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ON THE NON-EXISTENCE OF CERTAIN EULER PRODUCTS

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In a paper with the above title, T. M. Apostol and S. Chowla [1] proved the following result:

THEOREM 1. *For relatively prime integers h and k, $1 \leq h \leq k$, the series*

$$\sum_{n=0}^{\infty} \frac{1}{(kn+h)^s}$$

does not admit of an Euler product decomposition, that is, an identity of the form

$$\prod_p \left\{1+\frac{f_1(p)}{p^s}+\frac{f_2(p)}{p^{2s}}+\cdots\right\}=\sum_{n=0}^{\infty}\frac{1}{(kn+h)^s} \tag{1}$$

except when $h=k=1$; $h=1$, $k=2$. The series on the right is extended over all primes p and is assumed to be absolutely convergent for $R(s)>1$.

The proof given by Apostol and Chowla is simple and short. But we give here a very much shorter proof.

Proof of the Theorem. For identity (1) to hold, it is necessary that $h=1$. We assume this in what follows. We know that (1) holds for $k=1,2$. So we assume below that $k \geq 3$. Multiplying out, the left side of (1) gives the series

$$\sum \frac{f(n)}{n^s} \tag{2}$$

(absolutely convergent for $R(s)>1$), where $f(n)$ is the multiplicative function defined by

$$f\left(\prod p^a\right)=\prod f_a(p).$$

Identifying (2) with the series on the right side of (1), we obtain (recalling that $h=1$)

$$f(n)=\begin{cases}1, & n\equiv 1 (\text{mod } k)\\ 0, & \text{otherwise.}\end{cases} \tag{3}$$

Let p and q be two distinct primes which are $\equiv -1 (\text{mod } k)$. Since $k \geq 3$, (3)

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gives $f(p)=f(q)=0$, whereas since $pq\equiv 1(\text{mod } k)$, we have $f(pq)=1$. However, the multiplicativity of $f(n)$ shows that $1=f(pq)=f(p)f(q)=0$, a contradiction. This completes the proof of the theorem.

In fact, theorem (1) and result (3) are immediate consequences of the following theorem due to R. D. James and Ivan Niven [2]: Let M be a multiplicatively closed system of positive integers such that if $x\in M$ and $y\equiv x(\text{mod } n)(y>n)$ then $y\in M$; and let n denote the smallest positive integer which can be used to define M. Suppose further that A is the class of all integers relatively prime to n and B the class of all integers not belonging to A. Then M has unique factorization property if and only if $M\cap A=A$, $M\cap B=0$.

To derive the Apostol-Chowla theorem from this result, we take $M=\{kn+h,\ n=0,\ 1,2,\ldots\}$. Then for M to have unique factorization, it is clearly necessary that $h=1$. Now, applying the James-Niven theorem, we see that the multiplicative set $\{kn+1\}$ has unique factorization property if and only if $k=1$ or 2.

References

1. T. M. Apostol and S. Chowla, *On the Non-Existence of Certain Euler Products*, Det Kongelige Norske Videnskabers Selskab Forhandlinger Vol. **32** (1959), No. II, 65–67.

2. R. D. James and Ivan Niven, *Unique Factorization in Multiplicative Systems*, Proc. Amer. Math. Soc. **5** (1954); 834–838, MR16 336.

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