POLYTOPES, VALUATIONS, AND THE EULER RELATION

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1. Introduction. By a *d-polytope* we shall mean a d-dimensional convex polytope. We shall denote a j-dimensional face (or *j-face*) of a polytope by F^j. Every d-polytope P has proper j-faces for $0 \leqq j \leqq d-1$ and we shall also say that P is a d-face of itself. Observe that every face of a polytope is again a polytope. The collection of all convex polytopes shall be denoted by $\mathscr{P}$.

It is well known that for every d-polytope P, the Euler relation holds:

$$(1.1) \qquad f_0 - f_1 + f_2 - \ldots + (-1)^{d-1} f_{d-1} = 1 - (-1)^d,$$

where f_j denotes the number of j-faces of P.

Recently, G. C. Shephard has pointed out that other functions on polytopes (for example, mean widths, Steiner points, interior angles, etc.) satisfy identities remarkably similar to (1.1) (see **8** for additional background to this problem). All of these relations can be viewed as special cases of the following relation, $E(\epsilon)$, for a function ϕ:

$$(1.2) \qquad \sum_{j=0}^{d} (-1)^j \sum \phi(F^j) = \epsilon\phi(P),$$

where P is any d-polytope and the inner summation on the left is taken over all j-faces of P. If ϕ satisfies (1.2) for some ϵ and every polytope P, we will say that ϕ *satisfies an Euler relation.* In particular, if $\phi(P) = 1$ for every polytope P and $\epsilon = 1$, then (1.2) reduces to (1.1), the standard Euler relation.

Our aim in this paper is to call attention to the close relationship between functions which satisfy an Euler relation and valuations.

We shall say that ϕ is a *valuation* on a class $\mathscr{A}$ of sets if

$$(1.3) \qquad \phi(A) + \phi(B) = \phi(A \cup B) + \phi(A \cap B)$$

whenever A, B, $A \cup B$, and $A \cap B$ are all members of $\mathscr{A}$. For our purposes, the range of ϕ may be any vector space over the real numbers (see the remarks at the end of the paper, however). Generally, $\mathscr{A}$ will be either $\mathscr{P}$ or $\mathscr{S}$, the class of all simplices. For convenience, we set $\phi(\emptyset) = 0$.

A very similar notion, which we term a weak valuation, has been extensively studied by Hadwiger (see **2**, pp. 236–243, for the continuous, motion-invariant case and **3** for the general case). It would be of interest to know whether this notion is equivalent to that of a valuation.

Received May 3, 1967 and in revised form, July 11, 1967.

We prove three theorems relating these two concepts. The third of these, a rather surprising decomposition theorem, was first suggested by G. C. Shephard in a private communication.

(1.4) THEOREM. *Suppose that ϕ is a continuous function on $\mathscr{P}$ which satisfies $E(1)$ or $E(-1)$. Then ϕ is a valuation on $\mathscr{P}$.*

(1.5) THEOREM. *Suppose that ϕ is a valuation on $\mathscr{P}$. Then ϕ satisfies $E(\epsilon)$ on $\mathscr{P}$ if and only if ϕ satisfies $E(\epsilon)$ on $\mathscr{S}$.*

(1.6) THEOREM. *Suppose that ϕ is a valuation on $\mathscr{P}$. Then we can write ϕ as the sum of two valuations α and β, such that α satisfies $E(1)$ and β satisfies $E(-1)$.*

The second theorem appears more general than is the case, for it can be shown (see (5.7)) that ϵ can equal only 1 and -1 in an $E(\epsilon)$ relation.

Section 2 is devoted to proving a number of general results about valuations which are applied in § 3 to prove (1.4). In §§ 4 and 5, *derived valuations* are defined and their properties explored to prove (1.5) and (1.6). Analogues of the Dehn-Sommerville equations for functions satisfying $E(\epsilon)$ are proved in § 7, while in § 8 it is proved that if a continuous function satisfies an Euler relation, this is essentially the only linear relation which it can satisfy.

My thanks are due to Professor Shephard for originally suggesting this problem to me and to Professor Micha Perles for his careful and valuable criticism of an earlier version of this paper.

2. Properties of valuations. In this section we prove a number of results about valuations which will be of use later. First, some definitions.

We say that ϕ is a *weak valuation* on a class $\mathscr{A}$ of sets if

$$\phi(A \cup B) + \phi(A \cap B) = \phi(A) + \phi(B) \tag{2.1}$$

whenever A, B, $A \cup B$, and $A \cap B$ are all members of $\mathscr{A}$, and A and B have no common relatively interior points.

We say that ϕ is a *quasi-valuation* on $\mathscr{A} \subseteq \mathscr{P}$ if (2.1) holds whenever all four polytopes belong to $\mathscr{A}$, and every proper face of $A \cup B$ is a face of either A or B.

Finally, we define ϕ to be a *d-valuation* if (2.1) holds whenever A, B, and $A \cup B$ are d-polytopes. Note that this implies that $\dim(A \cap B) = d - 1$ or d. In an analogous fashion, we define a *weak d-valuation* and a *quasi-d-valuation*.

In general, if ϕ is a d-valuation, it is not true that ϕ is a d'-valuation if $d \neq d'$. However, if ϕ is a function, continuous with respect to the Hausdorff metric on convex sets (which we shall hereafter term simply as a *continuous* function), we have the following result.

(2.2) *Suppose that ϕ is a continuous function on $\mathscr{P}$ and suppose that ϕ is a d-valuation. Then ϕ is a k-valuation for $k \leqq d$.*

Proof. Let P, Q, and $P \cup Q$ be $(d-1)$-polytopes, and let P_ϵ and Q_ϵ be cylinders of height ϵ having bases P and Q, respectively. Then $P_\epsilon \cup Q_\epsilon$ and $P_\epsilon \cap Q_\epsilon$ are, likewise, cylinders of height ϵ over $P \cup Q$ and $P \cap Q$. Moreover, P_ϵ, Q_ϵ, and $P_\epsilon \cup Q_\epsilon$ are d-polytopes.

Since ϕ is a d-valuation, P_ϵ and Q_ϵ satisfy the valuation relation (2.1). By the continuity of ϕ, $\phi(P_\epsilon) \to \phi(P)$ as $\epsilon \to 0$ and, similarly, for the other polytopes. It thus follows that P and Q satisfy the valuation property.

As P and Q were arbitrary, ϕ is a $(d-1)$-valuation. Repeating the argument yields the result for any $k \leqq d$.

(2.3) *ϕ is a weak d-valuation on $\mathscr{P}$ if and only if ϕ is a d-valuation on $\mathscr{P}$.*

Proof. It is clear that any d-valuation is a weak d-valuation. For the converse statement, let P, Q, and $P \cup Q$ be d-polytopes and ϕ a weak d-valuation. We shall show that the valuation property holds for them. By the hypothesis on ϕ, we need consider only the case when $\dim P \cap Q = d$.

Assume that $P \cap Q$ has r *facets* (faces of dimension $d-1$) F such that $\operatorname{rel\,int} F \subseteq \operatorname{int} P$. We shall prove the result by means of induction on the number r.

The assertion is trivial if $r = 0$, for then $P \subseteq Q$. Assume that the statement is true for any $r < n$. We wish to prove it for $r = n$. Let F be a facet of $P \cap Q$ lying in the interior of P and let H be the associated hyperplane supporting $P \cap Q$ on F. We suppose that H^+ is the closed half-space determined by H such that $P \cap Q \subseteq H^+$ and let $P_1 = H^+ \cap P$ and $P_2 = H^- \cap P$. Observe that $Q \subseteq H^+$ and that $P_1 \cap Q = P \cap Q$.

Since $P_1 \cap Q$ has at most $r - 1$ facets interior to P_1, by our induction hypothesis we have that

$$\phi(P_1 \cup Q) + \phi(P_1 \cap Q) = \phi(P_1) + \phi(Q). \tag{2.4}$$

Since ϕ is assumed to be a weak d-valuation, we also have that

$$\phi(P_1 \cup P_2) + \phi(P_1 \cap P_2) = \phi(P_1) + \phi(P_2) \tag{2.5}$$

and

$$\phi(P_1 \cup Q) + \phi(P_2) = \phi(P_1 \cup P_2 \cup Q) + \phi((P_1 \cup Q) \cap P_2). \tag{2.6}$$

Using the fact that $P_1 \cup P_2 = P$, we can combine (2.4), (2.5), and (2.6) to show that

$$\phi(P \cup Q) + \phi(P_1 \cap Q) = \phi(P) + \phi(Q) + \phi(P_1 \cap P_2) - \phi((P_1 \cup Q) \cap P_2). \tag{2.7}$$

Since $P_1 \cap Q = P \cap Q$, our proof will be complete if we show that $P_1 \cap P_2 = (P_1 \cup Q) \cap P_2$. But this is easy since $P_2 \cap Q \subseteq P_2 \cap H = P_2 \cap P_1$. This completes the proof.

It would be of interest to know if the above result holds true for arbitrary convex sets. Of course, if ϕ is assumed to be continuous, a standard approximation argument, together with (2.3), may be used, but the general problem is unsettled.

It would be pleasant to report that it is also true that every quasi-valuation is a valuation. Unfortunately, this is false, as the following example (due to M. Perles) shows: define

$$(2.8) \qquad \lambda(P) = \begin{cases} \text{number of vertices of } P & \text{if } \dim P \geqq 1, \\ 2 & \text{if } \dim P = 0. \end{cases}$$

It is easily verified that $\lambda(P)$ is a quasi-valuation and equally easily verified that it is not a valuation. We can salvage the situation for continuous quasi-valuations, however.

(2.9) *Let ϕ be a continuous function on $\mathscr{P}$. Then ϕ is a quasi-d-valuation on $\mathscr{P}$ if and only if ϕ is a d-valuation on $\mathscr{P}$.*

Proof. It is clear that every d-valuation is a quasi-d-valuation.

Conversely, suppose that ϕ is a quasi-d-valuation, and let P, Q, and $P \cup Q$ be d-polytopes such that $\dim P \cap Q = d - 1$. It is easy to find a projective image P^* of P such that:

$P^* \cup Q$ is a polytope, every face of which is a face of either P^* or Q,
$P^* \cap Q = P \cap Q$, and
P^* is arbitrarily close to P in the Hausdorff metric
(see **5**, (2.10)). Since ϕ is a quasi-d-valuation, P^* and Q satisfy the valuation property. It then follows by the continuity of ϕ that P and Q also satisfy the valuation property. The result is then a consequence of (2.3).

We now return to valuations to state the following useful result.

(2.10) *Let ϕ be a k-valuation on $\mathscr{P}$ for all $k \leqq d$. If $P = P_1 \cup \ldots \cup P_n$, where $P, P_1, \ldots, P_n$ are all d-polytopes, then*

$$\phi(P) = \sum_{i=1}^{n} \phi(P_i) - \sum_{i<j} \phi(P_i \cap P_j) + \ldots + (-1)^{n-1}\phi(P_1 \cap \ldots \cap P_n).$$

The proof of this is completely analogous to the proof of the same result for Steiner points in (**4**, (10)).

(2.11) COROLLARY. *Suppose that ϕ is a k-valuation on $\mathscr{P}$ for all $k \leqq d$. If $\phi(S) = 0$ whenever S is a j-simplex and $j \leqq d$, then $\phi(P) = 0$ for any j-polytope P with $j \leqq d$.*

Proof. It is clear that any j-polytope P can be written as a finite union of j-simplices, $S_1, \ldots, S_n$, where $S_{i_1} \cap \ldots \cap S_{i_r}$ is a simplex for any set of indices $i_1, \ldots, i_r$. The result now follows immediately from (2.10).

Our final result is equivalent to (2.11).

(2.12) *Suppose that ϕ_1 and ϕ_2 are k-valuations of $\mathscr{P}$ for all $k \leqq d$. If $\phi_1(S) = \phi_2(S)$ whenever S is a j-simplex and $j \leqq d$, then $\phi_1(P) = \phi_2(P)$ for any j-polytope P with $j \leqq d$.*

3. Proof of the first theorem. Most of the machinery which we need to prove (1.4) is now ready, but one more lemma is needed.

(3.1) LEMMA. *Let ϕ be a function on $\mathscr{P}$ which satisfies $E(1)$ for a given odd value of d. Then ϕ is a quasi-d-valuation.*

Proof. Let P and Q be two d-polytopes such that $P \cup Q$ is a d-polytope and every proper face of $P \cup Q$ is a face of either P or Q. The remainder of the proof is largely devoted to keeping track of the various faces which are of interest. With this in mind we break up the set of all j-faces for $j \leqq d - 1$ which occur as a face of P, Q, $P \cup Q$ or $P \cap Q$ into five subsets:

$\mathscr{F}_1^j$ = j-faces which are faces of $P \cup Q$ and P, but not Q;
$\mathscr{F}_2^j$ = j-faces which are faces of $P \cup Q$ and Q, but not P;
$\mathscr{F}_3^j$ = j-faces which are faces of $P \cup Q$, P and Q;
$\mathscr{F}_4^j$ = j-faces which are faces of $P \cap Q$ and P, but not Q;
$\mathscr{F}_5^j$ = j-faces which are faces of $P \cap Q$ and Q, but not P.

It is clear that all of these sets are disjoint and that for $j < \dim(P \cap Q)$ we have that

$$\mathscr{F}^j(P \cap Q) = \mathscr{F}_3^j \cup \mathscr{F}_4^j \cup \mathscr{F}_5^j,$$
$$\mathscr{F}^j(P) = \mathscr{F}_1^j \cup \mathscr{F}_3^j \cup \mathscr{F}_4^j,$$
$$\mathscr{F}^j(Q) = \mathscr{F}_2^j \cup \mathscr{F}_3^j \cup \mathscr{F}_5^j,$$

where $\mathscr{F}^j(K)$ denotes the set of j-faces of the polytope K.

If $\dim(P \cap Q) = d - 1$, then

$$\mathscr{F}^{d-1}(P) = \mathscr{F}_1^{d-1} \cup \{P \cap Q\} \tag{3.2}$$

since $\mathscr{F}_3^{d-1} = \mathscr{F}_4^{d-1} = \mathscr{F}_5^{d-1} = \emptyset$. Similarly for $\mathscr{F}^{d-1}(Q)$. For all $j < d$, we also have that

$$\mathscr{F}^j(P \cup Q) = \mathscr{F}_1^j \cup \mathscr{F}_2^j \cup \mathscr{F}_3^j.$$

For each of the sets $\mathscr{F}_i^j$, let $\sum_i^j$ denote the sum $\sum\phi(F^j)$ taken over all members of $\mathscr{F}_i^j$; for each k-polytope X, let $\psi(X)$ denote $\sum_{j=0}^{k-1}(-1)^j\sum\phi(F^j)$, where the inner summation is taken over all j-faces of X. Using these notations, we see that if $\dim(P \cap Q) = d$, then since d is odd, we have that

$$2\phi(P \cup Q) = \psi(P \cup Q) = \sum_{j=0}^{d-1} (-1)^j\{\textstyle\sum_1^j + \sum_2^j + \sum_3^j\} = \tag{3.3}$$

$$\sum_{j=0}^{d-1} (-1)^j\{\textstyle\sum_1^j + \sum_3^j + \sum_4^j + \sum_2^j + \sum_3^j + \sum_5^j - \sum_3^j - \sum_4^j - \sum_5^j\} =$$

$$\psi(P) + \psi(Q) - \psi(P \cap Q) = 2[\phi(P) + \phi(Q) - \phi(P \cap Q)].$$

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If $\dim(P \cap Q) = d - 1$, then we can imitate the above proof except that

$$\sum_{\mathscr{F}^{d-1}(P\cap Q)} \phi(F^{d-1}) = \sum_{\mathscr{F}^{d-1}(P)} \phi(F^{d-1}) + \sum_{\mathscr{F}^{d-1}(Q)} \phi(F^{d-1}) - 2\phi(P \cap Q).$$

Noting that $\psi(P \cap Q) = 0$ since $\dim(P \cap Q) = d - 1$, we have that

$$(3.4) \quad 2\phi(P \cup Q) = \psi(P \cup Q) = \psi(P) + \psi(Q) - 2\phi(P \cap Q) = 2[\phi(P) + \phi(Q) - \phi(P \cap Q)].$$

From (3.3) and (3.4) it follows that ϕ is a quasi-d-valuation.

We are now ready to complete the proof of one part of the theorem. In fact, we shall prove a slightly stronger result.

(3.5) *Suppose that ϕ is a function continuous on $\mathscr{P}$ which satisfies $E(1)$ for each k-polytope if $k \leqq d$ (odd). Then ϕ is a k-valuation on $\mathscr{P}$ for all $k \leqq d$.*

Proof. By (3.1), it follows from the hypotheses that ϕ is a quasi-d-valuation. Since ϕ is continuous on $\mathscr{P}$, ϕ is a d-valuation by (2.9), and thus a k-valuation on $\mathscr{P}$ for $k \leqq d$ by (2.2). This completes the proof.

We can essentially duplicate the proof of (3.1) to show the following result.

(3.6) *Let ϕ be a function on $\mathscr{P}$ which satisfies $E(-1)$ for a given value of d (even). Then ϕ is a quasi-d-valuation.*

From here, it is again a short step to proving the following assertion.

(3.7) *Let ϕ be a continuous function on $\mathscr{P}$ which satisfies $E(-1)$ for all polytopes of dimension $k \leqq d$ (even). Then ϕ is a k-valuation on $\mathscr{P}$ for all $k \leqq d$.*

The proof of (1.3) is now complete. The result cannot be extended to discontinuous functions on $\mathscr{P}$ as the function $\lambda(P)$ defined by (2.8) shows. It may be verified that λ satisfies $E(1)$ for all polytopes (see **1**, (8.3.1) for details), but as we have already observed, λ is not a valuation.

4. Derived valuations. If ϕ is any valuation on $\mathscr{P}$, we define ϕ^*, the *valuation derived from* ϕ, on any d-polytope P by:

$$(4.1) \qquad \phi^*(P) = \sum_{j=0}^{d} (-1)^j \sum_{P^j \subseteq P} \phi(P^j).$$

The following theorem, from which we shall draw many useful corollaries, verifies that ϕ^* is a valuation.

(4.2) THEOREM. *Suppose that ϕ is a valuation on $\mathscr{P}$. Then ϕ^* defined from ϕ by* (4.1), *is a valuation.*

Proof. We shall show that ϕ^* is a weak valuation on $\mathscr{P}$ and then use (2.3) to conclude that ϕ^* is a valuation on $\mathscr{P}$.

Let P and Q be two d-polytopes such that $P \cup Q$ is convex and $P \cap Q$ is a face of both P and Q. We divide up the j-faces which appear in P, Q, $P \cup Q$, or $P \cap Q$ into the following disjoint classes:

$\mathscr{F}_1{}^j$ = j-faces of $P \cup Q$ which are j-faces of P and of Q;
$\mathscr{F}_2{}^j$ = j-faces of $P \cup Q$ which are j-faces of P but not Q;
$\mathscr{F}_3{}^j$ = j-faces of $P \cup Q$ which are j-faces of Q but not P;
$\mathscr{F}_4{}^j$ = j-faces of $P \cup Q$ which are j-faces of neither P nor Q;
$\mathscr{F}_5{}^j$ = j-faces of P which are j-faces of Q but not $P \cup Q$;
$\mathscr{F}_6{}^j$ = j-faces of P which are j-faces of neither Q nor $P \cup Q$;
$\mathscr{F}_7{}^j$ = j-faces of Q which are j-faces of neither P nor $P \cup Q$.

We also make the following conventions: $\sum_i{}^j$ denotes $\sum\phi(F^j)$, where the summation is taken over the class $\mathscr{F}_i{}^j$, and $\phi^j(R) = \sum\phi(R^j)$ if the summation is taken over all of the j-faces of the polytope R.

It is important to note the one-to-one correspondence among the classes $\mathscr{F}_4{}^j, \mathscr{F}_6{}^j, \mathscr{F}_7{}^j$, and $\mathscr{F}_5{}^{j-1}$ for $1 \leqq j \leqq d-1$ which arises in the following way: each $P^j \in \mathscr{F}_6{}^j$ corresponds to some $Q^j \in \mathscr{F}_7{}^j$ whose union $P^j \cup Q^j \in \mathscr{F}_4{}^j$ and whose intersection $P^j \cap Q^j \in \mathscr{F}_5{}^{j-1}$. Since ϕ is a valuation, by summing over all members of $\mathscr{F}_4{}^j$ we see that

$$\sum_4{}^j = \sum_6{}^j + \sum_7{}^j - \sum_5{}^{j-1}. \tag{4.3}$$

We further observe that for $j < d-1$,

$$\begin{aligned} \phi^j(P \cup Q) &= \sum_1{}^j + \sum_2{}^j + \sum_3{}^j + \sum_4{}^j, \\ \phi^j(P \cap Q) &= \sum_1{}^j + \sum_5{}^j, \\ \phi^j(P) &= \sum_1{}^j + \sum_2{}^j + \sum_5{}^j + \sum_6{}^j, \\ \phi^j(Q) &= \sum_1{}^j + \sum_3{}^j + \sum_5{}^j + \sum_7{}^j. \end{aligned}$$

Thus, if $1 \leqq j \leqq d-2$,

$$\begin{aligned} \phi^j(P \cup Q) &= \sum_1{}^j + \sum_2{}^j + \sum_3{}^j + \sum_4{}^j \\ &= (\sum_1{}^j + \sum_2{}^j + \sum_5{}^j + \sum_6{}^j) + (\sum_1{}^j + \sum_3{}^j + \sum_5{}^j + \sum_7{}^j) \\ &\qquad - (\sum_1{}^j + \sum_5{}^j) + (\sum_4{}^j - \sum_6{}^j - \sum_7{}^j) - \sum_5{}^j \\ &= \phi^j(P) + \phi^j(Q) - \phi^j(P \cap Q) - \sum_5{}^{j-1} - \sum_5{}^j. \end{aligned} \tag{4.4}$$

Similar reasoning for the two remaining cases of $j = 0$, $j = d-1$, yields:

$$\phi^0(P \cup Q) = \phi^0(P) + \phi^0(Q) - \phi^0(P \cap Q) - \sum_5{}^0 \tag{4.5}$$

and

$$\phi^{d-1}(P \cup Q) = \phi^{d-1}(P) + \phi^{d-1}(Q) - 2\phi(P \cap Q) - \sum_5{}^{d-2}. \tag{4.6}$$

It then follows from (4.4), (4.5), (4.6), and the valuation property for ϕ that

$$\begin{aligned} \sum_{j=0}^{d} (-1)^j \phi^j(P \cup Q) &= \sum_{j=0}^{d} (-1)^j \phi^j(P) + \sum_{j=0}^{d} (-1)^j \phi^j(Q) \\ - \sum_{j=0}^{d-1} (-1)^j \phi^j(P \cap Q) &- \sum_{j=0}^{d-2} (-1)^j \sum_5{}^j - \sum_{j=0}^{d-2} (-1)^j \sum_5{}^{j-1} + (-1)^{d-2} \sum_5{}^{d-2}. \end{aligned} \tag{4.7}$$

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As the last three terms sum to 0, we can rewrite (4.7) as

$$\phi^*(P \cup Q) = \phi^*(P) + \phi^*(Q) - \phi^*(P \cap Q). \tag{4.8}$$

Thus, ϕ^* is a weak valuation on $\mathscr{P}$, and hence a valuation. This concludes the proof.

The applications which we shall make of the preceding theorem depend upon the following easy observation.

(4.9) *A valuation ϕ satisfies $E(\epsilon)$ on $\mathscr{P}$ if and only if the valuation $\phi^* - \epsilon\phi$ vanishes identically on $\mathscr{P}$.*

With the two preceding results, we can now prove (1.5).

Proof. By (4.9), the result is equivalent to showing that $\phi^* - \epsilon\phi$ vanishes identically on $\mathscr{P}$ if and only if it vanishes identically on $\mathscr{S}$. But this statement is proved in (2.11) and the proof of the theorem is complete.

5. A decomposition theorem for valuations. Having defined derived valuations, we can now formulate (1.6) in more detail.

(5.1) THEOREM. *Let ϕ be a valuation defined on $\mathscr{P}$. Then there exist two valuations α and β such that:*
$\phi(P) = \alpha(P) + \beta(P)$ for all $P \in \mathscr{P}$,
α satisfies $E(1)$ on $\mathscr{P}$, and
β satisfies $E(-1)$ on $\mathscr{P}$.
In fact, we can compute α and β explicitly by: $\alpha(P) = \frac{1}{2}[\phi(P) + \phi^*(P)]$, $\beta(P) = \frac{1}{2}[\phi(P) - \phi^*(P)]$, where ϕ^* is the valuation derived from ϕ.

A preliminary lemma is needed which is of independent interest.

(5.2) LEMMA. *Let ϕ be any valuation defined on $\mathscr{P}$, let ϕ^* be derived from ϕ and ϕ^{**} derived from ϕ^*. Then $\phi^{**} = \phi$.*

Proof. By (4.2) we know that ϕ^{**} is a valuation and by (2.12) we know that $\phi = \phi^{**}$ on $\mathscr{P}$ if and only if $\phi = \phi^{**}$ on $\mathscr{S}$. Thus, let $\mathscr{P}$ be any d-simplex with vertices $v_0, \ldots, v_d$. Then

$$\phi^{**}(P) = \sum_{j=0}^{d} (-1)^j \sum_{P^j \subseteq P} \phi^*(P^j), \tag{5.3}$$

where

$$\phi^*(P^j) = \sum_{k=0}^{j} (-1)^k \sum_{P^k \subseteq P^j} \phi(P^k).$$

Since P is a d-simplex, each k-face of P is contained in exactly $\binom{d-k}{j-k}$ j-faces and it easily follows that:

$$\sum_{P^j \subseteq P} \phi^*(P^j) = \sum_{k=0}^{j} (-1)^k \binom{d-k}{j-k} \sum_{P^k \subseteq P} \phi(P^k). \tag{5.4}$$

Substituting (5.4) into (5.3) and collecting all the terms corresponding to k-faces of P yields:

$$(5.5) \qquad \phi^{**}(P) = \sum_{k=0}^{d} (-1)^k \sum_{P^k \subseteq P} \phi(P^k) \left[\sum_{j=k}^{d} (-1)^j \binom{d-k}{j-k} \right].$$

We observe that the bracketed sum on the right-hand side is 0 if $k < d$, and 1 if $k = d$. Hence,

$$(5.6) \qquad \phi^{**}(P) = (-1)^d \phi(P) \cdot (-1)^d = \phi(P).$$

Since P was an arbitrary d-simplex, the result follows.

We can now prove (5.1). Let α and β be defined as in (5.1) and let α^* be derived from α. It is clear that

$$\alpha^* = \tfrac{1}{2}[\phi^* + \phi^{**}] = \tfrac{1}{2}[\phi^* + \phi] = \alpha$$

from the preceding lemma. Hence, $\alpha^* - \alpha = 0$, which is equivalent to saying that α satisfies $E(1)$ on $\mathscr{P}$. A similar argument shows that β satisfies $E(-1)$. Since $\phi = \alpha + \beta$ by construction, the proof is complete.

As a corollary to (5.2), we also have the following result.

(5.7) *Suppose that ϕ is a valuation on $\mathscr{P}$ which satisfies $E(\epsilon)$. Then $\epsilon^2 = 1$.*

6. Further relations on simple and quasi-simple polytopes. Once a function is known to satisfy a relation $E(\epsilon)$ for every polytope, it is not difficult to show that it satisfies further relations analogous to the well-known Dehn-Sommerville equations (**9**, § 7.1). Although the Dehn-Sommerville equations were originally stated for simplicial polytopes (polytopes in which every facet is a simplex), for our purposes it is more convenient to work with their duals, the simple polytopes.

A polytope is termed *simple* [*quasi-simple*] if every vertex [edge] is contained in exactly d facets [$d - 1$ facets].

(6.1) *Let ϕ be a function defined on $\mathscr{P}$ which satisfies $E(\epsilon)$ on $\mathscr{P}$. Then for any simple d-polytope P, the following relation holds for each k, $1 \leqq k \leqq d$, where the inner summation on the left is taken over all of the j-faces of P, and the summation on the right is taken over all the k-faces of P*:

$$(6.2) \qquad \sum_{j=0}^{k} (-1)^j \binom{d-j}{k-j} \sum \phi(F^j) = \epsilon \sum \phi(F^k).$$

Proof. By assumption, for each k-face P^k of P,

$$(6.3) \qquad \sum_{j=0}^{k} (-1)^j \sum_{P^j \subseteq P^k} \phi(P^j) = \epsilon \phi(P^k).$$

Since P is simple, it is easily verified that each j-face of P is contained in exactly $\binom{d-j}{k-j}$ k-faces of P. Summing (6.3) over all k-faces of P yields (6.2), thus completing the proof.

Note that if ϕ is a function which vanishes on all polytopes of dimension less than k, then the proof of (6.1) is valid whenever each k-face of P is contained in exactly $d - k$ facets. In particular, since a function satisfying $E(-1)$ vanishes on 0-polytopes, the following result is true.

(6.4) COROLLARY. *If ϕ is a function satisfying $E(-1)$ on $\mathcal{P}$, then ϕ satisfies each relation* (6.2) *for every quasi-simplicial polytope P.*

It should be noted, however, that not all of the relations given in (6.1) are linearly independent. In fact, the same methods as were used by Shephard for the special cases of Steiner points and mean widths in (**6**, § 4) and (**7**, § 3), respectively, can be used to prove the following more general results.

(6.5) *Let ϕ be a function satisfying $E(1)$ on $\mathcal{P}$ which does not vanish on all 0-polytopes. Then exactly $[\frac{1}{2}(d + 1)]$ of the relations of the type* (6.2) *are linearly independent* (*namely, those corresponding to the odd values of k not exceeding d*).

(6.6) *Let ϕ be a function satisfying $E(-1)$ on $\mathcal{P}$ which does not vanish on all 1-polytopes. Then exactly $[d/2]$ of the relations of the type* (6.2) *are linearly independent* (*namely, those corresponding to the even values of k not exceeding d*).

7. A uniqueness theorem. The results of the last section show that if a function satisfies one of the Euler relations, it satisfies many more relations on the class of simple polytopes. However, such results are false on the class of all polytopes. In fact, in this section we shall prove that if a continuous function satisfies an Euler relation, this is the only linear relation of this type that the function can satisfy for every polytope. This result generalizes the theorem of Grünbaum (**1**, § 8.2) which states that the f-vectors of all d-polytopes satisfy essentially only one linear relation.

(7.1) THEOREM. *Suppose that ϕ is a continuous function satisfying $E(\epsilon)$ on $\mathcal{P}$ such that $\phi(P) = 0$ for every polytope P if* $\dim(P) < k$, *and such that $\phi(P_0) \neq 0$ for some k-polytope P_0. If for $d \geqq k$ every d-polytope Q satisfies*

$$(7.2) \qquad B_k{}^d \sum \phi(Q^k) + \ldots + B_{d-1}^d \sum \phi(Q^{d-1}) + B_d{}^d \phi(Q) = 0,$$

where the sums are taken over all faces of Q of the dimension indicated, then

$$B_j{}^d = (-1)^{j-k} B_k{}^d \quad \text{for } k \leqq j \leqq d-1, \qquad B_d{}^d = [(-1)^{k+1}\epsilon + (-1)^{d-k}] B_k{}^d.$$

Two preliminary lemmas are needed. In what follows, conv(A, B) denotes the *convex hull* of A and B; that is, the smallest convex set containing both A and B. Observe that if A and B are polytopes, then conv(A, B) is also a polytope.

(7.3) *Let ϕ be a valuation on $\mathcal{P}$, P a d-polytope, $v \in$ int P, and let $R^{j+1} =$ conv(v, P^j) for any j-face of P. Then*

$$(7.4) \qquad \phi(P) = \sum \phi(R^d) - \sum \phi(R^{d-1}) + \ldots + (-1)^d \phi(v).$$

https://doi.org/10.4153/CJM-1968-142-0 Published online by Cambridge University Press

Proof. For any polytope Q, let $\eta(Q) = \phi(\text{conv}(v, Q))$. It is easily checked that η is a valuation, and thus η^*, the derived valuation, is one as well. We can then rewrite (7.4) as

$$(7.5) \qquad \phi(P) = (-1)^{d-1}[\eta^*(P) - (-1)^d\phi(P)] + (-1)^d\phi(v)$$

or

$$(7.6) \qquad \eta^*(P) = \phi(v).$$

To prove this, a routine computation shows that $\eta^*(Q) = \phi(v)$ for any simplex Q having v as a vertex. Since P can be written as a union of simplices $S_1, \ldots, S_n$, such that each intersection of them is a simplex having v as a vertex, it follows from (2.10) that

$$(7.7) \quad \eta^*(P) = \sum\eta^*(S_i) - \sum\eta^*(S_i \cap S_j) + \ldots + (-1)^{n-1}\eta^*(S_1 \cap \ldots \cap S_n)$$
$$= \phi(v)\left[\binom{n}{1} - \binom{n}{2} + \ldots + (-1)^{n-1}\binom{n}{n}\right] = \phi(v).$$

This completes the proof of the lemma.

(7.8) *Let ϕ be a valuation on $\mathscr{P}$ as described in* (7.1). *Then for each $d \geqq k + 2$ there exists a d-polytope P and a point $v \in$ int P such that $\phi(P) - \sum\phi(R^d) \neq 0$, where $R^d =$ conv(v, P^{d-1}) and the summation is taken over all $(d - 1)$-faces of P.*

Proof. Let $\phi(P, v)$ denote the difference in question. Suppose that P_0 is a k-polytope such that $\phi(P_0) \neq 0$. Then from (2.10) it follows that there exists a k-simplex, S, such that $\phi(S) \neq 0$.

Let P be a d-simplex having S as a k-face and all of the rest of its vertices $v_{k+1}, \ldots, v_d$, near a fixed vertex, p, of S. Choose v near p as well.

By the continuity of ϕ, with suitable choices of $v, v_{k+1}, \ldots, v_d$, we can approximate $\phi(v, P)$ as closely as we like by assuming that all of them are identical with the vertex p of S. An easy computation then shows that $\phi(v, P)$ can be made arbitrarily close to $(k + 1 - d)\phi(S)$. Since $\phi(S) \neq 0$, the conclusion follows.

Proof of (7.1). We shall use induction on dimension. The assertion is immediate if $d = k + 1$. Suppose that the result is known for all dimensions less than d. We shall prove it for d. For convenience, we shall omit the superscript "d" on the coefficients of (7.2).

Let Q be a given $(d - 1)$-polytope and let P be a pyramid over Q. If we allow the apex of P to collapse to a point $v \in$ int Q, then by the continuity of ϕ, it follows that:

$$(7.9) \qquad B_k\sum\phi(Q^k) + \ldots + B_{d-1}\phi(Q) + B_d\phi(Q)$$
$$+ B_k\sum\phi(R^k) + \ldots + B_{d-1}\sum\phi(R^{d-1}) = 0,$$

where $R^j =$ conv(v, Q^{j-1}) for $j > 0$, and $R^0 = v$, by convention.

Similarly, if $P^{\#}$ is a bipyramid over Q and we collapse both vertices of $P^{\#}$ which do not lie in Q to the point v, then

$$(7.10)\qquad B_k\sum\phi(Q^k) + \ldots + B_{d-2}\sum\phi(Q^{d-2}) + B_d\phi(Q) + 2B_k\sum\phi(R^k) + \ldots + 2B_{d-1}\sum\phi(R^{d-1}) = 0.$$

Subtracting (7.10) from twice (7.9) yields

$$(7.11)\qquad B_k\sum\phi(Q^k) + \ldots + B_{d-2}\sum\phi(Q^{d-2}) + [2B_{d-1} + B_d]\phi(Q) = 0.$$

From our induction hypothesis, it then follows that

$$(7.12)\qquad B_j = (-1)^{j-k}B_k \quad \text{for } k \leqq j \leqq d-2$$

and

$$(7.13)\qquad 2B_{d-1} + B_d = [(-1)^{k+1}\epsilon + (-1)^{d-1-k}]B_k.$$

Because of (1.4), ϕ is a valuation and we see from (7.3) that

$$(7.14)\qquad B_k[\sum\phi(R^k) + \ldots + (-1)^{d-1-k}\sum\phi(R^{d-1})] = (-1)^{d-k-1}B_k\phi(Q).$$

Subtracting (7.9) from (7.10) and simplifying by means of (7.12), (7.13), and (7.14) yields

$$(7.15)\qquad (B_{d-1} + (-1)^{d-k}B_k)(-\phi(Q) + \sum\phi(R^{d-1})) = 0.$$

If we choose Q and v so that the right-hand bracketed term does not vanish, which we know can be done by (7.8), then

$$(7.16)\qquad B_{d-1} + (-1)^{d-k}B_k = 0.$$

This last equation, together with (7.13), completes the proof.

Continuity is a necessary hypothesis in the theorem as the function $\lambda(P)$ given in (2.8) shows, since, for all polytopes, λ satisfies $E(1)$ as well as the following linear relation: $\sum\lambda(P^0) - 2\lambda(P) = 0$.

Remarks. The results dealing solely with valuations (such as § 2) would be valid for any valuation taking values in an arbitrary abelian group with characteristic different from 2 (possibly with a topology if the result deals with continuity questions). Statements involving an Euler relation are true if the range of the valuation involved lies in a module over a commutative ring with identity. For the results of the last section, we require, in addition, that the ring have no zero divisors.

Professor Perles has pointed out that we do not need a valuation for the results of § 5 to hold. In fact, (5.1), (5.2), and (5.5) are all true under the weaker assumption that ϕ is an arbitrary function on $\mathscr{P}$ (although in that case, we cannot expect α and β in (5.1) to be valuations).

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