A CONVOLUTION TRANSFORM ADMITTING AN INVERSION FORMULA OF INTEGRO-DIFFERENTIAL TYPE

D. B. SUMNER

1. Introduction. Following Wiener's fundamental work on the Operational Calculus [**9**, pp. 557-584], Widder suggested [**5**, p. 219] that the inversion operator for the convolution transform

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)\,dt \tag{1}$$

should be a suitable interpretation of $E(D)$, where $D = d/dx$ and the function $E(s)$ is defined by

$$\frac{1}{E(s)} = \int_{-\infty}^{\infty} e^{-st}G(t)\,dt. \tag{2}$$

Two methods of interpreting $E(D)$ have been used. The first, which appeals only to real-variable methods, has been used by Widder and his collaborators [**8**, p. 659; **5**, p. 217; **4**] in cases where $E(s)$ is entire with real zeros. The substance of the method is to express $E(s)$ as

$$\lim_{n\to\infty} E_n(s),$$

where $E_n(s)$ is a product of n factors of the form

$$\left(1 - \frac{s}{\lambda}\right)e^{s/\lambda},$$

and where, under suitable convergence conditions,

$$\lim_{n\to\infty} E_n(D)\cdot f(x)$$

can be computed.

The second method, which uses the complex variable, has been applied by Widder and Hirschmann [**8**, p. 692], and by the present author [**2**, p. 174] in cases where the entire function $E(s)$ can be expressed as a Fourier integral. It is naturally of less general applicability.

At the conclusion of their article Widder and Hirschmann conjectured the existence of convolution transforms, whose inversion operators are of integro-differential form, and not purely differential, as in all the then known cases. Widder has followed this by using such integro-differential operators to invert the Lambert transform [**6**, p. 171] and the Fourier sine transform [**7**, p. 119].

Received October 17, 1951. This investigation was carried out while the author held a fellowship at the Summer Research Institute of the Canadian Mathematical Congress in 1951.

In the last-named work, he showed that the methods of [**8**] were still applicable, even though the function defined by the integral

$$\frac{E(s)}{F(s)} = \int_{-\infty}^{\infty} e^{-st} G(t)\, dt \tag{2'}$$

was meromorphic. In both applications (to the Lambert and to the Fourier sine transform), a distinction was made between the factors $F(D)$ and $[E(D)]^{-1}$ of the inversion operator.

The objects of this note are:

(i) to show that transforms requiring an integro-differential invertor occur in quite simple cases, the example treated being one in which the kernel is a function little more complicated in form than the classical Stieltjes case [**1**, p. **473**], where the kernel is $(1 + e^{-x})^{-1}$;

(ii) to prove that the order in which the factors of the invertor are applied is material.

We call $F(D)$ the differentiating factor, and $[E(D)]^{-1}$ the integrating factor. We use the methods of the complex variable, and interpret the factors of the invertor by expressing them as integrals.

2. Preliminary results. We consider the transform

$$f(x) = \int_{-\infty}^{\infty} H(x - t)\,\phi(t)\, dt, \tag{3}$$

where $H(x) = [e^{-2x} + 2e^{-x} \cos \pi\beta + 1]^{-1}$, and $0 < \beta < 1$. By adapting classical methods due to Widder [**3**, pp. **10-11**], after an exponential change of variable, the following theorem is easily seen to be true:

THEOREM 1. *Let $f(x)$ be defined by* (3), *and $\phi(t)$ be such that the integral converges for at least one value of x in the strip $|\Im x| < \pi(\beta + 1)$, then it converges for all such x, and converges uniformly in any compact subset of the strip, so that $f(x)$ is analytic in the strip.*

It is well known that

$$\int_{-\infty}^{\infty} e^{-st} H(t)\, dt = \frac{\pi}{\sin \pi s} \; \frac{\sin \pi\beta(1 - s)}{\sin \pi\beta} \qquad (0 < \Re s < 2). \tag{4}$$

We take the reciprocal of this function, with $s = D$, as the invertor for equation (3), and we write

$$F(D) = \frac{\sin \pi D}{\pi} = \lim_{\theta \to \pi} \frac{1}{2\pi i} [e^{i\theta D} - e^{-i\theta D}] = \lim_{\theta \to \pi} F_\theta(D),$$

$$[E(D)]^{-1} = \frac{\sin \pi\beta}{\sin \pi\beta(1 - D)} = \frac{\sin \pi\beta}{\pi\beta} \int_{-\infty}^{\infty} \frac{e^{-v} e^{vD}\, dv}{1 + e^{-v/\beta}}$$

and shall as usual interpret $e^{kD}(f)x$ as $f(x + k)$.

3. The inversion theorem. Our main result is the following theorem:

THEOREM 2. *Let $f(x)$ be defined by* (3), *with* $0 < \beta < 1$, *and $\phi(t)$ be such*

https://doi.org/10.4153/CJM-1953-015-x Published online by Cambridge University Press

that the integral converges for at least one x in the strip $|\Im x| < \pi(\beta + 1)$. *Then*

$$F(D) \cdot [E(D)]^{-1}f(x) = \tfrac{1}{2}[\phi(x+) + \phi(x-)],$$

whenever the right-hand side has a meaning.

We have

$$\begin{aligned}[E(D)]^{-1}f(x) &= \frac{\sin \pi\beta}{\pi\beta} \int_{-\infty}^{\infty} \frac{e^{-v}f(x+v)\,dv}{1+e^{-v/\beta}},\\ &= \frac{\sin \pi\beta}{\pi\beta} \int_{-\infty}^{\infty} \frac{e^{-v}\,dv}{1+e^{-v/\beta}} \int_{-\infty}^{\infty} H(x+v-t)\,\phi(t)\,dt,\\ &= \int_{-\infty}^{\infty} K(x-t)\,\phi\,(t)\,dt,\end{aligned}$$

where the interchange of the integrations is justified by the uniform convergence established in Theorem 1, and

$$K(r) = \frac{\sin \pi\beta}{\pi\beta} \int_{-\infty}^{\infty} \frac{e^{-v}H(r+v)\,dv}{1+e^{-v/\beta}}.$$

On considering the integral

$$\int \frac{1}{e^{-(r+w)}+1}\,\frac{e^{-w}\,dw}{1-e^{-w/\beta}}$$

taken round the rectangle with vertices $R \pm \pi i\beta$, $-R \pm \pi i\beta$, and making $R \to \infty$, we see that

$$K(r) = [1 + e^{-r}]^{-1}.$$

Thus

$$[E(D)]^{-1}f(x) = \int_{-\infty}^{\infty} \frac{\phi(t)\,dt}{1+e^{-(x-t)}}. \tag{5}$$

This is the classical Stieltjes transform (after an exponential change of variable); and if we denote it by $g(x)$, it is well known [**4**, p. 340] that

$$\lim_{\theta\to\pi} F_\theta(D) \cdot g(x) = \tfrac{1}{2}[\phi(x+) + \phi(x-)].$$

Theorem 2 is thus seen to be true.

We do not here allow the value $\beta = 0$, since the integral (4) then has the value $\pi(1 - s)/\sin \pi s$, whose reciprocal has no poles. It is easily seen that in this case, after suitable changes of variable, the equation (3) takes the form

$$h(x) = \int_0^{\infty} \frac{\psi(t)\,dt}{(x+t)^2}$$

which has been discussed by the present author [**2**].

4. Commutativity of the factors in the operator. That the order in which the factors $F(D)$ and $[E(D)]^{-1}$ were applied to the function defined by (3) is material, is brought out by the following example.

Let

$$\phi(t) = \begin{cases} e^t, & t > 0, \\ 0, & t \leqslant 0. \end{cases}$$

It is then easily seen that the corresponding

$$f(x) = \int_0^\infty H(x-t) \cdot e^t \, dt, \\ = \frac{e^x}{\sin \pi\beta} \cdot \text{arc tan}\left[\frac{\sin \pi\beta}{e^{-x} + \cos \pi\beta}\right];$$

while

$$F_\theta(D)f(x) = \frac{1}{\sin \pi\beta}\left[e^{x+i\theta} \text{arc tan}\left(\frac{\sin \pi\beta}{e^{-x-i\theta} + \cos \pi\beta}\right) - e^{x-i\theta} \text{arc tan}\left(\frac{\sin \pi\beta}{e^{-x+i\theta} + \cos \pi\beta}\right)\right] \to 0$$

as $\theta \to \pi$. The differentiating factor, if applied first, is thus seen to annihilate the function, a fact which makes pointless the subsequent application of the integrating factor.

References

1. T. J. Stieltjes, *Oeuvres complètes*, vol. 2.
2. D. B. Sumner, *An inversion formula for the generalized Stieltjes transform*, Bull. Amer. Math. Soc., *55* (1949), 174–183.
3. D. V. Widder, *The Stieltjes transform*, Trans. Amer. Math. Soc., *43* (1938), 7–60.
4. ———, *The Laplace transform* (Princeton, 1941).
5. ———, *Inversion formulas for convolution transforms*, Duke Math. J., *14* (1947), 217–249.
6. ———, *An inversion of the Lambert transform*, Math. Mag. (March-April, 1950), 171–182.
7. ———, *Symbolic inversions of the Fourier sine transform and of related transforms*, J. Indian Math. Soc., *14* (1950), 119–128.
8. D. V. Widder and I. I. Hirschmann Jr., *Generalized inversion formulae for convolution transforms*, Duke Math. J., *15* (1948), 659–696.
9. N. Wiener, *The operational calculus*, Math. Ann., *95* (1925-26), 557–584.

Hamilton College, McMaster University