of (L_P, μ_P) (cf. Proposition 9.8), for $P \in \mathscr{P}_\eta$, we have $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$. Thus L_η and μ_η have the required properties. □

Lemma 10.5. *Let η be a codimension zero point of X_0 of type 3, 4 or 5a. Let $P \in \eta$. Suppose there exists $L_{P,\eta}/F_{P,\eta}$ an unramified field extension of degree ℓ and $\mu_{P,\eta} \in L_{P,\eta}$ such that:*

(1) $N_{L_{P,\eta}/F_{P,\eta}}(\mu_{P,\eta}) = -\lambda$;
(2) $\mathrm{ind}(\alpha \otimes L_{P,\eta}) < \mathrm{ind}(\alpha)$;
(3) $\alpha \cdot (\mu_{P,\eta}) = 0 \in H^3(L_{P,\eta}, \mu_n^{\otimes 2})$;
(4) *if η is of type 3, $\lambda \in F_P^{*\ell}$ or $-\lambda \in F_P^{*\ell}$, then $\mathrm{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) < \mathrm{ind}(\alpha \otimes E_\eta)$.*

Then $\mathrm{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) < \mathrm{ind}(\alpha)/[E_\eta : F_\eta]$.

Proof. Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)$ as in Lemma 4.1. Then, by Lemma 4.2, $\mathrm{ind}(\alpha \otimes F_\eta) = \mathrm{ind}(\alpha' \otimes E_\eta)[E_\eta : F_\eta] = \mathrm{ind}(\alpha \otimes E_\eta)[E_\eta : F_\eta]$. Let $t = [E_\eta : F_\eta]$ and β be the image of α' in $H^2(\kappa(\eta), \mu_n)$.

Suppose η is of type 4. Then $\mathrm{ind}(\alpha \otimes F_\eta) < \mathrm{ind}(\alpha)$ and hence $\mathrm{ind}(\alpha \otimes E_\eta) = \mathrm{ind}(\alpha \otimes F_\eta)/t < \mathrm{ind}(\alpha)/t$. Thus $\mathrm{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) \leqslant \mathrm{ind}(\alpha \otimes E_\eta) < \mathrm{ind}(\alpha)/t$.

Suppose that η is of type 5a. Then α is unramified at η and hence $E_\eta = F_\eta$ and $t = 1$. The lemma is clear if $\alpha \otimes F_{P,\eta} = 0$. Suppose $\alpha \otimes F_{P,\eta} \neq 0$. Then $\beta \otimes \kappa(\eta)_P \neq 0$. Since $L_{P,\eta}$ is an unramified field extension, the residue field $L_P(\eta)$ of $L_{P,\eta}$ is a field extension of $\kappa(\eta)_P$ of degree ℓ. Since $\kappa(\eta)_P$ is a local field and $\mathrm{ind}(\beta)$ is divisible by ℓ, $\mathrm{ind}(\beta \otimes L_P(\eta)) < \mathrm{ind}(\beta)$ [CF67, p. 131]. In particular, $\mathrm{ind}(\alpha \otimes L_{P,\eta}) < \mathrm{ind}(\alpha)$.

Suppose that η is of type 3. Then $r\alpha \otimes E_\eta \neq 0$ and hence $r\alpha' \otimes E_\eta = r\alpha \otimes E_\eta \neq 0$. In particular, $r\beta \otimes E(\eta) \neq 0$ and $\mathrm{ind}(\alpha \otimes F_\eta) > t$. Suppose $\lambda \in F_P^{*\ell}$ or $-\lambda \in F_P^{*\ell}$. Then, by the choice of $L_{P,\eta}$, $\mathrm{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) < \mathrm{ind}(\alpha \otimes E_\eta) = \mathrm{ind}(\alpha)/t$. Suppose $\lambda \notin \pm F_P^{*\ell}$. Then $\lambda \notin \pm F_{P,\eta}^{*\ell}$. Since $L_{P,\eta}$ is a field extension of degree ℓ and $-\lambda$ is a norm from $L_{P,\eta}$, by Lemma 2.6, $L_{P,\eta} \simeq F_{P,\eta}(\sqrt[\ell]{\lambda})$. Since η is of type 3, $\nu_\eta(\lambda) = r\ell$ and $\lambda = \theta_\eta \pi_\eta^{r\ell}$ with $\theta_\eta \in F_\eta$ a unit at η. Let $\overline{\theta}_\eta$ be the image of θ_η in $\kappa(\eta)$. Then $\overline{\theta}_\eta \notin \kappa(\eta)_P^\ell$ and $L_P(\eta) = \kappa(\eta)_P(\sqrt[\ell]{\overline{\theta}_\eta})$. Since $\alpha \cdot (-\lambda) = 0$, by Lemma 4.7, $r\ell\alpha' = (E_\eta, \sigma_\eta, (-1)^{r\ell+1}\theta_\eta)$ and hence $r\ell\beta = (E(\eta), \sigma_0, (-1)^{r\ell+1}\overline{\theta}_\eta)$. Since $-\overline{\theta}_\eta$ is a norm from $L_P(\eta)$ and $L_P(\eta)/\kappa(\eta)_P$ is an extension of degree ℓ, $(-1)^{r\ell+1}\overline{\theta}_\eta$ is a norm from $L_P(\eta)$. Thus, by Lemma 3.3, $\mathrm{ind}(\beta \otimes E(\eta)_P \otimes L_P(\eta)) < \mathrm{ind}(\beta \otimes E(\eta))$. Thus

$$\begin{aligned}\mathrm{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta})) &= \mathrm{ind}(\alpha' \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta}))\\ &= \mathrm{ind}(\beta \otimes E(\eta)_P \otimes L_P(\eta))\end{aligned}$$

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$$< \operatorname{ind}(\beta \otimes E(\eta)) = \operatorname{ind}(\alpha' \otimes E_\eta)$$
$$= \operatorname{ind}(\alpha \otimes E_\eta) = \operatorname{ind}(\alpha)/t. \qquad \square$$

Proposition 10.6. *Let η be a codimension zero point of X_0 of type 3, 4 or 5a. For each $P \in \mathscr{P}_\eta$, let (L_P, μ_P) be chosen as in Proposition 9.8. Then there exist an unramified field extension L_η/F_η of degree ℓ and $\mu_\eta \in L_\eta$ such that:*

(1) $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$;
(2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$;
(3) $\operatorname{ind}(\alpha \otimes L_\eta) < \operatorname{ind}(\alpha)$;
(4) *for $P \in \mathscr{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and*

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell m},$$

for all $m \geqslant 1$.

Proof. Since η is of type 3, 4 or 5a, we have $\nu_\eta(\lambda) = r\ell$ for some integer r and $\lambda = \theta_\eta \pi_\eta^{r\ell}$ for some parameter π_η at η and $\theta_\eta \in F_\eta$ a unit at η. Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)$ as in Lemma 4.1. By Lemma 4.7, $r\ell\alpha' = (E_\eta, \sigma_\eta, (-1)^{r\ell+1}\theta_\eta)$. Let β be the image of α' in $H^2(\kappa(\eta), \mu_n)$ and $E(\eta)$ the residue field of E_η. Then $r\ell\beta = (E(\eta), \sigma_0, (-1)^{r\ell+1}\theta_0) \in H^2(\kappa(\eta), \mu_n)$, where σ_0 is the automorphism of $E(\eta)$ induced by σ_η and θ_0 is the image of θ_η in $\kappa(\eta)$.

Let S be a finite set of places of $\kappa(\eta)$ containing the places given by closed points of $\mathscr{P}_\eta$ and places ν of $\kappa(\eta)$ with $\beta \otimes \kappa(\eta)_\nu \neq 0$. Let $t = [E_\eta : F_\eta]$. For each $\nu \in S$, we now give a field extension $L_\nu/\kappa(\eta)_\nu$ of degree ℓ and $\mu_\nu \in L_\nu$ satisfying the conditions of Lemma 3.1 with $E_0 = E(\eta)$ and $d = \operatorname{ind}(\alpha)/t$.

Let $\nu \in S$. Then ν is given by a closed point P of η. If $P \in \mathscr{P}$, let $L_{P,\eta} = L_P \otimes F_{P,\eta}$ and $\mu_{P,\eta} = \mu_P \otimes 1 \in L_{P,\eta}$. Suppose that $P \notin \mathscr{P}$. Suppose that $\lambda \notin \pm F_P^{*\ell}$. Then $\lambda \notin \pm F_{P,\eta}^{*\ell}$. Let $L_{P,\eta} = F_{P,\eta}(\sqrt[\ell]{\lambda})$ and $\mu_{P,\eta} = -\sqrt[\ell]{\lambda}$. Suppose that $\lambda \in F_P^{*\ell}$ or $-\lambda \in F_P^{*\ell}$. Let $L_{P,\eta}/F_{P,\eta}$ be a cyclic unramified field extension of degree ℓ and $\mu_{P,\eta} \in L_{P,\eta}$ as in Lemma 9.3. Since $L_{P,\eta}/F_{P,\eta}$ is an unramified field extension of degree ℓ, π_η is a parameter in $L_{P,\eta}$ and the residue field $L_P(\eta)$ is a field extension of $\kappa(\eta)_P$ of degree ℓ. Let $L_\nu = L_P(\eta)$. Since $N_{L_{P,\eta}/F_{P,\eta}}(\mu_{P,\eta}) = -\lambda$, $\mu_{P,\eta} = \theta_{P,\eta}\pi_\eta^r$ for some $\theta_{P,\eta} \in L_{P,\eta}$ which is a unit at η. Let μ_ν be the image of $\theta_{P,\eta}$ in $L_\nu = L_P(\eta)$. Then $N_{L_\nu/\kappa(\eta)_\nu}(\mu_\nu) = -\theta_0$. Since the corestriction map $H^2(L_\nu, \mu_n) \to H^2(\kappa(\eta)_\nu, \mu_n)$ is injective, $r\beta \otimes L_\nu = (E_0 \otimes L_\nu, \sigma_0 \otimes 1, (-1)^r\mu_\nu)$. By Lemma 10.5, we have $\operatorname{ind}(\alpha \otimes (E_\eta \otimes F_{P,\eta}) \otimes L_{P,\eta}) < \operatorname{ind}(\alpha)/t$. Since $\alpha \otimes E_\eta = \alpha' \otimes E_\eta$, we have $\operatorname{ind}(\alpha' \otimes (E_\eta \otimes F_{P,\eta}) \otimes L_{P,\eta}) < \operatorname{ind}(\alpha)/t$. Since $\operatorname{ind}(\beta \otimes E_0 \otimes L_\nu) = \operatorname{ind}(\alpha' \otimes (E_\eta \otimes F_{P,\eta}) \otimes (L_{P,\eta}))$, $\operatorname{ind}(\beta \otimes E_0 \otimes L_\nu) < \operatorname{ind}(\alpha)/t$.

Since $\kappa(\eta)$ is a global field, by Lemma 3.1, there exist a field extension $L_0/\kappa(\eta)$ of degree ℓ and $\mu_0 \in L_0$ such that:

(1) $N_{L_0/k}(\mu_0) = -\theta_0$;
(2) $r\beta \otimes L_0 = (E(\eta) \otimes L_0, \sigma_0 \otimes 1, (-1)^r\mu_0)$;
(3) $\operatorname{ind}(\beta \otimes E(\eta) \otimes L_0) < \operatorname{ind}(\alpha)/t$;
(4) $L_0 \otimes \kappa(\eta)_P \simeq L_P(\eta)$ for all $P \in \mathscr{P}_\eta$;
(5) μ_0 is close to $\overline{\theta}_{P,\eta}$ for all $P \in \mathscr{P}_\eta$.

Then, by Lemma 4.8, there exist a field extension L_η/F_η of degree ℓ and $\mu \in L_\eta$ such that:

- the residue field of L_η is L_0;
- μ a unit in the valuation ring of L_η;

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- $\overline{\mu} = \mu_0$;
- $N_{L_\eta/F_\eta}(\mu) = -\theta_\eta$;
- $\alpha \cdot (\mu\pi_\eta^r) \in H^3(L_\eta, \mu_n^{\otimes 2})$ is unramified.

Since L_η is a complete discretely valued field with residue field L_0 a global field, $H^3_{nr}(L_\eta, \mu_n^{\otimes 2}) = 0$ [Ser97, p. 85] and hence $\alpha \cdot (\mu\pi_\eta^r) = 0$. Since L_η/F_η is unramified and $\alpha \otimes L_\eta = \alpha' \otimes L_\eta + (E_\eta \otimes L_\eta, \sigma_\eta, \pi_\eta)$, $\mathrm{ind}(\alpha \otimes L_\eta) \leqslant \mathrm{ind}(\alpha' \otimes E_\eta \otimes L_\eta)[E_\eta \otimes L_\eta : L_\eta] = \mathrm{ind}(\beta \otimes E(\eta) \otimes L_0)t < \mathrm{ind}(\alpha)$. Thus L_η and $\mu_\eta = \mu\pi_\eta^r \in L_\eta$ have the required properties. □

PROPOSITION 10.7. *Let η be a codimension zero point of X_0 of type 5b. Let (E_η, σ_η) be the lift of the residue of α at η and M_η be the unique subfield of E_η with M_η/F_η a cyclic extension of degree ℓ. For each $P \in \mathscr{P}_\eta$, let L_P and μ_P be as in Proposition 9.8. Then there exists $\mu_\eta \in M_\eta$ such that:*

(1) $N_{M_\eta/F_\eta}(\mu_\eta) = -\lambda$;
(2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(M_\eta, \mu_n^{\otimes 2})$;
(3) $\mathrm{ind}(\alpha \otimes M_\eta) < \mathrm{ind}(\alpha)$;
(4) *for $P \in \mathscr{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : M_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and*

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^m},$$

for all $m \geqslant 1$.

Proof. Let $E(\eta)$ and $M(\eta)$ be the residue fields of E_η and M_η at η. Since η is of type 5b, $M(\eta)$ is the unique subfield of $E(\eta)$ with $M(\eta)/\kappa(\eta)$ a cyclic field extension of degree ℓ. Let π_η be a parameter at η. Since η is of type 5, $\nu_\eta(\lambda) = r\ell$ and $\lambda = \theta_\eta\pi_\eta^{r\ell}$ for some $\theta_\eta \in F$ a unit at η. Let $\overline{\theta}_\eta$ be the image of θ_η in $\kappa(\eta)$. Let $P \in \mathscr{P}_\eta$. Suppose $M_\eta \otimes F_{P,\eta}$ is a field. Since $N_{M_\eta\otimes F_{P,\eta}/F_{P,\eta}}(\mu_P) = -\lambda = -\theta_\eta\pi_\eta^{r\ell}$, we have $\mu_P = \mu'_P\pi_\eta^r$ with $\mu'_P \in M_\eta \otimes F_{P,\eta}$ a unit at η and $N_{M_\eta\otimes F_{P,\eta}/F_{P,\eta}}(\mu'_P) = -\theta_\eta$. Suppose $M_\eta \otimes F_{P,\eta}$ is not a field. Then, by the choice of μ_P (cf. Proposition 9.8(11)), we have $\mu_P = \mu'_P\pi_\eta^r$, where $\mu'_P = (\theta'_1, \ldots, \theta'_\ell) \in M_\eta \otimes F_{P,\eta} = \prod F_{P,\eta}$ with each $\theta'_i \in F_{P,\eta}$ a unit at η. Let $\overline{\mu'}_P$ be the image of μ'_P in the residue field $M(\eta) \otimes \kappa(\eta)_P$ of $M_\eta \otimes F_{P,\eta}$ at η. Write $\alpha \otimes F_\eta = \alpha' + (E_\eta, \sigma_\eta, \pi_\eta)$ as in Lemma 4.1. Let β be the image of α' in $H^2(\kappa(\eta), \mu_n)$. Since $\alpha\cdot(-\lambda) = 0$, by Lemma 4.7, $r\ell\beta = (E(\eta), \sigma_\eta, (-1)^{r\ell+1}\overline{\theta}_\eta)$. Since $\alpha\cdot(\mu_P) = 0$ in $H^3(M_\eta\otimes F_{P,\eta}, \mu_n^{\otimes 2})$, once again by Lemma 4.7, $r\beta\otimes\kappa(\eta)_P = (E(\eta)\otimes M(\eta)\otimes\kappa(\eta)_P, \sigma_\eta, (-1)^r\overline{\mu'}_P)$. Since $\kappa(\eta)$ is a global field, by Corollary 3.6, there exists $\mu'_\eta \in M(\eta)$ such that:

(1) $N_{M(\eta)/\kappa(\eta)}(\mu'_\eta) = -\overline{\theta}_\eta$;
(2) $r\beta \otimes M(\eta) = (E(\eta) \otimes M(\eta), \sigma_\eta, (-1)^r{\mu'}_\eta)$;
(3) $\overline{\mu'}_P$ is close to μ'_η for all $P \in \mathscr{P}_\eta$.

Since M_η is complete, there exists $\tilde{\mu'_\eta} \in M_\eta$ such that $N_{M_\eta/F_\eta}(\tilde{\mu'_\eta}) = -\theta_\eta$ and the image of $\tilde{\mu'_\eta}$ in $M(\eta)$ is μ'_η. Let $\mu_\eta = \tilde{\mu'_\eta}\pi_\eta^r$. Since M_η/F_η is of degree ℓ, $\mathrm{ind}(\alpha \otimes M_\eta) < \mathrm{ind}(\alpha \otimes F_\eta)$ (cf. Remark 8.1). Thus μ_η has the required properties. □

PROPOSITION 10.8. *Let η be a codimension zero point of X_0 of type 6. For each $P \in \mathscr{P}_\eta$, let L_P and μ_P be as in Proposition 9.8. Then there exist an unramified field extension L_η/F_η of degree ℓ and $\mu_\eta \in L_\eta$ such that:*

(1) $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$;
(2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$;

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(3) $\operatorname{ind}(\alpha \otimes L_\eta) < \operatorname{ind}(\alpha)$;

(4) *for* $P \in \mathscr{P}_\eta$, *there is an isomorphism* $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ *and*

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell m},$$

for all $m \geqslant 1$.

Proof. Let $P \in \mathscr{P}_\eta$. Suppose $L_P \otimes F_{P,\eta}$ is a field. Let $L_P(\eta)$, $\overline{\theta}_{P,\eta} \in L_P(\eta)$, $\theta_0 \in \kappa(\eta)$ and β be as in the proof of Proposition 10.6. Then, as in the same proof, we have $N_{L_P(\eta)/\kappa(\eta)_P}(\overline{\theta}_P) = -\theta_0$ and $\operatorname{ind}(\beta \otimes E_0 \otimes L_P(\eta)) < \operatorname{ind}(\alpha)/[E_\eta : F_\eta]$. As in the proof of Proposition 10.7, we have $r\beta \otimes L_P(\eta) = (E_0 \otimes L_P(\eta), \sigma_0 \otimes 1, (-1)^r \overline{\theta}_P)$.

If L_P/F_P is not a field, by choice (cf. Proposition 9.8(12)), we have $\mu_P = (\theta_1\pi_\eta^r, \dots, \theta_\ell\pi_\eta^r)$. Since $\alpha \cdot (\mu_P) = 0$ in $H^3(L_P, \mu_n^{\otimes}) = \prod H^3(F_P, \mu_n^{\otimes 2})$, we have $\alpha \cdot (\theta_i\pi_\eta^r) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$. Thus, by Lemma 4.7, we have $r\beta \otimes \kappa(\eta)_P = (E_0, \sigma_0 \otimes 1, (-1)^r\overline{\theta}_i)$ for all i. Since $L_P(\eta) = \prod \kappa(\eta)_P$ and $\overline{\theta}_P = (\overline{\theta}_1, \dots, \overline{\theta}_\ell)$, we have $r\beta \otimes L_P(\eta) = (E_0 \otimes L_P(\eta), \sigma_0 \otimes 1, (-1)^r\overline{\theta}_P)$.

As in the proof of Proposition 10.6, we construct L_η and μ_η with the required properties. $\square$

Lemma 10.9. *Let* η *be a codimension zero point of* X_0 *and* P *a closed point on* η. *Suppose there exist* $\theta_\eta \in F_\eta$ *such that* $\alpha \cdot (\theta_\eta) = 0 \in H^3(F_\eta, \mu_n^{\otimes 2})$. *Then there exists* $\theta_P \in F_P$ *such that* $\alpha \cdot (\theta_P) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$, $\nu_\eta(\theta_P) = \nu_\eta(\theta_\eta)$ *and* $\theta_P^{-1}\theta_\eta \in F_{P,\eta}^{\ell m}$, *for all* $m \geqslant 1$.

Proof. Let π be a prime representing η at P. Since $X_0 \cup \operatorname{ram}_{\mathscr{X}}(\alpha)$ has normal crossings, there exists a prime δ at P such that the maximal ideal at P is generated by π and δ, and α is unramified at P, except possibly at π and δ. Since $F_{P,\eta}$ is a complete discretely valued field with π as a parameter, $\theta_\eta = w\pi^s$ for some $w \in F_\eta$ unit at η. Since the residue field $\kappa(\eta)_P$ of $F_{P,\eta}$ is a complete discretely valued field with $\overline{\delta}$ as a parameter, we have $\overline{w} = \overline{u}\overline{\delta}^r$ for some $u \in F_P$ unit at P. Let $\theta_P = u\delta^r\pi^s$. Then clearly $\nu_\eta(\theta_\eta) = \nu_\eta(\theta_P)$ and $\theta_P^{-1}\theta_\eta \in F_{P,\eta}^{\ell m}$, for all $m \geqslant 1$. Since $\alpha \cdot (\theta_P)$ is unramified at P, except possibly at π and δ, and $\alpha \cdot (\theta_P) = \alpha \cdot (\theta_\eta) = 0 \in H^3(F_{P,\eta}, \mu_n^{\otimes 2})$, by Corollary 5.5, $\alpha \cdot (\theta_P) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$. $\square$

11. The main theorem

Theorem 11.1. *Let* K *be a local field with residue field* κ *and* F *the function field of a curve over* K. *Let* D *be a central simple algebra over* F *of period* n, α *its class in* $H^2(F, \mu_n)$, *and* $\lambda \in F^*$. *If* $\alpha \cdot (-\lambda) = 0$ *and* n *is coprime to* $\operatorname{char}(\kappa)$, *then* $-\lambda$ *is a reduced norm from* D^*.

Proof. As in the proof of Theorem 4.12, we assume that $n = \ell^d$ for prime ℓ with $\ell \neq \operatorname{char}(\kappa)$ and F contains a primitive ℓth root of unity. We prove the theorem by induction on $\operatorname{ind}(D)$.

The case $\operatorname{ind}(D) = 1$ is clear. Assume that $\operatorname{ind}(D) > 1$.

Without loss of generality we assume that K is algebraically closed in F. Let X be a regular projective geometrically irreducible curve over K with $K(X) = F$. Let R be the ring of integers in K and κ its residue field. Let $\mathscr{X}$ be a regular proper model of F over R such that the union of $\operatorname{ram}_{\mathscr{X}}(\alpha)$, $\operatorname{supp}_{\mathscr{X}}(\lambda)$ and the special fiber X_0 of $\mathscr{X}$ is a union of regular curves with normal crossings. By Proposition 8.6, we assume that $\mathscr{X}$ has no special points, and there is no type 2 connection between codimension zero points of X_0 of type 3 or 5, and codimension zero points of X_0 of type 3, 4 or 5.

Let $\mathscr{P}$ be the set of nodal points of X_0. For each $P \in \mathscr{P}$, let L_P and μ_P be as in Proposition 9.8. Let η be a codimension zero point of X_0 and $\mathscr{P}_\eta = \mathscr{P} \cap \eta$. Let L_η and μ_η be as in Propositions 10.1, 10.3, 10.4, 10.6, 10.7 or 10.8 depending on the type of η. Then L_η/F_η is a field or the split extension of degree ℓ and $\mu_\eta \in L_\eta$ such that:

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(1) $N_{L_\eta/F_\eta}(\mu_\eta) = -\lambda$;
(2) $\alpha \cdot (\mu_\eta) = 0 \in H^3(L_\eta, \mu_n^{\otimes 2})$;
(3) $\mathrm{ind}(\alpha \otimes L_\eta) < \mathrm{ind}(\alpha)$;
(4) for $P \in \mathscr{P}_\eta$, there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^m},$$

for all $m \geqslant 1$.

Let $P \in \mathscr{X}$ be a closed point with $P \notin \mathscr{P}$. Then there is a unique codimension zero point η of X_0 with $P \in \eta$. We give a choice of an étale algebra L_P/F_P of degree ℓ and $\mu_P \in L_P^*$ such that:

(1) $N_{L_P/F_P}(\mu_P) = -\lambda$;
(2) $\mathrm{ind}(\alpha \otimes L_P) < \mathrm{ind}(\alpha)$;
(3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$;
(4) there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^m},$$

for all $m \geqslant 1$.

Suppose that η is of type 1. Let $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda}$. Then, by Lemma 6.2 and Proposition 10.1, L_P and μ_P have the required properties.

Suppose that η is of type 2. Suppose that there is a type 2 connection to a codimension zero point η' of X_0 of type 5. Let Q be the point of type 2 intersection η and η'. Suppose that $M_{\eta'} \otimes F_{Q,\eta'}$ not a field. Then, by choice (cf. Proposition 10.3), we have $L_\eta = \prod F_\eta$ and $\mu_\eta = (\theta_1, \ldots, \theta_\ell)$. Since $\alpha \cdot (\mu_\eta) = 0$, we have $\alpha \cdot (\theta_i) = 0$. For each i, $2 \leqslant i \leqslant \ell$, by Lemma 10.9, there exists $\theta_i^P \in F_P$ such that $\alpha \cdot (\theta_i^P) = 0 \in H^3(F_P, \mu_n^{\otimes 2})$ and $\theta_i^{-1}\theta_i^P \in F_{P,\eta}^{\ell^m}$, for all $m \geqslant 1$. Let $\theta_1^P = -\lambda(\theta_2^P \cdots \theta_\ell^P)^{-1}$. Then $L_P = \prod F_P$ and $\mu_P = (\theta_1^P, \ldots, \theta_\ell^P)$ have the required properties. Suppose that $M_{\eta'} \otimes F_{Q,\eta'}$ is a field or there is no type 2 connection from η to any point of type 5. Then, by choice (Proposition 10.4), we have $L_\eta = F_\eta(\sqrt[\ell]{\lambda})$ and $\mu_\eta = -\sqrt[\ell]{\lambda}$. Hence $L_P = F_P(\sqrt[\ell]{\lambda})$ and $\mu_P = -\sqrt[\ell]{\lambda} \in L_P$ have the required properties (cf. Lemma 6.2).

Suppose that η is not of type 1 or 2. Then, by choice, L_η/F_η is an unramified field extension of degree ℓ or the split extension of degree ℓ. Let $\hat{A}_P$ be the completion of the local ring at P and π a prime in $\hat{A}_P$ defining η at P. Since $P \notin \mathscr{P}$ and $\mathrm{ram}_{\mathscr{X}}(\alpha)$ is union of regular curves with normal crossings, there exists a prime $\delta \in \hat{A}_P$ such that α is unramified on $\hat{A}_P$, except possibly at π and δ. Further, $\lambda = w\pi^r\delta^s$ for some unit $w \in \hat{A}_P$. Since η is not of type 1 or 2, $\nu_\eta(\lambda) = r$ is divisible by ℓ. Thus, by Lemma 6.5, there exist an étale algebra L_P/F_P and $\mu_P \in L_P$ such that:

(1) $L_P \otimes F_{P,\eta} \simeq L_\eta \otimes F_{P,\eta}$;
(2) $\mathrm{ind}(\alpha \otimes L_P) < \mathrm{ind}(\alpha)$;
(3) $\alpha \cdot (\mu_P) = 0 \in H^3(L_P, \mu_n^{\otimes 2})$;
(4) there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and

$$\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^m},$$

for all $m \geqslant 1$.

Thus for every $x \in X_0$, we have chosen an étale algebra L_x/F_x of degree ℓ and $\mu_x \in L_x$ such that:

(1) $N_{L_x/F_x}(\mu_x) = -\lambda$;
(2) $\alpha \cdot (\mu_x) = 0 \in H^3(L_x, \mu_n^{\otimes 2})$;
(3) $\mathrm{ind}(\alpha \otimes L_x) < \mathrm{ind}(\alpha)$;
(4) for any branch (P, η), there is an isomorphism $\phi_{P,\eta} : L_\eta \otimes F_{P,\eta} \to L_P \otimes F_{P,\eta}$ and $\phi_{P,\eta}(\mu_\eta \otimes 1)(\mu_P \otimes 1)^{-1} \in (L_P \otimes F_{P,\eta})^{\ell^m}$, for all $m \geqslant 1$. Further, if η is a codimension zero point of X_0, then L_η/F_η is field or the split extension.

Let (P, η) be a branch. Since $\kappa(P)$ is a finite field, there exists t_P such that $\kappa(P)$ has no ℓ^{t_P}th primitive root of unity. Since $\kappa(\eta)_P$ is a complete discretely valued field with residue field $\kappa(P)$, $\kappa(\eta)_P$ has no ℓ^{t_P}th primitive root of unity. Since $F_{P,\eta}$ is a complete discretely valued field with residue field $\kappa(\eta)_P$, $F_{P,\eta}$ has no ℓ^{t_P}th primitive root of unity.

Let L/F be a degree ℓ extension as in Lemma 7.3. Then $\mathrm{ind}(\alpha \otimes L) < \mathrm{ind}(\alpha)$. Note that for every closed point P of X_0, the residue field $\kappa(P)$ at P is a finite field. Thus, for every closed point P of X_0, there exists $t_P \geqslant d$ such that there is no primitive ℓ^{t_P}th root of unity in $\kappa(P)$. Thus, by Proposition 7.5), there exist a field extension N/F of degree coprime to ℓ and $\mu \in L \otimes N$ such that:

- $N_{L\otimes N/N}(\mu) = -\lambda$; and
- $\alpha \cdot (\mu) = 0 \in H^3(L \otimes N, \mu_n^{\otimes 2})$.

Since $L \otimes N$ is also a function field of a curve over a local field, by induction hypotheses, μ is a reduced norm from $D \otimes L \otimes N$ and hence $-\lambda = N_{L\otimes N/N}(\mu)$ is a reduced norm from D. Since $N_{N/F}(-\lambda) = (-\lambda)^{[N:F]}$, $(-\lambda)^{[N:F]}$ is a norm from D. Since $[N : F]$ is coprime to ℓ, $-\lambda$ is a reduced norm from D. □

Corollary 11.2. *Let K be a local field with residue field κ and F the function field of a curve over K. Let Ω be the set of divisorial discrete valuations of F. Let D be a central simple algebra over F of period coprime to* $\mathrm{char}(\kappa)$ *and $\lambda \in F$. If λ is a reduced norm from $D \otimes F_\nu$ for all $\nu \in \Omega$, then λ is a reduced norm from D.*

Proof. Let n be the period of D and $\alpha \in H^2(F, \mu_n)$ be the class of D. Since λ is a reduced norm from F_ν for all $\nu \in \Omega_F$, $\alpha \cdot (\lambda) = 0$ in $H^3(F_\nu, \mu_n^{\otimes 2})$ for all $\nu \in \Omega$. Thus, by [Kat86, Proposition 5.2], $\alpha \cdot (\lambda) = 0$ in $H^3(F, \mu_n^{\otimes 2})$ and by Theorem 11.1, λ is a reduced norm from D. □

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